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On the rate of convergence for perforated plates with a small interior Dirichlet zone

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Abstract. The aim of the paper is to compare the asymptotic behavior of solutions of two boundary value problems for an elliptic equation posed in a thin periodically perforated plate. In the first problem, we impose homogeneous Dirichlet boundary condition only at the exterior lateral boundary of the plate, while at the remaining part of the boundary Neumann condition is assigned. In the second problem, Dirichlet condition is also imposed at the surface of one of the holes. Although in these two cases, the homogenized problem is the same, the asymptotic behavior of solutions is rather different. In particular, the presence of perturbation in the boundary condition in the second problem results in logarithmic rate of convergence, while for non-perturbed problem the rate of convergence is of power-law type.

Mathematics Subject Classification (2000). 35B25 · 35B27 · 35J25.

Keywords. Perforated plate · Asymptotics · Homogenization · Convergence rate.

1. Formulation of mixed boundary value problems

1.1. Perforated plate and layer

Let ω be a smooth domain in the plane $\mathbb{R}^2$ with one-dimensional boundary $\partial\omega$ and the compact closure $\overline{\omega} = \omega \cup \partial\omega$. Given a small parameter $h \in (0, 1]$, we introduce the thin entire plate (Fig. 1)

$$\Omega_h = \omega \times (-h/2, h/2) \subset \mathbb{R}^3. \tag{1.1}$$

We also consider a perforated plate $\Omega(h)$ (Fig. 2), which is defined as follows. Denote by $\mathbb{Q}$ the unit cube

$$\mathbb{Q} = \{x = (x_1, x_2, x_3) : |x_j| < 1/2,\ j = 1, 2, 3\}$$

and by $q_j^{\pm}$ the faces of this cube:

$$q_j^{\pm} = \{x \in \partial\mathbb{Q} : x_j = \pm 1/2\}. \tag{1.2}$$

Let Θ be a closed connected subset of the union $\mathbb{Q} \cup q_3^{\pm}$ (the faces $q_3^{\pm}$ are overshadowed in Fig. 3). We assume that Θ is the closure of a domain with Lipschitz boundary and that the complement of Θ in $\mathbb{Q}$ is a connected set with Lipschitz boundary.

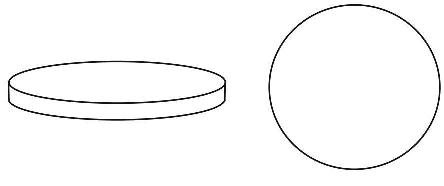

Fig. 1. The thin plate

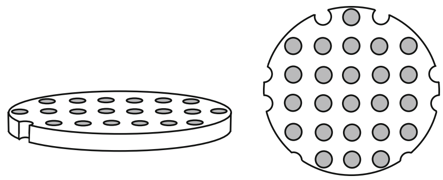

Fig. 2. The perforated plate

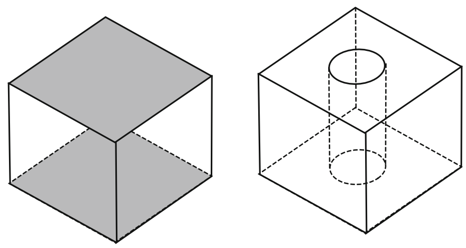

Fig. 3. Overshadowed faces $q_j^{\pm}$ of the unit cube $\mathbb{Q}$ and a closed set Θ in $\mathbb{Q}$

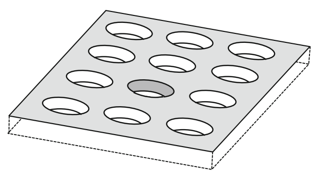

Fig. 4. The perforated layer

We use the notation

$$\Xi = \mathbb{Q} \setminus \Theta, \quad \varpi = \bigcup_{i=1,2} \bigcup_{\pm} q_i^{\pm}, \quad \mathcal{V} = \partial\Xi \setminus \varpi \tag{1.3}$$

and call ϖ and $\mathcal{V}$ respectively the lateral and staple boundary of the periodicity cell Ξ (Fig. 3). Notice that the cavity Θ does not touch ϖ (cf. Remark 4).

We introduce the perforated layer (Fig. 4)

$$\Pi = \left(\mathbb{R}^2 \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right) \setminus \bigcup_{\alpha \in \mathbb{Z}^2} \Theta^\alpha \tag{1.4}$$

and its h-compression

$$\Pi_h = \left\{x : \xi := h^{-1}x \in \Pi\right\}. \tag{1.5}$$

Here $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, and

$$\Theta^\alpha = \{x = (y, z) : (y - \alpha, z) \in \Theta\}. \tag{1.6}$$

We also define the rescaled sets

$$\Theta_h^\alpha = h\Theta^\alpha, \qquad \mathbb{Q}_h^\alpha = h(\mathbb{Q} + \alpha). \tag{1.7}$$

It is convenient to use the notation $x = (y, z)$ with $y = (y_1, y_2)$, $y_i = x_i$ for $i = 1, 2$, and $z = x_3$. The similar notation (η, ζ) will be used for the stretched coordinate system ξ in (1.5). The perforated plate

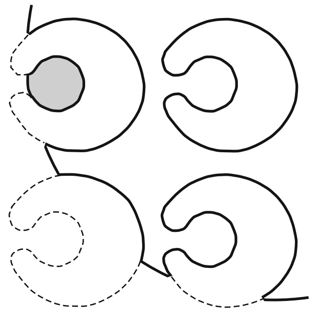

FIG. 5. Shaded fragment, although belongs to Ω_h, is not included in the perforated plate $\Omega(h)$

$\Omega(h)$ is then defined as the connected component with volume $O(h)$ of the set

$$\Omega_h \cap \Pi_h. \tag{1.8}$$

In other words, in $\Omega(h)$ we do not include small fragments (shaded in Fig. 5) which are cut away from the sets Θ_h^α by the lateral boundary $\Gamma_h = \partial\omega \times (-h/2, h/2)$ of the entire plate (1.1).

We assume the boundary $\partial\Omega(h)$ to be Lipschitz (cf. Remark 4).

1.2. Boundary value problems

In the domain $\Omega(h)$, we consider two mixed boundary value problems for the second-order divergence form scalar elliptic operator

$$L^h(x, \nabla_x) = -\nabla_x^\top a\left(h^{-1}x\right)\nabla_x, \tag{1.9}$$

where $\nabla_x = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)^\top$ is the gradient-operator regarded as a column of height 3, $\top$ stands for transposition and a is a matrix function of size 3×3, 1−periodic in the rapid variables $\xi_i = h^{-1}x_i$. We assume entries of a to be measurable functions in the cell Ξ while the matrix $a(\xi)$ is symmetric and satisfies

$$c_a|\tau|^2 \le \tau^\top a(\xi)\tau \le C_a|\tau|^2, \quad \tau\in\mathbb{R}^3, \quad C_a \ge c_a > 0, \tag{1.10}$$

for almost all $\xi\in\Xi$. In Sect. 5, we also assume some smoothness properties of the data. Then inequalities (1.10) ensure the uniform ellipticity of the operator (1.9). We denote

$$N^h(x, \nabla_x) = n^h(x)^\top a\left(h^{-1}x\right)\nabla_x \tag{1.11}$$

the Neumann boundary operator with n^h being the outward unit normal on the Lipschitz surface $\partial\Omega(h)$.

The first boundary value problem under consideration reads

$$L^h(x,\nabla_x)u^h(x) = f^h(x), \quad x\in\Omega(h), \tag{1.12}$$

$$N^h(x,\nabla_x)u^h(x) = 0, \quad x\in\Upsilon(h), \tag{1.13}$$

$$u^h(x) = 0, \quad x\in\Gamma(h), \tag{1.14}$$

where we have denoted

$$\Gamma(h) = \partial\Omega(h)\cap\Gamma_h, \qquad \Upsilon(h) = \partial\Omega(h)\setminus\overline{\Gamma(h)}. \tag{1.15}$$

In the case of insufficiently smooth data, the problem ought to be reformulated as the integral identity [1]

$$\left(a^h\nabla_x u^h, \nabla_x v^h\right)_{\Omega(h)} = \left(f^h, v^h\right)_{\Omega(h)}, \quad v^h\in\mathcal{H}^h, \tag{1.16}$$

where $a^h = a(h^{-1}x)$, $(\, , \,)_{\Omega(h)}$ is the natural inner product in the Lebesgue space $L^2(\Omega(h))$ and $\mathcal{H}^h = \mathring{H}^1(\Omega(h); \Gamma(h))$ is the Sobolev space of functions satisfying the Dirichlet condition (1.14).

To formulate the second boundary value problem under consideration, we suppose that the origin $\mathcal{O}$ lies inside the domain ω. Then there exists $h_0 > 0$ such that for all $h \in (0, h_0]$, the central periodicity cell

$$\Sigma_h^0 = \mathbb{Q}_h^0 \setminus \Theta_h^0 = \left\{ x : |x_j| < h/2, \ j = 1, 2, 3, \ h^{-1}x \notin \Theta \right\} \tag{1.17}$$

(cf. (1.7)) lies inside the perforated plate $\Omega(h)$. Let also $\gamma_h = \partial\Theta_h^0 \cap \mathbb{Q}_h^0$ denote the surface of the cavern. The boundary value problem in question is composed of the differential equation (1.12) in $\Omega(h)$ and the Dirichlet and Neumann conditions imposed on the surfaces $\Gamma_\bullet(h) = \Gamma(h) \cup \gamma_h$ and $\Upsilon_\bullet(h) = \Upsilon(h) \setminus \overline{\gamma_h}$, respectively. That is, we extend the Dirichlet conditions over the surface of the central cavern Θ_h^0. The differential form of this problem reads

$$L^h(x, \nabla_x) u_\bullet^h(x) = f^h(x), \quad x \in \Omega(h), \tag{1.18}$$

$$N^h(x, \nabla_x) u_\bullet^h(x) = 0, \quad x \in \Upsilon_\bullet(h), \tag{1.19}$$

$$u_\bullet^h(x) = 0, \quad x \in \Gamma_\bullet(h), \tag{1.20}$$

and corresponds to the integral identity

$$\left(a^h \nabla_x u_\bullet^h, \nabla_x v_\bullet^h\right)_{\Omega(h)} = \left(f^h, v_\bullet^h\right)_{\Omega(h)}, \quad v_\bullet^h \in \mathcal{H}_\bullet^h, \tag{1.21}$$

with

$$\mathcal{H}_\bullet^h = \mathring{H}^1(\Omega(h); \Gamma_\bullet(h)) \subset \mathcal{H}^h = \mathring{H}^1(\Omega(h); \Gamma(h)). \tag{1.22}$$

If $f^h \in L^2(\Omega(h))$, then both problems (1.16) and (1.21) admit unique solutions $u^h \in \mathcal{H}^h$ and $u_\bullet^h \in \mathcal{H}_\bullet^h$ which get additional differentiability properties under further smoothness assumptions.

The main purpose of the paper is to compare the asymptotic behavior of the solutions u^h and $u_\bullet^h$ of the above formulated *mixed* boundary value problems.

As for problem (1.12)–(1.14), the solution asymptotics is fully determined by the two-dimensional limit problem

$$-\nabla_y^\top A \nabla_y U(y) = F(y), \qquad y \in \omega, \tag{1.23}$$

$$U(y) = 0, \qquad y \in \partial\omega, \tag{1.24}$$

where $\nabla_y = (\partial/\partial y_1, \partial/\partial y_2)^\top$, A is a symmetric positive definite matrix in $\mathbb{R}^{2\times 2}$ constructed from the matrix function a in Ξ by means of the standard homogenization procedure (cf. [2, Sect. 6.1]), and F is a properly defined limit of f^h (see the next Section for the definition). The simplest but roughest formulation of available asymptotic results demonstrates the convergence of the three-dimensional solution u^h in a certain sense to the two-dimensional solution U. The same convergence occurs for the solution $u_\bullet^h$ with the enlarged Dirichlet zone $\Gamma_\bullet(h) \supset \Gamma(h)$, too. However, the asymptotic structure of the solution to problem (1.18)–(1.20) becomes much more elaborated and complicated due to the presence of a boundary layer in the vicinity of the origin. This boundary layer is described in terms of solutions of the following mixed boundary value problem in the perforated layer (1.16)

$$-\nabla_\xi^\top a(\xi) \nabla_\xi w(\xi) = \varphi(\xi), \ \xi \in \Pi, \tag{1.25}$$

$$-\nu(\xi)^\top a(\xi) \nabla_\xi \nu(\xi) = \psi(\xi), \ \xi \in \partial\Pi \setminus \overline{\gamma}, \tag{1.26}$$

$$w(\xi) = \beta(\xi), \ \xi \in \gamma, \tag{1.27}$$

where $\gamma = \partial\Theta \cap \partial\Pi$, and ν is the unit outward normal on $\partial\Pi$. Although the convergence itself dissembles the Dirichlet condition on the small surface piece γ_h, the appearance of the boundary layer and some other effects reduce the convergence rate $O\left(h^{1/2}\right)$ for u^h down to $O\left(|\ln h|^{-1}\right)$ for $u_\bullet^h$.

2. Preliminary description of the results

2.1. Homogenization of problems under consideration

In Sect. 3, we start with proving the Poincaré-Friedrichs inequality

$$\left\|u; L^2\left(\Omega\left(h\right)\right)\right\| \leq c \left\|\nabla_x u; L^2\left(\Omega\left(h\right)\right)\right\| \tag{2.1}$$

with a constant c independent of function $u \in \mathring{H}^1\left(\Omega\left(h\right); \Gamma\left(h\right)\right)$ and the geometrical parameter $h \in (0, h_0]$. Together with the positivity condition in (1.10) and the inclusion (1.22), the inequality (2.1) provides the common bound for the Sobolev norm of both solutions:

$$\left\|u^h; H^1\left(\Omega\left(h\right)\right)\right\|, \left\|u^h_\bullet; H^1\left(\Omega\left(h\right)\right)\right\| \leq C \left\|f^h; L^2\left(\Omega\left(h\right)\right)\right\|. \tag{2.2}$$

Furthermore, we prove the following convergence result involving some notation.

Definition 1. We say that a family of functions $\{f^h \in L^2(\Omega(h)) \;:\; h > 0\}$ converges weakly in $L^2(\Omega(h))$ to a function $f^0 \in L^2(\omega)$, as $h \to 0$, if

$$\frac{1}{h} \int\limits_{\Omega(h)} |f^h(x)|^2 \, \mathrm{d}x \leq C, \tag{2.3}$$

and for any $\varphi \in C^\infty(\mathbb{R}^3)$ it holds

$$\lim_{h \to 0} \frac{1}{h} \int\limits_{\Omega(h)} f^h(x)\varphi(x) \, \mathrm{d}x = S_0 \int\limits_{\omega} f^0(y)\varphi(y, 0) \, \mathrm{d}y \tag{2.4}$$

with $S_0 = \mathrm{meas}(\Xi) = \mathrm{meas}(\mathbb{Q} \setminus \Theta)$; here meas denotes the three-dimensional Lebesgue measure.

For example, it is easy to check that a family of functions $\{f^h \in L^2(\Omega(h)) \;:\; f^h(y,z) = \tilde{f}^h(y)$ for all $(y,z) \in \Omega(h)$ with $\tilde{f}^h \in L^2(\omega)\}$ converges in $L^2(\Omega(h))$, as $h \to 0$, if the family $\{\tilde{f}^h\}$ converges in $L^2(\omega)$.

It is also easy to see that if a family of continuous functions $\{f^h \;:\; h > 0\}$ converges to a function f^0 in the metric of uniform convergence, then f^h converges to $f^0(\cdot, 0)$ in $L^2(\Omega(h))$.

As was proved in [17], [18], any sequence $\{f^h \in L^2(\Omega(h)) \;:\; h > 0\}$ which satisfies estimate (2.3), contains a subsequence that converges weakly in $L^2(\Omega(h))$, as $h \to 0$.

Definition 2. We say that a family of functions $\{f^h \in L^2(\Omega(h)) \;:\; h > 0\}$ converges strongly in $L^2(\Omega(h))$ to a function $f^0 \in L^2(\omega)$, as $h \to 0$, if f^h converges to f^0 weakly in $L^2(\Omega(h))$, and

$$\lim_{h \to 0} \frac{1}{h} \|f^h;\; L^2(\Omega(h))\|^2 = S_0 \|f^0;\; L^2(\omega)\|^2.$$

Theorem 3. 1. *Let u^h be a solution of (1.12)–(1.14). Then, if f^h converges to f weakly in $L^2(\Omega(h))$, as $h \to 0$, then u^h converges strongly in $L^2(\Omega(h))$ to a solution of (1.23)–(1.24). Moreover,*

$$\lim_{h \to 0} \frac{1}{h} \int\limits_{\Omega(h)} \left|u^h(y,z) - U(y)\right|^2 \, \mathrm{d}y\mathrm{d}z = 0. \tag{2.5}$$

2. *A solution $u^h_\bullet$ of problem (1.18)–(1.20) converges strongly in $L^2(\Omega(h))$, as $h \to 0$, to the same limit U. Relation (2.5) also holds true.*

The coincidence of the limits has an elementary explanation: since the set γ_h shrinks to the isolated point $\mathcal{O} \in \omega$ and in view of the Sobolev embedding theorem, the trace at $\mathcal{O}$ of a function in $H^1(\omega)$ cannot be properly determined, the limit passage $h \to 0^+$ simply erases the Dirichlet condition on γ_h.

2.2. Asymptotics of solutions

In the preparatory Sect. 4, we formulate some results borrowed from [3,4] about solvability and decompositions at infinity of solutions of problem (1.25)–(1.27) in the perforated layer Π (Fig. 4). In Sect. 5, we employ a solution of this problem to compensate a discrepancy left in the Dirichlet condition on γ_h by the regular asymptotic terms (those terms compose the whole asymptotic form for u^h; compare the statements of Theorem 16 and Theorem 11). We emphasize that the energy solution of (1.25)–(1.27) with the finite Dirichlet integral $\left\|\nabla_\xi w; L^2(\Pi)\right\|^2$ stabilizes at infinity to a constant (see Sect. 4) and therefore, can loose the intrinsic decay property of a boundary layer so that we need to strive for and achieve the necessary decay in an artificial way.

The formal asymptotics of $u^h_\bullet(x)$, constructed in Sect. 5, does crucially differ from the asymptotics of $u^h(x)$. Apart from the boundary layer effect mentioned earlier, we have to use the unbounded, with the logarithmic singularity, solution of the limit problem (1.23), (1.24). In other words, we pose this problem in the punctured domain

$$\omega^{\odot} = \omega \setminus \mathcal{O}, \tag{2.6}$$

observing that the customary homogenization procedure leads to equation (1.23) in a point $y \in \omega$ only in the case when the Neumann condition (1.19) occurs on the surface $\{x \in \partial\Omega(h) : dist(x,(y,0)) < ch\}$ for a sufficiently small $h > 0$. The new solution takes the form

$$U(y) = U^0(y) + bG(y) \tag{2.7}$$

where $U^0 \in \mathring{H}^1(\omega; \partial\omega)$ is the energy solution of problem (1.23), (1.24), G is the Green function of the Dirichlet problem (1.23)–(1.24) with the singularity at the point $\mathcal{O}$, and the coefficient b should be chosen to provide the decay of the boundary layer $w(\xi)$. Owing to the representation

$$G(y) = -C_\Phi \ln|y| + O(1), \quad y \to \mathcal{O}, \tag{2.8}$$

and the evident relationship

$$\ln|y| = \ln|\eta| + \ln h, \tag{2.9}$$

the coordinate dilation $x \mapsto \xi = h^{-1}x$ brings the large parameter $|\ln h|$ into the resultant equation for b and finally gives the formula

$$b = -U(\mathcal{O})H, \quad H = (C_\Phi|\ln h| + C(\Xi,\omega,a))^{-1} \tag{2.10}$$

where $C(\Xi,\omega,a)$ is a certain number determined by the domains Ξ, ω, and the matrix a. In other words, the main asymptotic term of $u^h_\bullet(x)$ gets a complicated conglomerate structure and depends linearly on the parameter H in (2.10), i.e. implies a rational function of $|\ln h|$.

The effect of the rational dependence on $|\ln h|$ of solutions to the Dirichlet problem for a second-order scalar elliptic equation in a two-dimensional domain with a small hole was discovered in [5] (see also [6]). In paper [8] (see also [9]), the advanced asymptotic analysis of general boundary value problems for elliptic systems in domains with singular perturbed boundaries detected much more complicated asymptotic structures. In particular, it was established in [9, Chap. 2,4] that the rational dependence on $|\ln h|$ is governed by the logarithm in the fundamental solution (cf. (2.8)) or its substituter in the case of perturbed conical or corner point (see [10] and [11,20]). In contrast, the polynomial dependence on $|\ln h|$ in the lower-order asymptotic terms occurs much more often and may be maintained directly by specific right-hand sides in the boundary value problem.

Here we mention papers [21–28] where the rational dependence on $|\ln h|$ have been found in particular problems of mathematical physics and concomitant effects have been studied. We emphasize that asymptotic expansions [24,26,28] (see also [9, Chaps. 9, 10]) in spectral problems get extremely intricate structure involving the holomorphic dependence on the parameter H in (2.10). This complication may provoke mistakes (see, e.g., [29,30] together with a necessary correction and explanation in [26]).

Our investigation into the boundary layer phenomenon is based on the results in [3,4] describing the asymptotic behavior at infinity of solutions to problem (1.25)–(1.27). We emphasize that the results of this type in the perforated layer (1.4) are known only for scalar second-order equation, while in the intact layer $\mathbb{R}^2 \times (-1/2, 1/2)$ such asymptotics have also been constructed for the elasticity system [33] and the Stokes equations [34,35]. However, to the best knowledge of the authors, a rigorous study of elliptic systems and even scalar equations in locally perturbed three-dimensional entire plate (1.1) is not performed yet (see [36] for formal asymptotic analysis). Angular boundary layers for differential equations with rapidly oscillating coefficients were constructed in [37–39], but without observing any rational dependence on the logarithm.

Previously, elasticity problems in thin plates of rapidly varying thickness with periodic profile have been studied in [13–15] and then in several other works, under the assumption that the Neumann boundary condition is imposed on the upper and lower surfaces of the plate. In these works the homogenization result was obtained under additional assumptions on the plate geometry, these additional assumptions were partly weakened in the subsequent works. Under the weakest restrictions on the geometry, this problem was studied in [2, Chap. 6] in the periodic case, and in [19] in the locally periodic case. The corresponding non-stationary model has been considered in [12], where the homogenization problem for a non-stationary elasticity system in a thin plate with rapidly oscillating periodic profile has been investigated. The work [16] deals with non-linear elastic thin films having rapidly oscillating profile. By means of Γ-convergence approach, the authors derive the limit non-linear model.

There is an extensive mathematical literature on linear and non-linear elastic thin plates, films, and membranes. However, the discussion on this topic is out of the scope of this paper.

3. The results on convergence

This section is devoted to the proof of Theorem 3. It relies on the Γ-convergence arguments adapted to the convergence in variable spaces $L^2(\Omega(h))$, and Poincaré–Friedrichs inequality (2.1). This inequality is proved at the beginning of the section.

3.1. The Poincaré–Friedrichs inequality

We proceed with proving the inequality (2.1). To this end, we, first of all, extend a function $u^h \in \mathring{H}^1(\Omega(h); \Gamma(h))$ onto the perforated thin layer Π_h (see (1.5)) by setting $u^h = 0$ for $x \in \Pi_h \setminus \Omega(h)$. Then, we consider a cell $\Xi_h^\alpha = \mathbb{Q}_h^\alpha \setminus \Theta_h^\alpha$ (cf. (1.3)–(1.6)) and write down the decomposition

$$u^h(x) = u_\alpha^{h\perp}(x) + \overline{u}_\alpha^h, \quad \overline{u}_\alpha^h = (\mathrm{meas}_3(\Xi_h^\alpha))^{-1} \int_{\Xi_h^\alpha} u^h(x)\,\mathrm{d}x. \tag{3.1}$$

Owing to the evident orthogonality condition

$$\int_{\Xi_h^\alpha} u_\alpha^{h\perp}(x)\,\mathrm{d}x = 0,$$

the Poincaré inequality ensures that

$$\left\|u_\alpha^{h\perp}; L^2(\Xi_h^\alpha)\right\|^2 \le ch^2 \left\|\nabla_x u_\alpha^{h\perp}; L^2(\Xi_h^\alpha)\right\|^2 = ch^2 \left\|\nabla_x u^h; L^2(\Xi_h^\alpha)\right\|^2. \tag{3.2}$$

Hence, applying the continuous extension operator (see, e.g., [1])

$$H^1(\Xi) \ni v \mapsto \widehat{v} \in H^1(\mathbb{Q})$$

in the stretched coordinates $\xi = h^{-1}x$ (cf. (1.5)), we obtain the functions $\widehat{u}_\alpha^{h\perp}$ and $\widehat{u}_\alpha^h = \widehat{u}_\alpha^{h\perp} + \overline{u}_\alpha^h$ in $H^1(\mathbb{Q})$ such that $\widehat{u}_\alpha^h = u$ on Ξ_α^h and

$$\begin{aligned}\left\|\nabla_x \widehat{u}_\alpha^h; L^2(\mathbb{Q}_h^\alpha)\right\|^2 &= \left\|\nabla_x \widehat{u}_\alpha^{h\perp}; L^2(\mathbb{Q}_h^\alpha)\right\|^2 \\ &\le \widehat{c}\left(\left\|\nabla_x u_\alpha^{h\perp}; L^2(\Xi_h^\alpha)\right\|^2 + h^{-2}\left\|u_\alpha^{h\perp}; L^2(\Xi_h^\alpha)\right\|^2\right) \\ &\le C\left\|\nabla_x u; L^2(\Xi_h^\alpha)\right\|^2.\end{aligned}$$

Since the cavity Θ_h^α lies inside the cube $\mathbb{Q}_h^\alpha$ and does not touch its lateral faces $q_{1h}^{\pm\alpha}$ and $q_{2h}^{\pm\alpha}$ (cf. (1.2)), the local extension procedure described above, serves for the whole perforated layer Π_h while the obtained function $\widehat{u}^h$ meets the estimate

$$\left\|\nabla_x \widehat{u}^h; L^2(\Pi_h)\right\|^2 \le C\left\|\nabla_x u^h; L^2(\Omega(h))\right\|^2 \tag{3.3}$$

and has a support in the set

$$\left\{x = (y,z) : \operatorname{dist}(y,\omega) \le \sqrt{2}h, \quad |z| \le h/2\right\}. \tag{3.4}$$

Thus, Friedrich's inequality in two variables $y = (y_1, y_2)$ yields

$$\begin{aligned}\left\|u^h; L^2(\Omega(h))\right\|^2 &= \left\|u^h; L^2(\Pi_h)\right\|^2 = \left\|\widehat{u}^h; L^2(\Pi_h)\right\|^2 \le \left\|\widehat{u}^h; L^2\left(\mathbb{R}^2 \times (-h/2, h/2)\right)\right\|^2 \\ &\le c\left\|\nabla_x \widehat{u}^h; L^2\left(\mathbb{R}^2 \times (-h/2, h/2)\right)\right\|^2 \le C\left\|\nabla_x u^h; L^2(\Omega(h))\right\|^2.\end{aligned} \tag{3.5}$$

Here c is a positive constant depending on $\operatorname{diam}(\omega) + \sqrt{2}h$ only (see, e.g., [1]) and therefore, c can be fixed independently of $h \in (0, 1]$. Thus, the inequality (2.1) is proved.

Remark 4. Inequality (2.1) remains valid without the restriction $\Theta \subset \mathbb{Q} \cup q_3^+ \cup q_3^-$ (Sect. 1). To derive it for a general perforated plate one may employ the tetris procedure developed in [19,40]. Our requirement of the Lipschitz boundary $\partial\Xi$ is also superfluous; cf. [7] where examples of non-Lipschitz domains with or without the compact embedding $H^1(\Xi) \subset L^2(\Xi)$ are listed and certain sufficient conditions are derived. However, in order to simplify the presentation, we here avoid those possible generalizations.

The asymptotic ansatz for the solution $u^h(x)$ of the problem (1.12)–(1.14) in the perforated plate $\Omega(h)$ looks as follows:

$$u^h(x) = U(y) + h\sum_{i=1}^{2} Y_i\left(h^{-1}x\right)\frac{\partial U}{\partial y_i}(y) + \cdots \tag{3.6}$$

where Y_i is the standard asymptotic corrector, i.e., a periodic solution of the following Neumann problem in the periodicity cell

$$-\nabla_\xi^\top a(\xi)\nabla_\xi Y_i(\xi) = \nabla_\xi^\top a(\xi) e_{(i)}, \ \xi \in \Xi, \tag{3.7}$$

$$\nu(\xi)^\top a(\xi)\nabla_\xi Y_i(\xi) = -\nu(\xi)^\top a(\xi) e_{(i)}, \ \xi \in \mathcal{V}, \tag{3.8}$$

where $e_{(i)}$ is the unit vector of the y_i-axis and ν the unit outward normal on the staple boundary $\mathcal{V}$ of the cell (see (1.3)). As usual, we do not write down explicitly the periodicity conditions on the lateral side ϖ of the cell; however, the variational formulation of problem (3.7), (3.8),

$$\left(a\nabla_\xi Y_i, \nabla_\xi V\right)_\Xi = -\left(a e_{(i)}, \nabla_\xi V\right)_\Xi, \ V \in H_{per}^1(\Xi), \tag{3.9}$$

relies upon the Sobolev space $H_{per}^1(\Xi)$ of functions, 1-periodic in y_1 and y_2. Due to the formula

$$\int_\Xi \nabla_\xi^\top a(\xi)\,\mathrm{d}\xi = \int_{\mathcal{V}} \nu(\xi)^\top a(\xi)\,\mathrm{d}s_\xi,$$

problem (3.9) admits a solution $Y_i \in H^1_{per}(\Xi)$ which becomes unique under the orthogonality condition

$$\int\limits_{\Xi} Y_i(\xi)\,\mathrm{d}\xi = 0. \tag{3.10}$$

If the data a and $\mathcal{V}$ are smooth, the solution Y_i gets additional differentiability properties and can be regarded as the classical solution of problem (3.7), (3.8).

The (2×2)-matrix A in (1.23) has the following entries:

$$A_{jk} = \int\limits_{\Xi} e_{(j)}^{\top}\left(a(\xi)\,e_{(k)} + a(\xi)\,\nabla_\xi Y_k(\xi)\right)\mathrm{d}\xi =$$
$$= \int\limits_{\Xi}\left(e_{(j)} + \nabla_\xi Y_j(\xi)\right)^{\top} a(\xi)\left(e_{(k)} + \nabla_\xi Y_k(\xi)\right)\mathrm{d}\xi. \tag{3.11}$$

It is obvious that A is symmetric and positive definite (see, e.g., [2, Sect. 6.1]).

3.2. Proof of Theorem 3

In order to show that solutions u^h of problem (1.12)–(1.14) and solutions $u^h_\bullet$ of problem (1.18)–(1.20) have the same limit as $h \to 0$, it is convenient to apply the Γ-convergence argument. Denote

$$J^h(u) = \frac{1}{h}\int\limits_{\Omega(h)}\left(a^h(x)\nabla_x u(x)\cdot\nabla_x u(x) - 2f^h(x)u(x)\right)\mathrm{d}x, \qquad u \in H^1(\Omega(h)).$$

The function u^h, which solves problem (1.12)–(1.14), provides a unique minimum in the following minimization problem

$$\min_{u\in \mathring{H}^1(\Omega(h),\Gamma(h))} J^h(u). \tag{3.12}$$

We extend $J^h(u)$ to the space $L^2(\Omega(h))$ by setting $J^h(u) = +\infty$ for $u \in L^2(\Omega(h)) \setminus \mathring{H}^1(\Omega(h),\Gamma(h))$.

Let $\{w^h\}$ be a sequence of functions from $H^1(\Omega(h),\Gamma(h))$ such that $J^h(w^h) \le C$. We are going to show that for such a sequence the upper bound holds

$$\|w^h;\, H^1(\Omega(h))\| \le C\sqrt{h}. \tag{3.13}$$

To this end, we combine the upper bound $J^h(w^h) \le C$ with (2.3). This yields

$$\int\limits_{\Omega(h)} a^h(x)\nabla_x w^h(x)\cdot\nabla_x w^h(x)\,\mathrm{d}x \le C\|f^h;\, L^2(\Omega(h))\|\,\|w^h;\, L^2(\Omega(h))\| + Ch$$
$$\le C\sqrt{h}\|w^h;\, L^2(\Omega(h))\| + Ch.$$

Using (2.1) we obtain

$$\int\limits_{\Omega(h)} a^h(x)\nabla_x w^h(x)\cdot\nabla_x w^h(x)\,\mathrm{d}x \le C\sqrt{h}\|\nabla w^h;\, L^2(\Omega(h))\| + Ch,$$

which implies the desired bound (3.13).

We extend the functions f^h into Ω_h by setting $f^h = 0$ for $x \in \Omega_h \setminus \Omega(h)$ and keep for the extended function the same notation f^h.

Under our assumptions on the geometry of Θ, the extension result proved in [31, Theorem 2.1] can be easily adapted to the case of functions defined in $\Omega(h)$ so that there exists a family of extension operators $E_h : \mathring{H}^1(\Omega(h), \Gamma(h)) \mapsto \mathring{H}^1(\Omega_h, \Gamma_h)$, such that $(E_h w)(x) = w(x)$ for $x \in \Omega(h)$, and

$$\|E_h w;\, L^2(\Omega_h)\| \le C\|w;\, L^2(\Omega(h))\|, \qquad \|\nabla_x(E_h w);\, L^2(\Omega_h)\| \le C\|\nabla_x w;\, L^2(\Omega(h))\|$$

with a constant C that does not depend on $h \in (0, h_0]$. Indeed, it suffices to reflect the function w and the domain $\Omega(h)$ with respect to the upper surface $\{x : z = h/2\}$ of Π_h and then extend the resulting function periodically in z. Due to our assumptions on the geometry, Theorem 2.1 from [31] applies to the extended function and yields the desired estimates. For the sake of brevity, the notation w^h is kept for the extended function $E_h w^h$.

We denote

$$\widehat{w}^h(y) = \frac{1}{h}\int\limits_{-h/2}^{h/2} (E_h w^h)(y,z)\,\mathrm{d}z, \qquad \widehat{f}^h(y) = \frac{1}{h}\int\limits_{-h/2}^{h/2} f^h(y,z)\,\mathrm{d}z.$$

Clearly, $\|\widehat{f}^h;\, L^2(\omega)\| \le h^{-1}\|f^h;\, L^2(\Omega_h)\| \le C$. It also readily follows from (2.4) that $\widehat{f}^h$ converges weakly in $L^2(\omega)$ to $S_0 f(\cdot)$, as $h \to 0$.

Assume now that a family $\{w^h\}_{h\in(0,h_0]}$, $w^h \in \mathring{H}^1(\Omega(h), \Gamma(h))$, satisfies the upper bound $J^h(w^h) \le C$. Then (3.13) holds, and the extended function belongs to $\mathring{H}^1(\Omega_h, \Gamma_h)$ and meets the estimate

$$\frac{1}{h}\|w^h;\, \mathring{H}^1(\Omega_h, \Gamma_h)\|^2 \le C. \tag{3.14}$$

By the Poincare inequality, we have

$$\frac{1}{h}\int\limits_{\Omega_h} |w^h(y,z) - \widehat{w}^h(y)|^2\,\mathrm{d}z\mathrm{d}y \le Ch\|w^h;\, \mathring{H}^1(\Omega_h, \Gamma_h)\|^2 \le Ch^2. \tag{3.15}$$

Applying the Jenssen inequality, we obtain

$$\int\limits_{\omega} |\nabla_y \widehat{w}^h(y)|^2 \le \frac{1}{h}\int\limits_{\Omega_h} |\nabla_x w^h(x)|^2\,\mathrm{d}x \le C.$$

Since $\widehat{w}^h|_{\partial\omega} = 0$, then the family $\{\widehat{w}^h\}_{h\in(0,h_0]}$ is compact in $L^2(\omega)$. Therefore, for a subsequence,

$$\widehat{w}^h \longrightarrow w^0, \qquad \text{as } h \to 0, \quad \text{in } L^2(\omega).$$

Combining this convergence with (3.15) yields

$$\lim_{h\to 0}\ \frac{1}{h}\int\limits_{\Omega_h} |w^h(y,z) - w^0(y)|^2\,\mathrm{d}z\mathrm{d}y \;=\; 0.$$

This implies that

$$\lim_{h\to 0}\ \frac{1}{h}\int\limits_{\Omega_h} w^h(x) f^h(x)\mathrm{d}x \;=\; \int\limits_{\omega} w^0(y) f(y)\mathrm{d}y.$$

We now recall the definition of Γ-convergence. This definition is adapted to the problem under consideration.

Given a family of functionals $\mathcal{J}^h : L^2(\Omega(h)) \mapsto \mathbb{R} \cup \{+\infty\}$, we say that $\mathcal{J}^h$ Γ-converges in $L^2(\Omega(h))$ to a functional $\mathcal{J}^0 : L^2(\omega) \mapsto \mathbb{R} \cup \{+\infty\}$ if

1. For any sequence $\{v^h \in L^2(\Omega(h))\}$ such that v^h converges weakly in $L^2(\Omega(h))$ to a function $v^0 \in L^2(\omega)$, it holds

$$\liminf_{h\to 0} \mathcal{J}^h(v^h) \geq \mathcal{J}^0(v^0). \tag{3.16}$$

For any $v \in L^2(\omega)$ there is a sequence $\{v^h\}$, $v^h \in L^2(\Omega(h))$ such that

$$\limsup_{h\to 0} \mathcal{J}^h(v^h) \leq \mathcal{J}^0(v^0). \tag{3.17}$$

Proposition 5. *The family J^h Γ-converges in $L^2(\Omega(h))$ to a functional $J^0 : L^2(\omega) \mapsto \mathbb{R} \cup \{+\infty\}$ with*

$$J^0(w) = \begin{cases} \int\limits_\omega A\nabla_y w \cdot \nabla_y w \, \mathrm{d}y - 2\int\limits_\omega wf \, \mathrm{d}y, & \text{if } w \in H_0^1(\omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

Proof. We first justify the liminf inequality (3.16). Consider a family w^h, $w^h \in L^2(\Omega(h))$, such that w^h converges weakly in $L^2(\Omega(h))$ to some function $w^0 \in L^2(\omega)$, and $\liminf\limits_{h\to 0} J^h(w^h) < +\infty$. Then for a subsequence $h_k \to 0$ we have

$$\lim_{k\to\infty} J^{h_k}(w^{h_k}) = \liminf_{h\to 0} J^h(w^h).$$

Clearly, $w^{h_k} \in \mathring{H}^1(\Omega(h), \Gamma(h))$. As was proved in (3.13), $\|w^{h_k}; \mathring{H}^1(\Omega(h), \Gamma(h))\| \leq C\sqrt{h}$. Thus $w^0 \in H_0^1(\omega)$, and

$$\lim_{k\to\infty} \frac{1}{h} \int\limits_{\Omega(h)} w^{h_k}(x) f^{h_k}(x) \, \mathrm{d}x = \int\limits_\omega w^0(y) f(y) \, \mathrm{d}y.$$

By the standard Γ-convergence argument (see [32]), it can be shown that

$$\liminf_{k\to\infty} \frac{1}{h} \int\limits_{\Omega(h)} a^{h_k} \nabla_x w^{h_k}(x) \cdot \nabla_x w^{h_k}(x) \, \mathrm{d}x \geq \int\limits_\omega A\nabla_y w^0(y) \cdot \nabla_y w^0(y) \, \mathrm{d}y,$$

and the desired liminf inequality follows.

Since the functional $J^0(w)$ is continuous in $H^1(\omega)$ norm, it suffices to prove the limsup inequality (3.17) for a dense in $\mathring{H}^1(\omega)$ set. For $w^0 \in C_0^\infty(\omega)$, letting

$$w^h(x) = w^0(y) + hY\left(\frac{x}{h}\right) \nabla_y w^0(y), \qquad x \in \Omega(h),$$

we derive the limsup inequality. This completes the proof of Proposition 5. □

The first statement of Theorem 3 is now straightforward. Indeed, since the family of minimizers u^h of the functionals J^h admits a subsequence which converges strongly in $L^2(\Omega(h))$, then, after taking a subsequence, u^h converges to a minimizer of the limit functional J^0. The required statement is now a consequence of the uniqueness of this minimizer.

In order to prove the second statement, we introduce the functional

$$J^h_\bullet(w) = \begin{cases} J^h(w), & \text{if } w \in \mathring{H}^1(\Omega(h), \Gamma_\bullet(h)), \\ +\infty, & \text{otherwise.} \end{cases}$$

We are going to show that

$$\Gamma - \lim_{h\to 0} J^h_\bullet = J^0. \tag{3.18}$$

Since, by the definition of J^h and $J^h_\bullet$, we have $J^h_\bullet(w) \geq J^h(w)$ for any $w \in L^2(\Omega(h))$, then we should only prove the limsup inequality. Using the continuity of J^0 in $H_0^1(\omega)$ and the standard lower-semicontinuity arguments, we conclude that it is sufficient to prove the limsup inequality for $w^0 \in C_0^\infty(\omega)$.

Let ϕ be a function in $C^\infty(\mathbb{R})$ such that $0 \le \phi \le 1$, $\phi(s) = 0$ for $s \le 3/2$, and $\phi(s) = 1$ for $s > 2$. Denote $\phi^h(x) = \phi(\log h/\log|y|)$ and $w^h = w^0(y) + hY\left(h^{-1}x\right)\nabla w^0(y)$. Straightforward computations show that

$$\lim_{h\to 0}\frac{1}{h}\|(w^h - w^h\phi^h);\, H^1(\Omega(h))\|^2 = 0.$$

Therefore,

$$\lim_{h\to 0}\frac{1}{h}\int\limits_{\Omega(h)} a^h\nabla_x(w^h\phi^h)\cdot\nabla_x(w^h\phi^h)\,\mathrm{d}x = \lim_{h\to 0}\frac{1}{h}\int\limits_{\Omega(h)} a^h\nabla_x w^h\cdot\nabla_x w^h\,\mathrm{d}x = \int\limits_{\omega} A\nabla_y w^0\cdot\nabla_y w^0\,\mathrm{d}y,$$

which yields (3.18). The second statement of Theorem 3 is an immediate consequence of (3.18). □

If the domain ω is smooth, and the function $f^h = f$ is also smooth and does not depend on h, then the limit function U is smooth in ω. Substituting the first three terms of expansion (3.6) for a solution of problem (1.12)–(1.14), and using estimate (2.2) and the maximum principle, after straightforward computations we conclude that

$$\frac{1}{h}\int\limits_{\Omega(h)} |u^h(x) - U(y)|^2\,\mathrm{d}x \le Ch^2, \tag{3.19}$$

and

$$\frac{1}{h}\int\limits_{\Omega(h)} |\nabla u^h(x) - \nabla\left(U(y) + hY\left(\frac{x}{h}\right)\cdot\nabla U(y)\right)|^2\,\mathrm{d}x \le Ch.$$

The asymptotics of $u^h_\bullet$ is much more tricky. It is studied in the following sections.

4. The problem in perforated layer

In this section we, based on results in [3,4], formulate a number of statements for the mixed boundary value problem in perforated layer Π (see 1.4). First, we give a weighted inequality of Friedrichs' type that assures the solvability of problem (1.25)–(1.27) in the intrinsic energy space $\widehat{W}$ (Lemma 6). Then we present an asymptotic expansion, as $|y|\to\infty$, of the energy solution $w \in \widehat{W}$ and construct a solution z of the homogeneous problem that has the logarithmic growth at infinity. This allows us to introduce the logarithmic capacity (see Remark 9) and to determine the coefficient c_w in the asymptotics of w (formulae (4.9) and (4.18)).

4.1. Solvability of the problems in weighted spaces

As a consequence of the Hardy inequality with logarithm

$$\int\limits_0^1 \rho^{-1}|\ln\rho|^{-2}|W(\rho)|^2\,\mathrm{d}\rho \le 4\int\limits_0^1 \rho\left|\frac{\mathrm{d}W}{\mathrm{d}\rho}(\rho)\right|^2\mathrm{d}\rho, \qquad W\in C^1_c[0,1), \tag{4.1}$$

the weighted inequality

$$\left\|\left((1+\rho)^{-1}\left(1+(\ln\rho)_+\right)^{-1}w\right); L^2(\Pi)\right\|^2 \le c_R\left(\left\|\nabla_\xi w; L^2(\Pi)\right\|^2 + \left\|w; L^2(\mathbb{B}_R\cap\Pi)\right\|^2\right), \quad w\in C^1_c\left(\overline{\Pi}\right), \tag{4.2}$$

is proved in [4]; here $(t)_+ = \frac{1}{2}(t+|t|)$ is the positive part of $t\in\mathbb{R}$,

$$\mathbb{B}_R = \left\{\xi = (\eta,\zeta)\in\mathbb{R}^3 : \rho := |\eta| < R\right\}$$

is a ball, and the radius $R \geq \frac{1}{2}\sqrt{3}$ is chosen in such a way that $\Xi \subset \mathbb{B}_R$. Due to the Dirichlet condition (1.27) on the surface γ, we have

$$\left\|w; L^2\left(\mathbb{B}_R \cap \Pi\right)\right\|^2 \leq c_R \left\|\nabla_\xi w; L^2\left(\mathbb{B}_R \cap \Pi\right)\right\|^2.$$

By W we denote Hilbert space obtained as a completion of $C_c^\infty\left(\overline{\Pi} \diagdown \gamma\right)$ (infinitely differentiable functions with compact supports) with respect to the norm

$$\left(\left\|\nabla_\xi w; L^2\left(\Pi\right)\right\|^2 + \left\|\left(1+\rho\right)^{-1}\left(1+\left(\ln\rho\right)_+\right)^{-1} w; L^2\left(\Pi\right)\right\|^2\right)^{1/2}. \tag{4.3}$$

A completion of $C_c^\infty\left(\overline{\Pi}\right)$ in the same norm is denoted by $\widehat{W}$.

The following facts can be readily verified. First, the space W is the completion of $C_c^\infty\left(\overline{\Pi}\diagdown\gamma\right)$ with respect to the Dirichlet integral norm $\left\|\nabla_\xi w; L^2\left(\Pi\right)\right\|$. Second, a constant function is an element of $\widehat{W}$. Furthermore, the following assertion is derived in [4] with the help of the Riesz representation theorem.

Lemma 6. *Let $\varphi \in L^2_{loc}\left(\Pi\right)$, $\psi \in L^2_{loc}\left(\partial\Pi\diagdown\gamma\right)$ satisfy*

$$\left(1+\rho\right)\left(1+\left(\ln\rho\right)_+\right)\varphi \in L^2\left(\Pi\right), \qquad \left(1+\rho\right)\left(1+\left(\ln\rho\right)_+\right)\psi \in L^2\left(\partial\Pi\diagdown\gamma\right), \tag{4.4}$$

and let β be the trace on γ of a function $B \in H^1\left(\Pi \cap \mathbb{B}_R\right)$. The following variational problem corresponding to (1.25)–(1.27), has a unique solution: to find $w \in \widehat{W}$ such that $w - B = 0$ on γ, and the integral identity

$$\left(a\nabla_\xi w, \nabla_\xi v\right)_\Pi = \left(\varphi, v\right)_\Pi + \left(\psi, v\right)_{\partial\Pi}, \ v \in W, \tag{4.5}$$

is valid. Its solution meets the estimate

$$\left\|w; \widehat{W}\right\| \leq c\left(N\left(\varphi,\psi\right) + \left\|B; H^1\left(\Pi\cap\mathbb{B}_R\right)\right\|\right), \tag{4.6}$$

where $\left\|w; \widehat{W}\right\|$ is the norm (4.3), and $N\left(\varphi,\psi\right)$ is the sum of norms of the expressions indicated in (4.4).

4.2. Asymptotics of the solution

In what follows, we suppose the boundary $\partial\Pi$ to be of the Hölder class $C^{2,\alpha}$, $\alpha > 0$. This is to apply directly a result in [4] which was actually obtained under this additional smoothness assumption, although the scheme developed there works for the Lipschitz boundary $\partial\Pi$ as well.

We now demand better decay and differentiability properties of the data outside the ball $\mathbb{B}_R$. First,

$$\left(1+\rho\right)^\varkappa \varphi \in L^2\left(\Pi\right) \text{ with a certain } \varkappa \in (0,1) \tag{4.7}$$

and second, ψ is the trace on $\partial\Pi\diagdown\mathbb{B}_R$ of a function Ψ such that

$$\left(1+\rho\right)^\varkappa \Psi \in L^2\left(\Pi\diagdown\mathbb{B}_R\right), \ \ \left(1+\rho\right)^\varkappa \nabla_\xi\Psi \in L^2\left(\Pi\diagdown\mathbb{B}_R\right). \tag{4.8}$$

The latter means that $\psi \in H^{1/2}\left(\partial\Pi\diagdown\mathbb{B}_R\right)$ and thus, a general result in the theory of elliptic differential equations (see, e.g., [41, Chap. 2]) places the solution w given by Lemma 6, into $H^2_{loc}\left(\overline{\Pi}\diagdown\mathbb{B}_{3R/2}\right)$. Furthermore, Theorem 3.1 [4] delivers the decomposition

$$w\left(\xi\right) = c_w + \widetilde{w}\left(\xi\right) \tag{4.9}$$

and the following formulae for the remainder:

$$\begin{aligned} \left(1+\rho\right)^{-\tau}\widetilde{w} \in L^2\left(\Pi\right), \qquad \left(1+\rho\right)^{1-\tau}\nabla_\xi\widetilde{w} \in L^2\left(\Pi\right)^3, \\ \left(1+\rho\right)^{1-\tau}\nabla^2_\xi\widetilde{w} \quad \in L^2\left(\Pi\diagdown\mathbb{B}_R\right)^{3\times 3}, \ \text{ with any } \tau > 0. \end{aligned} \tag{4.10}$$

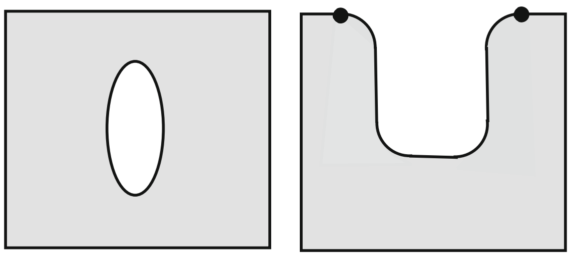

FIG. 6. **a** Interior cavity, **b** non-interior hole

Remark 7. If $\partial\Theta$ has a void intersection with $\partial\mathbb{Q}$ (Fig. 6a) and in addition to (4.7), (4.8), β is the trace on $\gamma = \partial\Theta$ of a function $B \in H^2(\Pi\setminus\mathbb{B}_R)$, then, by the same argument as above, $w \in H^2(\Pi)$. In the case of a hole (Fig. 6b) the boundary $\mathcal{V}$ of the cell Ξ has the collision line for the Dirichlet and Neumann conditions. The presence of square root singularity at this line (see [42] and [43, Chaps. 1, 11]) prevents the solution $w \in H^1_{loc}(\overline{\Pi})$ from being an element of the space $H^2_{loc}(\overline{\Pi})$. With just this reason we write the last inclusion in (4.10) for the remainder $\widetilde{w}$ in the set where the boundary $\partial\Pi$ is smooth and carries the Neumann condition only. □

Note that $(1+\rho)^{-\tau} c_w \notin L^2(\Pi)$ and hence, the inclusions in (4.10) maintain a certain decay of $\widetilde{w}$. In other words, the solution w of problem (1.25)–(1.27) stabilizes to the constant c_w but gets the natural decay properties of boundary layers if and only if

$$c_w = 0. \tag{4.11}$$

To specify condition (4.11), we need auxiliary constructions. Let Φ denote the fundamental solution for the differential operator $-\nabla_y^\top A \nabla_y$ in $\mathbb{R}^2$ (see, e.g., [10]),

$$\Phi(y) = -C_\Phi \ln r + \phi(\vartheta), \tag{4.12}$$

where C_Φ is a positive constant, (r, ϑ) the polar coordinate system in the $y-$plane, and ϕ a smooth function of mean zero on the unit circle $\mathbb{S}^1$.

Remark 8. If $A^{1/2}$ is the positive square root of the symmetric and positive definite matrix A, the coordinate change

$$y \mapsto \mathfrak{y} = A^{-1/2} y$$

transforms the differential operator in (1.23) into the Laplacian:

$$\Delta_\mathfrak{y} = \nabla_\mathfrak{y}^\top \nabla_\mathfrak{y} = \nabla_\mathfrak{y}^\top A^{-1/2} A A^{-1/2} \nabla_\mathfrak{y} = \nabla_y^\top A \nabla_y.$$

Hence,

$$\Phi(y) = -(2\pi)^{-1} |\det A|^{-1/2} \ln |\mathfrak{y}|$$

and in particular, $C_\Phi = (2\pi)^{-1} |\det A|^{-1/2}$. □

The characteristic size of a cell $\Xi^\alpha = \{\xi = (\eta, \zeta) : (\eta - \alpha, \zeta) \in \Xi\}$ in the perforated layer (1.5) is infinitesimal as $|\alpha| \to \infty$ in comparison with the distance $O(|\alpha|)$ from the central cell. This primitive observation adduced in [4,33,35,38,44] to employ the common homogenization procedure for constructing asymptotics at infinity of solutions in periodic media, however, a justification of the derived asymptotic forms remains the most involved statement in all these papers.

4.3. Calculation of the constant c_Π

First of all, let us attempt to find out a solution to the homogeneous ($\varphi = 0$, $\psi = 0$, $\beta = 0$) problem (1.25)–(1.27) in the form

$$Z(\xi) = X(\eta)\left\{\Phi(\eta) + \sum_{i=1}^{2} Y_i(\xi)\frac{\partial \Phi}{\partial \eta_i}(\eta) + \mathfrak{Y}(\Phi;\xi,\eta)\right\} + \widehat{Z}(\xi), \tag{4.13}$$

where $X \in C^\infty(\mathbb{R}^2)$ is a cut-off function, $X(\eta) = 1$ for $|\eta| > 2R$ and $X(\eta) = 0$ for $|\eta| < R$, Y_i is the standard corrector satisfying the relation (3.7), (3.8), (3.10), and $\mathfrak{Y}$ is the "junior" corrector, i.e., the linear combination

$$\mathfrak{Y}(\Phi;\xi,\eta) = \sum_{i,k=1}^{2} Y_{ik}(\xi)\frac{\partial^2\Phi}{\partial\eta_i\partial\eta_k}(\eta) \tag{4.14}$$

satisfying the problem

$$\begin{aligned} -\nabla_\xi^\top a(\xi)\nabla_\xi\mathfrak{Y}(\Phi;\xi,y) &= \left(\nabla_y^\top,0\right)a(\xi)\left(\nabla_y^\top,0\right)^\top\Phi(y) \\ &\quad + \sum_{i=1}^{2}\left(\nabla_\xi^\top a(\xi)\left(\nabla_y^\top,0\right)^\top Y_i(\xi) + \left(\nabla_y^\top,0\right)a(\xi)\nabla_\xi Y_i(\xi)\right)\frac{\partial\Phi}{\partial y_i}(y), \quad \xi\in\Xi, \end{aligned} \tag{4.15}$$

$$\nu(\xi)^\top a(\xi)\nabla_\xi\mathfrak{Y}(\Phi;\xi,y) = -\nu(\xi)^\top a(\xi)\left(\nabla_y^\top,0\right)^\top\sum_{i=1}^{2}Y_i(\xi)\frac{\partial\Phi}{\partial y_i}(y), \quad \xi\in\mathcal{V},$$

with the periodicity conditions on the lateral side ϖ of the cell (see (1.3)). This problem has a periodic in η solution since Φ satisfies the equation $-\nabla_y^\top A\nabla_y\Phi(y) = 0$, $y \in \mathbb{R}^2\diagdown\mathcal{O}$, with the matrix A defined in (3.11). We emphasize especially that in (4.14) and (4.15), the standard convention in homogenization is accepted: the argument of Y_i and Y_{ik} is regarded as a fast variable while Φ depends on the slow variable. We do not need to pay any attention to the cut-off function X because its derivatives have compact supports while we are interested in the behavior of the function (4.13) as $|\eta| \to +\infty$.

We insert representation (4.13) into the homogeneous problem (1.25)–(1.27) to derive a differential equation and boundary conditions for the remainder $\widehat{Z}$, the right-hand sides of which will be denoted by $\widehat{\varphi}$, $\widehat{\psi}$ and $\widehat{\beta}$. Note that $\widehat{\beta} = 0$ due to the presence of the cut-off function X in (4.13). In view of relations (3.7), (3.8) and (4.15), we see that outside the ball $\mathbb{B}_{2R}$, $\widehat{\varphi}$ and $\widehat{\psi}$ are linear combinations with periodic coefficients of third-order derivatives of the fundamental solution (4.12) which are of order $|\eta|^{-3}$ at infinity. That is why $\widehat{\varphi}$ and $\widehat{\psi}$ meet conditions (4.4) and (4.7), (4.8) with any $\varkappa \in (0,1)$. Hence, the solution $\widehat{Z} \in W$ exists and admits the asymptotic form

$$\widehat{Z}(\xi) = C_\Pi + \widetilde{Z}(\xi), \tag{4.16}$$

where the remainder $\widetilde{Z}$ meets conditions (4.10). The constant C_Π in (4.16) depends only on the shape of the periodicity cell Ξ and the matrix a.

Remark 9. In the layer $\Lambda = \mathbb{R}^2 \times (-1/2, 1/2)$ with the only cylindrical hole $\Theta = \theta \times (-1/2, 1/2)$ (Fig. 7), the mixed boundary value problem for the Laplace equation

$$-\Delta_\xi Z(\xi) = 0, \quad \xi \in \Lambda\diagdown\overline{\Theta},$$

$$\frac{\partial}{\partial\zeta}Z(\eta,\pm 1/2) = 0, \quad \eta\in\mathbb{R}^2\diagdown\overline{\theta}, \qquad Z(\eta,\zeta) = 0, \quad \eta\in\partial\theta, \quad \zeta\in(-1/2,1/2),$$

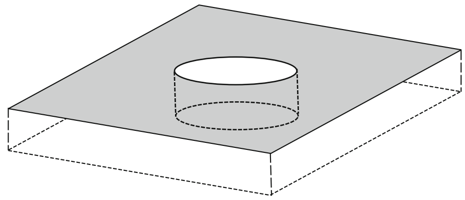

Fig. 7. Layer Λ with cylindrical hole Θ

has the explicit solution depending on the longitudinal coordinates $\eta = (\eta_1, \eta_2)$ only,

$$Z(\xi) = \frac{1}{2\pi}\left(\ln\frac{1}{|\eta|} - \ln\frac{1}{R_e(\theta)}\right) + O\left(|\eta|^{-1}\right),$$

where $-\ln R_e(\theta)$ and $R_e(\theta)$ are the logarithmic capacity and the exterior conformal radius of the domain $\theta \subset \mathbb{R}^2$, respectively (see [45,46]). Analogously, we call the number

$$c_{\log}(\gamma, \Pi) = C_\Phi^{-1} C_\Pi \tag{4.17}$$

(see (4.12) and (4.16)) the logarithmic capacity of the (central) cavity Θ^0 in the perforated layer Π. $\square$

Since $X\Phi$ does not belong to the function space W (integrals in (4.3) diverge for $w = X\Phi$), formulae (4.13), (4.16) deliver a non-trivial solution Z to the homogeneous problem (1.25)–(1.27). We shall use this particular solution for developing asymptotic structures in Sect. 5. Moreover, according to the generalization [4,44] of the method [47] for periodic media, the detected solution furnishes the following integral representation of the constant c_w in (4.9).

$$c_w = \int\limits_{\Pi} \varphi(\xi)\, Z(\xi)\, \mathrm{d}\xi + \int\limits_{\partial\Pi\setminus\gamma} \psi(\xi)\, Z(\xi)\, \mathrm{d}s_\xi - \int\limits_{\gamma} \beta(\xi)\, \nu(\xi)^\top a(\xi)\, \nabla_\xi Z(\xi)\, \mathrm{d}s_\xi. \tag{4.18}$$

Note that the first two integrals in (4.18) converge due to the inclusion (4.10), while owing to our assumption on β, the third integral is convergent in spite of square root singularities of $\nabla_\xi Z(\xi)$ at the collision line (Remark 7).

In the paper [4], the full asymptotic expansion of solution to the boundary value problem in the perforated layer is constructed and accurate estimates for the remainders are derived. In particular, the results ensure the inclusions

$$(1+\rho)^{-\tau}\left(Z - XZ^0 - C_\Pi\right) \in L^2(\Pi), \quad (1+\rho)^{1-\tau}\, \nabla_\xi\left(Z - XZ^0\right) \in L^2(\Pi)^3 \tag{4.19}$$

where τ is arbitrary positive and

$$Z^0(\xi) = \Phi(\eta) + \sum_{i=1}^{2} Y_i(\xi)\, \frac{\partial\Phi}{\partial\eta_i}(\eta). \tag{4.20}$$

We emphasize that it is not possible to enlarge the weight exponents $-\tau$ and $1-\tau$ in (4.19) because asymptotic terms, next to (4.20), are of the form (4.14) and

$$C_\Pi^{(1)}\frac{\partial\Phi}{\partial\eta_1}(\eta) + C_\Pi^{(2)}\frac{\partial\Phi}{\partial\eta_2}(\eta)$$

with some constants $C_\Pi^{(i)}$. Moreover, due to [4], the following pointwise estimates are valid:

$$\begin{aligned} &\left|Z(\xi) - X(\xi)\, Z^0(\xi) - C_\Pi\right| \le c\,(1+|\xi|)^{-1}, \\ &\left|\nabla_\xi Z(\xi) - X(\xi)\, \nabla_\xi Z^0(\xi)\right| \le c\,(1+|\xi|)^{-2}. \end{aligned} \tag{4.21}$$

5. Constructing the formal asymptotics

Here and in the sequel, we assume that the boundary $\partial\omega$ is smooth.

The boundary layer $w\left(h^{-1}x\right)$ is intended to compensate for the discrepancy left in the Dirichlet condition (1.20) on γ_h by the principal asymptotic term of the solution expansion. If - for the moment - we accept for $u^h_\bullet(x)$ the same asymptotic ansatz (3.6) as for $u^h(x)$, then the main term of the discrepancy becomes equal to $U(\mathcal{O})$. Therefore, the boundary layer has to be a solution to the problem (1.25)–(1.27) with $\varphi = 0$, $\psi = 0$ and $\beta(\xi) = -U(\mathcal{O})$. Since constants live in the function space $\widehat{W}$ with norm (4.3), the unique solution $w \in \widehat{W}$ of this problem in Π is nothing but $-U(\mathcal{O})$. If $U(\mathcal{O}) \neq 0$, the function w does not decay at infinity and hence, cannot be chosen as a boundary layer term. In any case augmenting the ansatz (3.6) with the constant term $w(\xi) = -U(\mathcal{O})$ does not reduce the discrepancy because now the discrepancy $-U(\mathcal{O})$ of the same order h^0 appears in the Dirichlet condition (1.20) on Γ_h.

Based on the original idea [5], we ought to widen the notion of a solution to the limit problem (1.23), (1.24), in particular, to formulate this problem in the punctured domain (2.6). An alternative explanation [9, §2.4] of the further modification of the asymptotic ansatz relies on the evident fact: the sum

$$cZ(\xi) - U(\mathcal{O}) \quad \text{with } c \in \mathbb{R}$$

still compensates for the main discrepancy and since the solution Z of the homogeneous problem (1.25)–(1.27) has logarithmical growth at infinity in the perforated layer, we need to deal with solutions of the limit problem (1.23), (1.24) which also have logarithmic behavior near the point $\mathcal{O}$ (see Remark 8).

5.1. The asymptotic procedure and the rational dependence of ln *h*

We explain in detail the asymptotic procedure in the case when singularities of the involved asymptotic terms are situated outside $\Omega(h)$. In the general case considered in Sect. 5.2, the procedure is similar but requires additional cut-off functions. Notice that the coefficient b in the linear combination (2.7) is chosen in such a way (see 2.10) that the boundary layer decays at infinity.

We introduce the solution G of the problem

$$-\nabla_y^\top A \nabla_y G(y) = \delta(y), \quad y \in \omega, \tag{5.1}$$

$$G(y) = 0, \quad y \in \partial\omega, \tag{5.2}$$

where δ is the Dirac mass and the equation (5.1) is understood in the distributional sense. Clearly, G is a particular Green function of problem (1.23), (1.24) with logarithmic singularity at the point $\mathcal{O}$. The decomposition

$$G(y) = \Phi(y) + G^0(y) \tag{5.3}$$

is valid with the fundamental solution (4.12) and the smooth regular part G^0 (cf. (2.8)). The function (5.3) belongs to $L^2(\omega)$ but lives outside the intrinsic energy space $\mathring{H}^1(\omega;\partial\omega)$. It can be readily verified (cf. [27, §6.4]) that any solution $U \in L^2(\omega)$ of problem (1.23), (1.24) stated in the punctured domain $\omega^\odot$ in (2.6), takes the form (2.7) where $U^0 \in H^2(\omega) \cap \mathring{H}^1(\omega;\partial\omega)$ is the energy solution of (1.23), (1.24) in ω with $F \in L^2(\omega)$, and the coefficient $b \in \mathbb{R}$ is arbitrary.

In the asymptotic ansatz (3.6), we substitute the singular solution (2.7) for the smooth solution. To make our further consideration much more transparent, we, for a while, suppose that the cavity Θ in the cell envelopes the line segment $\{\xi : \eta = 0,\ |\zeta| \le 1\}$ (Figs. 3b, 6b) so that the singularities of the function

$$y \mapsto U(y) + h\sum_{i=1}^{2} Y_i\left(h^{-1}x\right)\frac{\partial U}{\partial y_i}(y) \tag{5.4}$$

fall outside the domain $\Omega(h)$.

Near the Dirichlet zone γ_h, where $r = |y| = O(h)$ and hence, $\rho = |\eta| = O(1)$, the main asymptotic term of the function (5.4), owing to (2.7) and (4.12), (5.3), reads:

$$U^0(\mathcal{O}) + b\left(-C_\Phi \ln r + \phi(\vartheta) + G^0(\mathcal{O}) + h\sum_{i=1}^{2} Y_i\left(h^{-1}x\right)\frac{\partial \Phi}{\partial y_i}(y)\right)$$
$$= U^0(\mathcal{O}) + b\left(-C_\Phi \ln(h\rho) + \phi(\vartheta) + G^0(\mathcal{O}) + \sum_{i=1}^{2} Y_i(\xi)\frac{\partial \Phi}{\partial \eta_i}(\eta)\right). \tag{5.5}$$

Note that the relation (2.9) is taken into account. Implying the main discrepancy in the Dirichlet conditions on γ_h, the expression (5.5) with the minus sign appears as the right-hand side in (1.27):

$$\beta(\xi) = -U^0(\mathcal{O}) + b\left(C_\Phi \ln h - G^0(\mathcal{O})\right) - bZ^0(\xi), \quad \xi \in \gamma, \tag{5.6}$$

while Z^0 is determined in (4.20). The discrepancy of order h^{-2} and h^{-1} is also left in the equation (1.12) and the Neumann boundary condition (1.13), respectively. In this way, the right-hand sides in the problem for the boundary layer w must be chosen as follows:

$$\begin{aligned} \varphi(\xi) &= b\nabla_\xi^\top a(\xi)\nabla_\xi Z^0(\xi), \quad \xi \in \Pi, \\ \psi(\xi) &= -b\nu(\xi)\, a(\xi)\nabla_\xi Z^0(\xi), \quad \xi \in \partial\Pi \setminus \overline{\gamma}. \end{aligned} \tag{5.7}$$

We emphasize that both the differential operators in (5.7) abolish the constants $U^0(\mathcal{O})$, $G^0(\mathcal{O})$ and $bC_\Phi|\ln h|$.

The solution w of problem (1.25)–(1.27) with the right-hand sides (5.7), (5.6) can be found in the explicit form:

$$w(\xi) = -U^0(\mathcal{O}) - b\left(C_\Phi|\ln h| + G^0(\mathcal{O})\right) + b\left(Z(\xi) - Z^0(\xi)\right), \tag{5.8}$$

where Z is a solution of the homogeneous problem constructed in Sect. 4. By virtue of formulas (4.19), the solution (5.8) decays at infinity if and only if

$$-U^0(\mathcal{O}) - b\left(C_\Phi|\ln h| + G^0(\mathcal{O})\right) + bC_\Pi = 0. \tag{5.9}$$

In other words, we fix the coefficient b in (2.7)

$$b = -\frac{U^0(\mathcal{O})}{C_\Phi|\ln h| + G^0(\mathcal{O}) - C_\Pi} =: -HU^0(\mathcal{O}) \tag{5.10}$$

(cf. formula (2.10) with $C(\Xi, \omega, a) = G^0(\mathcal{O}) - C_\Pi$). We mention that $\ln h = -|\ln h|$ because $h \in (0, 1]$, $C_\Phi > 0$ (see Remark 8), and the denominator in (5.10) stays positive for $h \in (0, h_0]$ with some $h_0 > 0$.

5.2. General case

The same condition (5.10) for the decay of the boundary layer term holds true even without the above assumption on the cell $\Xi = \mathbb{Q} \setminus \Theta$. However, the argumentation must be modified since we cannot use the sum (5.4) in the whole domain $\Omega(h)$ due to singularities of U and $\nabla_y U$ at the point $\mathcal{O}$. To smooth these singularities away, we multiply the sum with the cut-off function

$$X_h(y) = X\left(h^{-1}y\right) \tag{5.11}$$

where $X \in C^\infty\left(\mathbb{R}^2\right)$ has been used in (4.13). Notice that X_h is obtained from X by the coordinate dilation $y \mapsto \eta = h^{-1}y$ so that X_h is equal to 1 everywhere in $\Omega(h)$ with exception of the $Rh-$neighborhood of γ_h and $X_h = 0$ on the surface γ_h of the cavity Θ_h^0 in the central cell Ξ_h^0. As a result, the ansatz for the solution $u_\bullet^h(x)$ of the problem (1.18)–(1.20) looks as follows:

$$u_\bullet^h(x) = X_h(y)\left(U(y) + h\sum_{i=1}^{2} Y_i\left(h^{-1}x\right)\frac{\partial U}{\partial y_i}(y)\right) + w\left(h^{-1}x\right) + \cdots \tag{5.12}$$

The cut-off function X_h makes the first terms in (5.12) vanish on γ_h and therefore, the Dirichlet condition (1.27) becomes homogeneous, that is, $\beta = 0$. At the same time, the last expression in (5.5) is now multiplied with $X_h(y) = X(\eta)$ that brings the right-hand sides

$$\varphi(\xi) = b\nabla_\xi^\top a(\xi)\nabla_\xi \mathfrak{Z}(\xi) \quad \text{and} \quad \psi(\xi) = -b\nu(\xi)^\top a(\xi)\nabla_\xi \mathfrak{Z}(\xi) \tag{5.13}$$

into (1.25) and (1.26), respectively, while

$$\mathfrak{Z}(\xi) = X(\eta)\left(U^0(\mathcal{O}) - b\left(C_\Phi\left|\ln h\right| + G^0(\mathcal{O})\right) + bZ^0(\xi)\right) \tag{5.14}$$

(cf. (5.6)). Again we write the solution explicitly

$$w(\xi) = bZ(\xi) - b\mathfrak{Z}(\xi). \tag{5.15}$$

The function (5.15) coincides with (5.8) outside the ball $\mathbb{B}_{2R}$ and thus, the conclusion on the decay of the boundary layer term remains the same as above.

All the terms in the asymptotic ansatz (5.12) have been found including the coefficient (5.10) in (2.7).

Remark 10. An alternative way to derive the formula (5.10) is to apply the integral representation (4.18) for the coefficient c_w in (4.9) and (4.11). □

6. The justification of the asymptotics

In this section, we justify the asymptotics of a solution to problem (1.18)–(1.20). Recall that the crucial feature of this problem is the presence of a small additional Dirichlet zone γ_h. It should be noted that an estimate for the discrepancy in the asymptotics of a solution to problem (1.12)–(1.14) is well known (see Theorem 11).

In Sect. 6.2, we prove auxiliary weighted inequalities that take into account the Dirichlet condition imposed on the boundary of the central periodicity cell (1.17). The most technical part of the paper (Sect. 6.3) is estimating the discrepancies left in equations (1.18)–(1.20) when substituting the intermediate approximate solution $u_\bullet^h$, see (6.19). To this end, we find out bounds for all terms on the right-hand side of (6.24) i.e. for all terms on the right-hand side of the integral identity for the difference $\mathcal{R}_\bullet^h = u_\bullet^h - U_\bullet^h$, see (6.18). Using various arguments, we derive the bound $ch^{2\alpha+1}$, $\alpha > 0$, for the expression $\left(a\nabla_x\mathcal{R}_\bullet^h, \mathcal{R}_\bullet^h\right)_{\Omega(h)}$ which readily leads to estimate (6.38) in Theorem 16. In the final statement on the asymptotics of $u_\bullet^h$, we clean up the approximate solution and drop lower-order terms that were involved into $U_\bullet^h$ due to a technical reason only.

6.1. Assumptions on the problem data

We assume that the boundary $\partial\omega$ of the domain $\omega \in \mathbb{R}^2$ is smooth, e.g., of the Hölder class C^{2,α_ω}, $\alpha_\omega > 0$, and $f^h(x) = f\left(y, h^{-1}x\right)$ where f is 1−periodic in the variables η_1, η_2 and belongs to $L^2\left(\omega; C^{2,\alpha}_{per}(\Xi)\right)$,

$$\alpha \in (0, 1/2]. \tag{6.1}$$

The norm in the Hölder space $C^{0,\alpha}(\Xi)$ is given as follows

$$\left\|v; C^{0,\alpha}(\Xi)\right\| = \sup_{\xi\in\Xi}|v(\xi)| + \sup_{\xi,\mathfrak{x}\in\Xi}\left(|\xi - \mathfrak{x}|^{-\alpha}|v(\xi) - v(\mathfrak{x})|\right) \tag{6.2}$$

and $C^{0;\alpha}_{per}(\Xi)$ stands for the subspace of periodic functions. The abstract Lebesgue space $L^2(\omega;\mathfrak{L})$ with a Banach space $\mathfrak{L}$ has the norm

$$\left\|f;L^2(\omega;\mathfrak{L})\right\| = \left(\int\limits_\omega \|f(y,\cdot);\mathfrak{L}\|^2\,\mathrm{d}y\right)^{1/2}. \tag{6.3}$$

Assuming $a \in C^{2,\alpha_\Xi}_{per}(\Xi)^{3\times3}$ and $\mathcal{V} \in C^{2,\alpha_\Xi}$ with $\alpha_\Xi \in (\alpha,1)$, we refer to [48] and obtain that $Y_i \in C^{2,\alpha_Y}(\Xi)$ with any $\alpha_Y \in (0,\alpha_\Xi)$. Moreover, if we denote

$$F(y) = \int\limits_\Xi f(y,\xi)\,\mathrm{d}\xi, \qquad \text{then} \;\; F \in L^2(\omega) \tag{6.4}$$

and thus, $U \in H^2(\omega) \cap \mathring{H}^1(\omega;\partial\omega)$ and

$$\left\|U;H^2(\omega)\right\| \le c\left\|F;L^2(\omega)\right\| \le c\left\|f;L^2\left(\omega;C^{2,\alpha}_{per}(\Xi)\right)\right\|. \tag{6.5}$$

The above observations show that the asymptotic terms detached in (3.6) fall into the Sobolev space $H^1(\Omega(h))$.

An accurate estimate of the remainder

$$\mathcal{R}^h(x) = u^h(x) - U(y) - h\sum_{i=1}^{2} Y_i\left(h^{-1}x\right)\frac{\partial U}{\partial y_i}(y) \tag{6.6}$$

in the asymptotic ansatz (3.6) is known for the solution of the problem (1.12)–(1.14) (see, e.g., [2, Sect. 6.1]) and we give the corresponding theorem without a proof.

Theorem 11. *Under the above assumption, the solution $u^h \in \mathring{H}^1(\Omega(h);\Gamma(h))$ of the problem (1.12)–(1.14) and the solution $U \in H^2(\omega) \cap \mathring{H}^1(\omega;\partial\omega)$ of the problem (1.23)–(1.24) are in the relationship*

$$\left\|\mathcal{R}^h;L^2(\Omega(h))\right\| + \left\|\nabla_x\mathcal{R}^h;L^2(\Omega(h))\right\| \le ch^{\alpha+1/2}\left\|f;L^2\left(\omega;C^{0,\alpha}_{per}(\Xi)\right)\right\| \tag{6.7}$$

where $\mathcal{R}^h$ is the difference (6.6) and the constant c in (6.7) does not depend on the parameter $h \in (0,1]$ and the function $f^h(x) = f\left(y,h^{-1}x\right)$, $f \in L^2\left(\omega;C^{0,\alpha}_{per}(\Xi)\right)$.

Note that

$$\begin{gathered}\left\|U;H^1(\Omega(h))\right\| \le C_U\left(\mathrm{mes}_3\,\Omega(h)\right)^{1/2} \le ch^{1/2},\\ \left\|Y_i\partial U/\partial y_i;H^1(\Omega(h))\right\| \le ch^{-1}h^{1/2} = ch^{-1/2}\end{gathered} \tag{6.8}$$

while the factor h^{-1} in the second line appears due to differentiation of the function Y_i in the rapid variables. Hence, the estimate (6.7) indeed justifies both terms in the asymptotic expansion.

6.2. Auxiliary inequalities

We now turn to derive an estimate of the remainder in the asymptotic expansion of the solution $u^h_\bullet$ to the problem (1.18)–(1.20) with the Dirichlet zone extended over the small surface γ_h. To do so, we need several weighted inequalities.

Lemma 12. *For any function $U^0 \in H^2(\omega)$, the inequality*

$$\left\|r^{-1}(1+|\ln r|)^{-1}\nabla_y U^0;L^2(\omega)\right\| + \left\|r^{-2}(1+|\ln r|)^{-1}\left(U^0 - U^0(\mathcal{O})\right);L^2(\omega)\right\| \le c\left\|U^0;H^2(\omega)\right\| \tag{6.9}$$

is valid, where $r = \mathrm{dist}(y,\mathcal{O})$ and c is independent of U.

Proof. Bearing the completion argument in mind, we suppose that $U^0 \in C^\infty(\overline{\omega})$. Moreover, owing to the Sobolev embedding theorem

$$\left|U^0(\mathcal{O})\right| \leq c\left\|U^0; H^2(\omega)\right\|,$$

we subtract $U^0(\mathcal{O})$ from U^0 and multiply the result with a cut-off function $\chi \in C_c^\infty\left(\mathbb{B}_{r(\omega)}\right)$ which is equal to one in the disk $\mathbb{B}_{r(\omega)/2} = \{y : r < r(\omega)/2\}$ with $r(\omega) = \sup\{s : \mathbb{B}_s \subset \omega\}$. Clearly,

$$\left\|\chi\left(U^0 - U^0(\mathcal{O})\right); H^2(\omega)\right\| \leq c\left|\left|U^0; H^2(\omega)\right|\right|.$$

Since the estimate (6.9) for $(1-\chi)\left(U^0 - U^0(\mathcal{O})\right)$ is evident because all weights are separated from zero outside $\mathbb{B}_{r(\omega)/2}$, we further deal with $\chi\left(U^0 - U^0(\mathcal{O})\right)$ denoted still by U^0.

As has been commented to (4.2), the Hardy inequality with logarithm readily provides the necessary estimate of the first norm in (6.9), namely it suffices to separate polar variables r and φ in the disk $\mathbb{B}_{r(\omega)}$, apply the Hardy inequality with $\rho = r(\omega)^{-1} r$ and integrate over the unit circle $\mathbb{S}^1 \ni \varphi$.

In the same way we process the second norm while employing the following variant of the Hardy inequality

$$\int_0^1 \rho^{-3}(1+|\ln\rho|)^{-2}|W(\rho)|^2 \mathrm{d}\rho \leq c\int_0^1 \rho^{-1}(1+|\ln\rho|)^{-2}\left|\frac{\mathrm{d}W}{\mathrm{d}\rho}(\rho)\right|^2 \mathrm{d}\rho,\ W \in C^1[0,1],\ W(0) = 0. \tag{6.10}$$

Let us write down a proof of (6.10). By the Newton–Leibnitz formula, we have

$$\begin{aligned}\int_0^1 \rho^{-3}(1+|\ln\rho|)^{-2}|W(\rho)|^2\,\mathrm{d}\rho &= 2\int_0^1 \rho^{-3}(1+|\ln\rho|)^{-2}\int_0^\rho \frac{\mathrm{d}W}{\mathrm{d}r}(r)\,W(r)\,\mathrm{d}r\mathrm{d}\rho\\ &= 2\int_0^1 \frac{\mathrm{d}W}{\mathrm{d}r}(r)\,W(r)\int_r^1 \rho^{-3}(1+|\ln\rho|)^{-2}\,\mathrm{d}\rho\mathrm{d}r\\ &\leq c\int_0^1 \left|\frac{\mathrm{d}W}{\mathrm{d}r}(r)\right||W(r)|\,r^{-1}(1+|\ln r|)^{-2}\int_r^1 \rho^{-2}\mathrm{d}\rho\mathrm{d}r.\end{aligned}$$

Note that the function $r(1-\ln r)^2$ is monotone for $r \in \left[0, e^{-1}\right]$. Since the last integral is equal to $r^{-1} - 1 \leq r^{-1}$, we complete the proof by applying the Schwarz inequality in the obtained relation

$$\int_0^1 \rho^{-3}(1+|\ln\rho|)^{-2}|W(\rho)|^2\,\mathrm{d}\rho \leq c\int_0^1 r^{-1/2}(1+|\ln r|)^{-1}\left|\frac{\mathrm{d}W}{\mathrm{d}r}(r)\right| r^{-3/2}(1+|\ln r|)^{-1}|W(r)|\,\mathrm{d}r.$$

□

We also shall need estimates in the thin boundary strip

$$\omega_h^{⋒} = \{y \in \omega : \mathrm{dist}(y, \partial\omega) \leq c_{⋒} h\} \tag{6.11}$$

and the corresponding part of the perforated plate

$$\Omega^{⋒}(h) = \left\{x \in \Omega(h) : y \in \omega_h^{⋒}\right\}. \tag{6.12}$$

□

Lemma 13. *For any functions* $U \in \mathring{H}^1(\omega; \partial\omega)$, $V \in H^1(\omega)$ *and* $u^h \in \mathring{H}^1(\Omega(h); \Gamma(h))$*, there hold the inequalities*

$$\left\| U; L^2\left(\omega_h^{\Cap}\right)\right\| \le ch \left\|\nabla_y U; L^2\left(\omega_h^{\Cap}\right)\right\|, \tag{6.13}$$

$$\left\| V; L^2\left(\omega_h^{\Cap}\right)\right\| \le ch^{1/2} \left\|V; H^1(\omega)\right\|, \tag{6.14}$$

$$\left\| u^h; L^2\left(\Omega^{\Cap}(h)\right)\right\| \le ch \left\|u^h; H^1(\Omega(h))\right\|, \tag{6.15}$$

with constants independent of the parameter $h \in (0,1]$ *and the functions.*

Proof. First, one applies the one-dimensional Hardy inequality

$$\int_0^T t^{-2} |W(t)|^2 \, \mathrm{d}t \le 4 \int_0^T \left|\frac{\mathrm{d}W}{\mathrm{d}t}(t)\right|^2 \mathrm{d}t, \quad W \in C_c^\infty(0,+\infty), \; T > 0, \tag{6.16}$$

for the function U restricted to a neighborhood of $\partial\omega$ and written in the local coordinates (t,s) where $t = \operatorname{dist}(y, \partial\omega)$ and s is the arc length on $\partial\omega$. Since the Jacobian of the change $y \mapsto (t,s)$ is bounded and separated from zero, integrating (6.16) in s gives (6.13) because $t^{-2} \ge c_{\Cap}^{-2} h^{-2}$.

Second, in the same way one uses the formula

$$\varepsilon^{-1} \int_0^\varepsilon |V(t,s)|^2 \, dt \le c \int_0^T \left(\left|\frac{\partial V}{\partial t}(t,s)\right|^2 + |V(t,s)|^2 \right) \mathrm{d}t, \quad \varepsilon \in \left(0, \frac{T}{2}\right)$$

obtained by integrating in $\tau \in (0,\varepsilon)$ the standard trace inequality

$$|V(t,s)|^2 \le c \int_\tau^{\tau+T/2} \left(\left|\frac{\partial V}{\partial t}(t,s)\right|^2 + |V(t,s)|^2 \right) \mathrm{d}t.$$

Third, we extend u^h from $\Omega(h)$ onto the enlarged intact plate (3.4) (see Sect. 3) and use the inequality (6.13) in the longitudinal coordinates while noticing that the distance from $x \in \Omega^{\Cap}(h)$ to the lateral side of (3.4), where the extension vanishes, is still $O(h)$. □

6.3. Estimating the discrepancy of the approximate solution

We now construct an intermediate asymptotic approximation to the solution $u_\bullet^h$ of problem (1.18)–(1.20) while employing several cut-off functions. In addition to (5.11), we introduce the function $\mathcal{X}_h \in C_c^\infty(\omega)$ which vanishes in the $ch-$neighborhood of the boundary $\partial\omega$, equals one outside the $2ch-$neighborhood and admits the restrictions

$$0 \le \mathcal{X}_h(y) \le 1, \quad |\nabla_y \mathcal{X}_h(y)| \le ch^{-1}. \tag{6.17}$$

Multiplying with $\mathcal{X}_h$ all the terms on the right of (5.12), we make them to meet the Dirichlet condition (1.20) on $\Gamma(h) = \Gamma_\bullet(h) \setminus \gamma_h$. We emphasize that the Dirichlet condition on γ_h is fulfilled because X_h has been put on first terms and w satisfies the homogeneous ($\beta = 0$) boundary condition (1.27) (see the end of Sect. 5).

By construction, the difference

$$\mathcal{R}_\bullet^h = u_\bullet^h - \mathcal{U}_\bullet^h \in \mathring{H}^1(\Omega(h), \Gamma_\bullet(h)) \tag{6.18}$$

with

$$\mathcal{U}_\bullet^h(x) = \mathcal{X}_h(y) \left(X_h(y) \left(U(y) + h \sum_{i=1}^2 Y_i\left(h^{-1}x\right) \frac{\partial U}{\partial y_i}(y) \right) + w\left(h^{-1}x\right) \right). \tag{6.19}$$

Hence, we may take $\mathcal{R}^h_\bullet$ as the test function $v^h_\bullet$ in (1.21). Subtracting $\left(a\nabla_x\mathcal{U}^h_\bullet, \nabla_x\mathcal{R}^h_\bullet\right)_{\Omega(h)}$ from both sides of the specified integral identity, we arrive at the formula

$$\left(a\nabla_x\mathcal{R}^h_\bullet, \nabla_x\mathcal{R}^h_\bullet\right)_{\Omega(h)} = \left(f^h, \mathcal{R}^h_\bullet\right)_{\Omega(h)} - \left(a\nabla_x\mathcal{U}^h_\bullet, \nabla_x\mathcal{R}^h_\bullet\right)_{\Omega(h)}. \tag{6.20}$$

By Friedrich's inequality (3.5) and the positivity condition (1.10), the left-hand side of (6.20) is bounded from below by $c\left\|\mathcal{R}^h_\bullet; H^1\left(\Omega\left(h\right)\right)\right\|^2$ with a constant $c > 0$ that does not depend on h. Therefore, our immediate objective becomes to process and estimate the right-hand side of (6.20). In particular, we shall transfer the cut-off functions in (6.19) onto $\mathcal{R}^h_\bullet$ and the next lemma contains an estimate of the product $X_h\mathcal{X}_h\mathcal{R}^h_\bullet$.

Lemma 14. *If* $\mathcal{R}^h_\bullet \in \mathring{H}^1\left(\Omega\left(h\right); \Gamma_\bullet\left(h\right)\right)$, *then* $\mathfrak{R}^h_\bullet = X_h\mathcal{X}_h\mathcal{R}^h_\bullet \in \mathring{H}^1\left(\Omega\left(h\right); \Gamma_\bullet\left(h\right)\right)$ *satisfies the estimate*

$$\left(1+|\ln h|\right)^{-2}\left\|\nabla_x\mathfrak{R}^h_\bullet; L^2\left(\Omega\left(h\right)\right)\right\| + \left\|r^{-1}\left(1+|\ln r|\right)^{-1}\mathfrak{R}^h_\bullet; L^2\left(\Omega\left(h\right)\right)\right\| \leq c\left\|\nabla_x\mathcal{R}^h_\bullet; L^2\left(\Omega\left(h\right)\right)\right\|, \tag{6.21}$$

where c is independent of $\mathcal{R}^h_\bullet$ and $h \in (0, h_0]$.

Proof. We readily have

$$\left\|r^{-1}\left(1+|\ln r|\right)^{-1}\mathfrak{R}^h_\bullet; L^2\left(\Omega\left(h\right)\right)\right\|^2 \leq \left\|r^{-1}\left(1+|\ln r|\right)^{-1}\mathcal{R}^h_\bullet; L^2\left(\Omega\left(h\right)\right)\right\|^2,$$

$$\left\|\nabla_x\mathfrak{R}^h_\bullet; L^2\left(\Omega\left(h\right)\right)\right\|^2 \leq c\Big(\left\|\nabla_x\mathcal{R}^h_\bullet; L^2\left(\Omega\left(h\right)\right)\right\|^2 + \left\|\mathcal{R}^h_\bullet\nabla_x X_h; L^2\left(\Omega\left(h\right)\right)\right\|^2 + \left\|\mathcal{R}^h_\bullet\nabla_x\mathcal{X}_h; L^2\left(\Omega\left(h\right)\right)\right\|^2\Big).$$

By definition of the cut-off function $\mathcal{X}_h$ and the inequality (6.15), we see that

$$\left\|\mathcal{R}^h_\bullet\nabla_x\mathcal{X}_h; L^2\left(\Omega^{\Cap}\left(h\right)\right)\right\|^2 \leq ch^{-2}\left\|\mathcal{R}^h_\bullet; L^2\left(\Omega^{\Cap}\left(h\right)\right)\right\|^2 \leq c\left\|\mathcal{R}^h_\bullet; H^1\left(\Omega\left(h\right)\right)\right\|^2.$$

Since $r \leq ch$ for $x \in \operatorname{supp}|\nabla_x X_h|$ (see comments to formula (5.11)), we obtain

$$\left\|\mathcal{R}^h_\bullet\nabla_x\mathcal{X}_h; L^2\left(\Omega\left(h\right)\right)\right\|^2 \leq ch^{-2}\left\|\mathcal{R}^h_\bullet; L^2\left(\Omega\left(h\right)\cap\operatorname{supp}|\nabla_x X_h|\right)\right\|^2 \leq c\left(1+|\ln h|\right)^2\left\|r^{-1}\left(1+|\ln r|\right)^{-1}\mathcal{R}^h_\bullet; L^2\left(\Omega\left(h\right)\right)\right\|^2.$$

Thus, to conclude with the inequality (6.21), it suffices to verify that

$$\left\|r^{-1}\left(1+|\ln r|\right)^{-1}\mathcal{R}^h_\bullet; L^2\left(\Omega\left(h\right)\right)\right\|^2 \leq c\left\|\nabla_x\mathcal{R}^h_\bullet; L^2\left(\Omega\left(h\right)\right)\right\|^2. \tag{6.22}$$

This relation follows from the one-dimensional Hardy inequality (4.1). Indeed, denoting by $\widehat{\mathcal{R}}^h_\bullet \in H^1\left(\mathbb{R}^2\times(-h/2, h/2)\right)$ the extension of $\mathcal{R}^h_\bullet \in \mathring{H}^1\left(\Omega\left(h\right); \Gamma_\bullet\left(h\right)\right)$ constructed in Sect. 3, we, similarly to the proof of Lemma 12, observe that

$$\begin{aligned}\left\|r^{-1}\left(1+|\ln r|\right)^{-1}\mathcal{R}^h_\bullet; L^2\left(\Omega\left(h\right)\right)\right\|^2 &\leq \left\|r^{-1}\left(1+|\ln r|\right)^{-1}\widehat{\mathcal{R}}^h_\bullet; L^2\left(\mathbb{R}^2\times(-h/2, h/2)\right)\right\|^2 \\ &\leq c\left\|\nabla_x\widehat{\mathcal{R}}^h_\bullet; L^2\left(\mathbb{R}^2\times(-h/2, h/2)\right)\right\|^2 \\ &\leq c\left\|\nabla_x\widehat{\mathcal{R}}^h_\bullet; L^2\left(\Omega\left(h\right)\right)\right\|^2.\end{aligned} \tag{6.23}$$

□

We rewrite (6.20) as follows:

$$\left(a\nabla_x\mathcal{R}^h_\bullet, \nabla_x\mathcal{R}^h_\bullet\right)_{\Omega(h)} = I_{\mathcal{X}} + I_X + I_w + I_U + \widetilde{I} \tag{6.24}$$

and specify terms on the right-hand side while estimating them in parallel. □

First, we take into account commutators due to the passage of $\mathcal{X}_h$ from $\mathcal{U}^h_\bullet$ onto $\mathcal{R}^h_\bullet$, namely

$$I_{\mathcal{X}} = \left(a \left(U + h \sum_{i=1}^{2} Y_i \frac{\partial U}{\partial y_i} + w \right) \nabla_x \mathcal{X}_h, \nabla_x \mathcal{R}^h_\bullet \right)_{\Omega(h)} - \left(a \nabla_x \left(U + h \sum_{i=1}^{2} Y_i \frac{\partial U}{\partial y_i} + w \right), \mathcal{R}^h_\bullet \nabla_x \mathcal{X}_h \right)_{\Omega(h)}. \tag{6.25}$$

By estimates (6.13)–(6.15) and formulas (2.7), (2.10), (5.3), (6.17), we obtain

$$\begin{aligned} |I_{\mathcal{X}}| \leq\ & ch^{-1} \Big\{ \Big[h^{1/2} h^{3/2} \left(\left\| U^0; H^2(\omega) \right\| + |b| \right) + h^{3/2} \left\| w; L^2\left(\Pi_{h^{-1}R_\omega}\right) \right\| \Big] \left\| \nabla_x \mathcal{R}^h_\bullet; L^2(\Omega(h)) \right\| \\ & + h^{1/2} h^{1/2} \left(\left\| U^0; H^2(\omega) \right\| + |b| \right) \\ & + h^{3/2} h^{-1} \left\| \nabla_\xi w; L^2\left(\Pi_{h^{-1}R_\omega}\right) \right\| \left\| \mathcal{R}^h_\bullet; L^2\left(\Omega(h) \cap \operatorname{supp} |\nabla_x \mathcal{X}_h|\right) \right\| \Big\} \\ \leq\ & c \Big\{ h^2 \left(\left\| U^0; H^2(\omega) \right\| + \left| U^0(\mathcal{O}) \right| \right) + h^{3/2} h^{-\tau/2} \Big(\left\| (1+\rho)^{-\tau} w; L^2\left(\Pi_{h^{-1}R_\omega}\right) \right\| \\ & + \left\| (1+\rho)^{1-\tau} \nabla_\xi w; L^2\left(\Pi_{h^{-1}R_\omega}\right) \right\| \Big) \Big\} \left\| \mathcal{R}^h_\bullet; H^1(\Omega(h)) \right\| \\ \leq\ & c \left(h^1 + h^{(3-\tau)/2} \right) \left\| f; L^2\left(\omega; C^{0,\alpha}_{per}(\Xi)\right) \right\| \left\| \mathcal{R}^h_\bullet; H^1(\Omega(h)) \right\|. \end{aligned} \tag{6.26}$$

Let us explain the calculation (6.26). The factors h^{-1} and $h^{1/2}$ come from differentiation of $\mathcal{X}_h$ and integration in z, respectively. The integration in (6.25) is to be performed only over the set $\Omega(h) \cap \operatorname{supp} |\nabla_x \mathcal{X}_h|$ which belongs to the set (6.12) under a proper choice of $c_⋒$ in (6.11). Hence, estimating U and $\nabla_y U$, $\nabla^2_y U$ was done by applying the inequalities (6.13) and (6.14), respectively, while recalling that $|Y_i(\xi)| \leq c$ and $|\nabla_x Y_i(\xi)| \leq ch^{-1}$. The inequality (6.15) gives the multiplier h in the estimation of $\left\| \mathcal{R}^h_\bullet; L^2(\operatorname{supp} |\nabla_x \mathcal{X}_h| \cap \Omega(h)) \right\|$. The factors $h^{3/2}$ and h^{-1} on norms of the boundary layer term w are caused by $dx = h^3 d\xi$ and $\nabla_x = h^{-1} \nabla_\xi$. The integration set $\Pi_{h^{-1}R_\omega} = \left\{ \xi : |\eta| \geq h^{-1} R_\omega \right\}$ envelopes the set $\{\xi : x = h\xi \in \operatorname{supp} |\nabla_x \mathcal{X}_h| \cap \Omega(h)\}$ where $h^{-\varkappa}(1+\rho)^{\varkappa} \geq R^{\varkappa}_\omega$. We emphasize that formulas (4.9), (4.10) and (5.15), (5.14) ensure the estimate

$$\left\| (1+\rho)^{\varkappa} w; L^2(\Pi) \right\| + \left\| (1+\rho)^{1+\varkappa} \nabla_\xi w; L^2(\Pi) \right\| \leq c \left(\left| U^0(\mathcal{O}) \right| + |b| \right) \quad \text{with any } \varkappa \in (0,1), \tag{6.27}$$

which was also used in (6.26) together with the evident relations

$$\left| U^0(\mathcal{O}) \right| \leq c \left\| U^0; H^2(\omega) \right\| \leq c \left\| F; L^2(\omega) \right\| \leq c \left\| f; L^2\left(\omega; C^{0,\alpha}_{per}(\Xi)\right) \right\|, \tag{6.28}$$

(cf. (6.4), (6.5)).

We further have

$$I_X = \left(a \left(U^0 - U^0(\mathcal{O}) + b \left(G^0 - G^0(\mathcal{O}) \right) \right) \nabla_x X_h, \nabla_x \left(\mathcal{X}_h \mathcal{R}^h_\bullet \right) \right)_{\Omega(h)} - \left(a \nabla_x \left(U^0 + bG^0 \right), \mathcal{X}_h \mathcal{R}^h_\bullet \nabla_x X_h \right)_{\Omega(h)}. \tag{6.29}$$

Here integration occurs over the set $\operatorname{supp} |\nabla_x \mathcal{X}_h| \cap \Omega(h)$ with diameter $O(h)$ and volume $O(h^3)$ (see the comment to (5.11)) and therefore, the cut-off function $\mathcal{X}_h$ can be omitted in (6.29). By Lemma 12 and formulas (5.3), (5.10), (6.28), we derive

$$\begin{aligned}|I_X| \le\ & ch^{-1}h^{1/2}\Big\{\Big(h^2(1+|\ln h|)\left\|r^{-2}(1+|\ln r|)^{-1}\left(U^0-U^0(\mathcal{O})\right);L^2(\omega)\right\|^2\\ &+h^1h^{3/2}|b|\Big)\left\|\nabla_x\mathcal{R}^h_\bullet;L^2(\Omega(h))\right\|\\ &+\Big(h(1+|\ln h|)\left\|r^{-1}(1+|\ln r|)^{-1}\nabla_yU^0;L^2(\omega)\right\|+h^{3/2}|b|\Big)\left\|\mathcal{R}^h_\bullet;L^2(\Omega(h)\cap\operatorname{supp}|\nabla_xX_h|)\right\|\Big\}\\ \le\ & ch^{3/2}(1+|\ln h|)^2\left(\left\|U^0;H^2(\omega)\right\|+|b|\right)\left\|\mathcal{R}^h_\bullet;H^1(\Omega(h))\right\|. \end{aligned} \tag{6.30}$$

We mention that the factors h^{-1} and $h^{1/2}$ have the same origin as in (6.26) and norms of the regular part G^0 of the Green function are processed with the help of the relations

$$\left|G^0(y)-G^0(\mathcal{O})\right|\le cr,\quad \left|\nabla_yG^0(y)\right|\le c.$$

Finally, the norm $\left\|\mathcal{R}^h_\bullet;L^2(\Omega(h)\cap\operatorname{supp}|\nabla_xX_h|)\right\|$ is bounded according to (6.23) which results in the additional factor $1+|\ln h|$.

Some commutators are absent in (6.29), in particular, we have forgotten the fundamental solution $b\Phi$ (cf. (2.7), (5.3)) and have subtracted the integral $\left(a\left(U^0(\mathcal{O})+bG^0(\mathcal{O})\right)\nabla_xX_h,\nabla_x\left(\mathcal{X}_h\mathcal{R}^h_\bullet\right)\right)_{\Omega(h)}$. We insert those into the expression

$$\begin{aligned}I_w=&-\left(a\nabla_xw,\nabla_x\left(\mathcal{X}_h\mathcal{R}^h_\bullet\right)\right)_{\Omega(h)}\\ &-\left(a\nabla_xX_h\left(U^0(\mathcal{O})+bG^0(\mathcal{O})+b\left(\Phi+h\sum_{i=1}^2Y_i\frac{\partial\Phi}{\partial y_i}\right)\right),\nabla_x\left(\mathcal{X}_h\mathcal{R}^h_\bullet\right)\right)_{\Omega(h)}.\end{aligned} \tag{6.31}$$

Notice that we have included into I_w the term

$$-b\left(a\nabla_xX_h\left(\Phi+h\sum_{i=1}^2Y_i\frac{\partial\Phi}{\partial y_i}\right),\nabla_x\left(\mathcal{X}_h\mathcal{R}^h_\bullet\right)\right)_{\Omega(h)}$$

which will be recalled while constructing I_U in (6.32).

Going over to the stretched coordinates $\xi=h^{-1}x$ and observing that $\Phi(y)=\Phi(\eta)+C_\Phi\ln h$ and the function $\xi\mapsto W^h_\bullet(\xi)=\mathcal{X}_h(h\eta)\mathcal{R}^h_\bullet(h\xi)$ has a compact support, we found out that

$$\left(a\nabla_\xi w,\nabla_\xi W^h_\bullet\right)_\Pi+\left(a\nabla_\xi\left(X\left(U^0(\mathcal{O})+b\left(G^0-C_\Phi\ln h+\Phi+\sum_{i=1}^2Y_i\frac{\partial\Phi}{\partial y_i}\right)\right)\right),\nabla_\xi W^h_\bullet\right)_\Pi=0$$

according to our definition of the boundary layer term w in the end of Sect. 5 (cf. (5.13)–(5.15), (5.11) and (4.13)).

In other words, the expression (6.31) vanishes.

We now consider the terms

$$I_U=\left(f_h,X_h\mathcal{X}_h\mathcal{R}^h_\bullet\right)_{\Omega(h)}-\left(a\nabla_x\left(U^0+bG^0+h\sum_{i=1}^2Y_i\frac{\partial}{\partial y_i}\left(U^0+bG^0\right)\right),\nabla_x\left(X_h\mathcal{X}_h\mathcal{R}^h_\bullet\right)\right)_{\Omega(h)}, \tag{6.32}$$

$$\widetilde{I}=\left((1-X_h\mathcal{X}_h)f_h,\mathcal{R}^h_\bullet\right)_{\Omega(h)}=\left((1-\mathcal{X}_h)f_h,\mathcal{R}^h_\bullet\right)_{\Omega(h)}+\left((1-X_h)f_h,\mathcal{R}^h_\bullet\right)_{\Omega(h)}=:\widetilde{I}_{\mathcal{X}}+\widetilde{I}_{X_h}. \tag{6.33}$$

We proceed with the following observation on the functions $f \in L^2\left(\omega; C^{0,\alpha}_{per}(\Xi)\right) \subset L^2\left(\omega; L^\infty(\Xi)\right)$ and $f^h(x) = f\left(y, h^{-1}x\right)$:

$$\left\|f^h; L^2(\Omega(h))\right\|^2 = \sum_\alpha \int_{\Xi_h^\alpha} \left|f\left(y, h^{-1}x\right)\right|^2 \mathrm{d}x \leq \sum_\alpha \int_{\Xi_h^\alpha} \|f(y,\cdot); L^\infty(\Xi)\|^2 \mathrm{d}x$$
$$\leq ch \sum_\alpha \int_{q_h^\alpha} \|f(y,\cdot); L^\infty(\Xi)\|^2 \mathrm{d}y = ch\left\|f; L^2(\omega; L^\infty(\Xi))\right\|^2. \tag{6.34}$$

Here f^h is extended by zero from $\Omega(h)$ onto Π_h, $\sum_\alpha$ denotes the summation with $\alpha = (\alpha_1, \alpha_2)$ over all periodicity cells Ξ_h^α intersecting $\Omega(h)$, $\Xi_h^\alpha = \{x : (y - h\alpha, z) \in \Xi_h\}$, $\Xi_h = \left\{x : h^{-1}x \in \Xi\right\}$ (cf. (1.3), (1.6) and (1.5)) while $q_h^\alpha = \{y : |y_i - \alpha_i h| < h/2,\ i = 1, 2\}$.

Dealing with (6.33), we take into account the position of the sets $\operatorname{supp}(1 - X_h)$ and $\operatorname{supp}(1 - \mathcal{X}_h)$, and employ the weighted inequalities (6.21) and (6.13) to conclude that

$$\begin{aligned}
\left|\widetilde{I}_X\right| &\leq c\left\|f^h; L^2(\Omega(h))\right\|\left\|\mathfrak{R}^h_\bullet; L^2(\operatorname{supp}(1 - X_h))\right\| \\
&\leq ch^{1/2}\left\|f; L^2(\omega; L^\infty(\Xi))\right\| h(1 + |\ln h|)\left\|r^{-1}(1 + |\ln r|)^{-1}\mathfrak{R}^h_\bullet; L^2(\Omega(h))\right\| \\
&\leq ch^{3/2}(1 + |\ln h|)\left\|f; L^2\left(\omega; C^{0,\alpha}_{per}(\Xi)\right)\right\|\left\|\nabla_x \mathcal{R}^h_\bullet; L^2(\Omega(h))\right\|, \\
\left|\widetilde{I}_{\mathcal{X}}\right| &\leq c\left\|f^h; L^2(\Omega(h))\right\|\left\|\mathfrak{R}^h_\bullet; L^2(\operatorname{supp}(1 - \mathcal{X}_h))\right\| \\
&\leq ch^{1/2}\left\|f; L^2(\omega; L^\infty(\Xi))\right\| h\left\|\nabla_x \mathcal{R}^h_\bullet; L^2(\Omega(h))\right\| \\
&\leq ch^{3/2}\left\|f; L^2\left(\omega; C^{0,\alpha}_{per}(\Xi)\right)\right\|\left\|\nabla_x \mathcal{R}^h_\bullet; L^2(\Omega(h))\right\|.
\end{aligned} \tag{6.35}$$

Dealing with the remaining term (6.32) on the right of (6.24), we write down

$$\begin{aligned}
&a(\xi)\nabla_x\left(U^0(y) + bG^0(y) + h\sum_{i=1}^2 Y_i(\xi)\left(\frac{\partial U^0}{\partial y_i}(y) + b\frac{\partial G^0}{\partial y_i}(y)\right)\right) \\
&= a(\xi)(\mathbf{I}_y + \nabla_\xi Y(\xi))\left(\nabla_y U^0(y) + b\nabla_y G^0(y)\right) \\
&\quad + h\sum_{i,k=1}^2 a_{\cdot,k}(\xi) Y_i(\xi)\left(\frac{\partial^2 U^0}{\partial y_i \partial y_k}(y) + b\frac{\partial^2 G^0}{\partial y_i \partial y_k}(y)\right)
\end{aligned}$$

with $\nabla_y = (\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, 0)^t$, $Y(\xi) = (Y_1(\xi), Y_2(\xi), 0)$, and $\mathbf{I}_y$ being a 3×3 diagonal matrix whose diagonal entries are equal to 1, 1, 0, respectively. Thus,

$$\begin{aligned}
I_U &= \widehat{I}_U + \sum_{i,k=1}^2 I_U^{ik} \\
&= \left(f_h, X_h \mathcal{X}_h \mathcal{R}^h_\bullet\right)_{\Omega(h)} - \left(a(\mathbf{I}_y + \nabla_\xi Y)\left(\nabla_y U^0 + b\nabla_y G^0\right), \nabla_x\left(X_h \mathcal{X}_h \mathcal{R}^h_\bullet\right)\right)_{\Omega(h)} \\
&\quad + h\left(\sum_{i,k=1}^2 a_{\cdot,k} Y_i\left(\frac{\partial^2 U^0}{\partial y_i \partial y_k} + b\frac{\partial^2 G^0}{\partial y_i \partial y_k}\right), \nabla_x\left(X_h \mathcal{X}_h \mathcal{R}^h_\bullet\right)\right)_{\Omega(h)}
\end{aligned}$$

and

$$\begin{aligned}
\left|I_U^{ik}\right| &\leq chh^{1/2}\left(\left\|U^0; H^2(\omega)\right\| + |b|\right)\left\|\nabla_x \mathfrak{R}^h_\bullet; L^2(\Omega(h))\right\| \\
&\leq ch^{3/2}(1 + |\ln h|)\left\|f; L^2\left(\omega; C^{0,\alpha}_{per}(\Xi)\right)\right\|\left\|\mathcal{R}^h_\bullet; H^1(\Omega(h))\right\|.
\end{aligned} \tag{6.36}$$

Moreover, integrating by parts and applying the relations (3.7), (3.8) for the corrector terms Y_i, we obtain

$$\widehat{I}_U = \left(f + \sum_{i,k=1}^{2} e_k^\top a\left(e_{(i)} + \nabla_\xi Y_i\right)\left(\frac{\partial^2 U^0}{\partial y_i \partial y_k} + b\frac{\partial^2 G^0}{\partial y_i \partial y_k}\right), \mathfrak{R}_\bullet^h \right)_{\Omega(h)}$$
$$= -\sum_{i,k=1}^{2}\left(\left(A_{ik} - \sum_{i,k=1}^{2} e_{(k)}^\top a\left(e_{(i)} + \nabla_\xi Y_i\right)\right)\left(\frac{\partial^2 U^0}{\partial y_i \partial y_k} + b\frac{\partial^2 G^0}{\partial y_i \partial y_k}\right), \mathfrak{R}_\bullet^h\right)_{\Omega(h)}.$$

We emphasize that according to the standard definition (3.11) of the homogenized matrix $A = (A_{ik})$, the underbraced function has null mean over the periodicity cell $\Xi \ni \xi$. Now, to conclude with the estimate

$$|I_U| \le ch^{\alpha+1/2}\left\|f; L^2\left(\omega; C^{0,\alpha}_{per}(\Xi)\right)\right\| \left\|\mathcal{R}_\bullet^h; H^1(\Omega(h))\right\|, \tag{6.37}$$

we use the following known Lemma (see [2, Sect. 6.2.1], [43, Proposition 2.6]).

Lemma 15. *Let* $\mathcal{R} \in \mathring{H}^1(\Omega(h); \Gamma(h))$, $Z \in L^2\left(\omega; C^{0,\alpha}(\Xi)\right)$, $\alpha \in (0,1)$ *and*

$$\int_\Xi Z(y,\xi)\,\mathrm{d}\xi = 0 \quad \text{for a.e. } y \in \omega.$$

Then the following inequality is valid:

$$\left|\int_{\Omega(h)} Z\left(y, h^{-1}x\right)\mathcal{R}(x)\,\mathrm{d}x\right| \le ch^{1/2}h^\alpha \left\|Z; L^2\left(\omega; C^{0,\alpha}(\Xi)\right)\right\| \left\|\mathcal{R}; H^1(\Omega(h))\right\|.$$

□

6.4. The results on asymptotics

We now collect the obtained estimates for terms on the right-hand side of (6.24). Since $\tau > 0$ is arbitrary in (6.26) (see Sect. 4) and the bounds in (6.30), (6.35) have the factor $h^{3/2}(1+|\ln h|)^q$, in view of the assumption (6.1). We emphasize that the restriction $\alpha \le 1/2$ is meaningful. Although any $\alpha \in (0,1)$ is permitted in Lemma 15, the bound in the estimate (6.26) has the factor h^1 and therefore, lifting the smoothness of $f \in L^2\left(\omega; C^{0,\alpha}_{per}(\Xi)\right)$ does not make the final estimate better.

Theorem 16. *Under the above assumptions, the difference between the solution* $u_\bullet^h \in \mathring{H}^1(\Omega(h); \Gamma_\bullet(h))$ *of problem* (1.18)–(1.20) *and the global approximation* (6.19) *admits the following estimate*

$$\left\|u_\bullet^h - \mathcal{U}_\bullet^h; H^1(\Omega(h))\right\| \le ch^{\alpha+1/2}\left\|f; L^2\left(\omega; C^{0,\alpha}_{per}(\Xi)\right)\right\|, \tag{6.38}$$

where c *is a constant independent of the parameter* $h \in (0,1]$ *and the function* $f^h(x) = f\left(y, h^{-1}x\right)$, $f \in L^2\left(\omega; C^{0,\alpha}_{per}(\Xi)\right)$.

Let us erase the cut-off functions in the approximation function (6.19). First of all, by the estimates (6.13)–(6.15) we obtain

$$\left\|(1-\mathcal{X}_h)\left(U + h\sum_{i=1}^{2} Y_i \frac{\partial U}{\partial y_i}\right); H^1(\Omega(h))\right\|$$
$$\le ch^{1/2}\left(h^{-1}\left\|U; L^2\left(\omega_h^{\Cap}\right)\right\| + \left\|\nabla_y U; L^2\left(\omega_h^{\Cap}\right)\right\| + h\left\|\nabla_y^2 U; L^2\left(\omega_h^{\Cap}\right)\right\|\right)$$
$$\le ch\left\|f; L^2\left(\omega; C^{0,\alpha}_{per}(\Xi)\right)\right\|.$$

Moreover, taking into account the structure of the boundary layer term (5.15) and the pointwise estimates for the rate of decay in (4.21), we see that

$$
\begin{aligned}
\left\|(1-\mathcal{X}_h)\,w;H^1(\Omega(h))\right\|^2 &\le c\int\limits_{\Omega^{\Cap}(h)}\left(h^{-2}|w(\xi)|^2+|\nabla_x w(\xi)|^2\right)\mathrm{d}x\\
&\le ch^{-2}|b|^2\int\limits_{\Omega^{\Cap}(h)}(1+|\xi|)^{-2}\,\mathrm{d}x\le ch^{-2}|b|^2\left(1+h^{-1}\right)^{-2}\operatorname{mes}_3\Omega^{\Cap}(h)\\
&\le ch^2\left\|f;L^2\left(\omega;C^{0,\alpha}_{per}(\Xi)\right)\right\|^2.
\end{aligned}
$$

Hence, the cut-off function $\mathcal{X}_h$ can be excluded from approximate solution (6.19) without any loss of the accuracy. It is also the case for the cut-off function X_h. Indeed, the function U has logarithmic singularity and does not belong to $H^1(\Omega(h))$. However we have

$$
\begin{aligned}
\left\|(1-X_h)\,U;L^2(\Omega(h))\right\|^2 &\le ch^{1/2}\left(\int\limits_0^{ch}\left(1+\frac{|\ln r|}{|\ln h|}\right)^2 r\,\mathrm{d}r\right)^{1/2}\left\|f;L^2\left(\omega;C^{0,\alpha}_{per}(\Xi)\right)\right\|\\
&\le ch^{3/2}\left\|f;L^2\left(\omega;C^{0,\alpha}_{per}(\Xi)\right)\right\|,\\
h\left\|X_hY_i\frac{\partial U}{\partial y_i};L^2(\Omega(h))\right\| &\le chh^{1/2}\left(\int\limits_{ch}^{C}\left(1+|\ln h|^{-2}r^{-2}\right)r\,\mathrm{d}r\right)^{1/2}\left\|f;L^2\left(\omega;C^{0,\alpha}_{per}(\Xi)\right)\right\|\\
&\le ch^{3/2}\left\|f;L^2\left(\omega;C^{0,\alpha}_{per}(\Xi)\right)\right\|,\\
\left\|X_hw;L^2(\Omega(h))\right\| &\le ch^{-\tau+3/2}\left\|(1+|\xi|)^{-\tau}w;L^2(\Pi_h)\right\|\le ch^{-\tau+3/2}|b|\\
&\le ch\left\|f;L^2\left(\omega;C^{0,\alpha}_{per}(\Xi)\right)\right\|.
\end{aligned}
$$

Hence, the proximity of $u^h_\bullet$ and U is established in the L^2-norm. We now formulate the final result.

Theorem 17. *There hold the estimates*

$$
\begin{aligned}
\left\|u^h_\bullet - X_h\left(U+h\sum_{i=1}^{2}Y_i\frac{\partial U}{\partial y_i}\right)-w;H^1(\Omega(h))\right\| &\le ch^{\alpha+1/2}\left\|f;L^2\left(\omega;C^{0,\alpha}_{per}(\Xi)\right)\right\|,\\
\left\|u^h_\bullet - U;L^2(\Omega(h))\right\| &\le ch^{\alpha+1/2}\left\|f;L^2\left(\omega;C^{0,\alpha}_{per}(\Xi)\right)\right\|,
\end{aligned}
$$

where $U=U^0+bG$ *is the solution of problem* (1.23), (1.24) *in the punctured domain* (2.6)*,* b *is the coefficient* (5.10)*,* Y_i *are the asymptotic corrector components* (*see* (3.7)–(3.9)) *and* w *is the boundary layer term* (5.15) *determined in Sect.* 4.

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