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PERIODIC HOMOGENIZATION OF NONLOCAL OPERATORS WITH A CONVOLUTION-TYPE KERNEL*

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Abstract. The paper deals with a homogenization problem for a nonlocal linear operator with a kernel of convolution type in a medium with a periodic structure. We consider the natural diffusive scaling of this operator and study the limit behavior of the rescaled operators as the scaling parameter tends to 0. More precisely we show that in the topology of resolvent convergence the family of rescaled operators converges to a second order elliptic operator with constant coefficients. We also prove the convergence of the corresponding semigroups both in L^2 space and the space of continuous functions and show that for the related family of Markov processes the invariance principle holds.

Key words. homogenization, nonlocal operators, jump Markov processes, semigroup convergence

AMS subject classifications. 45E10, 60J75, 35B27, 45M05

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1. Introduction.

Recently there has been an increasing interest in the integral operators with a kernel of convolution type. These operators appear in many applications, such as models of population dynamics and the continuous contact model, where they describe the evolution of the density of a population; see, for instance, [4, 5, 8, 11, 12] for the details. In these papers only the case of spatially homogeneous dispersal kernel has been investigated. We focus in this paper on the spatially inhomogeneous dispersal kernel depending both on the displacement $y-x$ and on the starting and the ending positions $x, y \in \mathbb{R}^d$. We also mention here that the convolution-type nonlocal operators describe the evolution of jump Markov processes (see, for instance, [5]). If we compare convolution-type operators with the second order elliptic differential operators being the generators of diffusion process we can observe that some of the properties of these two classes of operators are quite similar while the others are rather distinct.

In this connection it is interesting to understand which homogenization results obtained for elliptic differential operators with rapidly oscillating coefficients (see, for instance, [6, 10]) remain valid for nonlocal convolution-type operators and which are not. In this paper we study a homogenization problem for convolution-type operators in a periodic medium.

We consider an integral convolution-type operator of the form

$$(Lu)(x) = \lambda(x) \int_{\mathbb{R}^d} a(x-y)\mu(y)(u(y)-u(x))dy. \tag{1}$$

Here $\lambda(x)$ and $\mu(y)$ are bounded positive periodic functions characterizing the properties of the medium, and $a(z)$ is the jump kernel being a positive integrable function such that $a(-z) = a(z)$. The detailed assumptions are given in the next section.

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We then make a diffusive scaling of this operator

$$(L^\varepsilon u)(x) = \varepsilon^{-d-2}\lambda\Big(\frac{x}{\varepsilon}\Big)\int_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\mu\Big(\frac{y}{\varepsilon}\Big)(u(y)-u(x))dy, \tag{2}$$

where ε is a positive scaling factor. Our goal is to study the homogenization problem for operators L^ε, that is, to characterize the limit behavior of L^ε as $\varepsilon \to 0$.

Homogenization theory of differential operators is a well-developed field, and there is a vast literature on this topic; we mention here the monographs [2] and [10]. In contrast with differential operators, the homogenization theory for nonlocal operators is not so well-developed. In the existing mathematical literature there are several works devoted to homogenization of integro-differential equations with Levy-type operators, where an essential progress has been achieved. In particular, in [1] the periodic homogenization problem for equations with the Levy operator has been considered in the framework of viscosity solutions; the work [15] deals with homogenization of nonlinear equations with Levy-type operators.

In [9] jump-diffusions with periodic coefficients driven by stable Lévy processes with stability index $\alpha > 1$ were considered. It was shown that the limit process is an α-stable Lévy process with an averaged jump-measure.

The paper [13] deals with scaling limits of the solutions to stochastic differential equations with stationary coefficients driven by Poisson random measures and Brownian motions. The annealed convergence theorem is proved, in which the limit exhibits a diffusive or superdiffusive behavior, depending on the integrability properties of the Poisson random measure. It is important in this paper that the diffusion coefficient does not degenerate.

In the recent work [14] the homogenization problem for a Feller diffusion process with jumps generated by an integro-differential operator has been studied under the assumptions that the corresponding generator has rapidly periodically oscillating diffusion and jump coefficients, and under additional regularity conditions. It should be noted that the generators considered in [14] have a nonzero diffusion part which improves the compactness properties of the corresponding resolvent.

In contrast with the abovementioned papers we consider here the homogenization problem for integral operators with an integrable kernel that oscillates both in x and y variables.

The goal of the present work is to prove a homogenization result for the operators L^ε. More precisely, we are going to show that the family L^ε converges to a second order divergence form elliptic operator with constant coefficient in the so-called G-topology, that is, for any $m > 0$ the family of operators $(-L^\varepsilon + m)^{-1}$ converges strongly in $L^2(\mathbb{R}^d)$ to the operator $(-L^0 + m)^{-1}$, where $L^0 = \Theta^{ij}\frac{\partial^2}{\partial x^i \partial x^j}$ with a positive definite constant matrix Θ.

As a consequence of this convergence we obtain the convergence of the corresponding semigroups.

Under additional regularity assumptions on the functions $a(x)$, $\lambda(x)$, and $\mu(x)$ the operators L and L^ε act in the space $C_0(\mathbb{R}^d)$, where $C_0(\mathbb{R}^d)$ stands for the Banach space of continuous functions vanishing at infinity with the norm $\|f\| = \sup|f(x)|$. Also these operators generate Markov processes with trajectories in the space of cádlág functions with values in $\mathbb{R}^d$; we denote this space by $D_{\mathbb{R}^d}[0,\infty)$. Our second aim is to show that under the mentioned conditions the homogenization result is also valid in the space $C_0(\mathbb{R}^d)$ and that under the diffusive scaling the invariance principle holds for the family of rescaled processes.

The methods used in the paper rely on asymptotic expansion techniques and constructing first and second order periodic correctors. Notice that, in contrast with the case of differential operators, in our case the kernel of integral operator in the auxiliary cell problem differs from the kernel in the original problem. This is an interesting feature of the studied nonlocal operators.

Another crucial feature of the nonlocal operators considered here is the noncompactness of their resolvent. In this connection we cannot use the techniques based on the compactness of the family of solutions. Instead we construct an ansatz that approximates the solution in L^2 and C_0 norms. In order to justify the solvability of the cell problem we use the fact that the corresponding auxiliary periodic operator can be represented as the sum of a bounded coercive operator and a compact operator.

The paper is organized as follows. In section 2 we provide the detailed setting of the problem and formulate our main results.

Then in section 3 we introduce a number of auxiliary periodic problems, define correctors, and prove some technical statements.

Section 4 is devoted to the proof of the homogenization result both in the space $L^2(\mathbb{R}^d)$ (Theorem 1) and in the space $C_0(\mathbb{R}^d)$ (Theorem 2).

Section 5 deals with the convergence of the corresponding semigroups. We also prove in this section the invariance principle (Theorem 3).

2. Problem setup and main results. In this section we provide all the conditions on the coefficients of operator L and then formulate our main results.

For the function $a(z)$ we assume that

$$a(z) \in L^1(\mathbb{R}^d) \cap L^2_{\mathrm{loc}}(\mathbb{R}^d), \ \ a(z) \ge 0; \ \ a(-z) = a(z), \tag{3}$$

and

$$\|a\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} a(z)\, dz = a_1 > 0; \qquad \int_{\mathbb{R}^d} |z|^2 a(z)\, dz < \infty. \tag{4}$$

Functions $\lambda(x), \mu(x)$ are periodic and bounded from above and from below:

$$0 < \alpha_1 \le \lambda(x), \ \mu(x) \le \alpha_2 < \infty. \tag{5}$$

In what follows we identify periodic functions with functions defined on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. The operator L is a bounded (not necessary symmetric) operator in $L^2(\mathbb{R}^d)$. Indeed, letting

$$(L^- u)(x) = \lambda(x) \int_{\mathbb{R}^d} a(x-y)\mu(y)u(y)\, dy,$$

we have for any $u \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned}
\|(L^- u)\|^2_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \lambda^2(x) a(x-y)\mu(y)u(y)a(x-z)\mu(z)u(z)\, dzdydx \\
&\le \alpha_2^4 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x-y)|u(y)|a(x-z)|u(z)|\, dzdydx \\
&= \alpha_2^4 \int_{\mathbb{R}^d} a(y)\, dy \int_{\mathbb{R}^d} a(z)\, dz \int_{\mathbb{R}^d} |u(x+y)|\, |u(x+z)| dx \\
&\le a_1^2 \alpha_2^4 \|u\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Therefore, L^- can be extended to a bounded operator acting from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. This implies the boundedness of L. We denote the space of bounded operators from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ by $\mathcal{L}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$.

Let us consider the family of operators

$$(L^\varepsilon u)(x) = \frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\lambda\Big(\frac{x}{\varepsilon}\Big)\mu\Big(\frac{y}{\varepsilon}\Big)\big(u(y)-u(x)\big)dy. \tag{6}$$

Since for any $\varepsilon > 0$ the bounded operator L^ε is symmetric and nonpositive in the space $L^2(\mathbb{R}^d, \nu_\varepsilon)$ with $\nu_\varepsilon(x) = \frac{\mu(\frac{x}{\varepsilon})}{\lambda(\frac{x}{\varepsilon})}$, then by the spectral theorem

$$\|(m - L^\varepsilon)^{-1}\|_{L^2(\mathbb{R}^d,\nu_\varepsilon)} \le \frac{1}{m}$$

for any $m > 0$. Under our assumption (5) we have

$$\gamma_1 \|f\|^2_{L^2(\mathbb{R}^d)} \le \|f\|^2_{L^2(\mathbb{R}^d,\nu_\varepsilon)} \le \gamma_2 \|f\|^2_{L^2(\mathbb{R}^d)}$$

with

$$0 < \gamma_1 \le \nu_\varepsilon(x) \le \gamma_2 < \infty.$$

Therefore, for any $m > 0$ the operators $(L^\varepsilon - m)^{-1}$ are bounded in $L^2(\mathbb{R}^d)$ uniformly in ε.

We are interested in the limit behavior of the operators L^ε as $\varepsilon \to 0$. Since the norm of L^ε in $L^2(\mathbb{R}^d)$ tends to infinity, the limit operator, if it exists, need not be bounded. We are going to show that the operators L^ε converge in the topology of resolvent convergence. Let us fix an arbitrary $m > 0$ and define u^ε as the solution of the equation

$$(L^\varepsilon - m)u^\varepsilon = f, \quad \text{ i.e., } \; u^\varepsilon = (L^\varepsilon - m)^{-1} f, \tag{7}$$

with $f \in L^2(\mathbb{R}^d)$. Denote by L^0 the following operator in $L^2(\mathbb{R}^d)$:

$$L^0 u = \Theta^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} = \Theta \cdot \nabla\nabla u, \quad \mathcal{D}(L^0) = H^2(\mathbb{R}^d) \tag{8}$$

with a symmetric positive definite matrix $\Theta = \{\Theta^{ij}\},\ i,j = 1,\dots,d$, expressed (in (38)) in terms of a corrector $\varkappa_1$ solving in turn the cell problem (24). Here and in what follows we assume the summation over repeated indices; $\Theta\cdot\nabla\nabla u$ stands for $\Theta^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j}$. Notice that L^0 is a nonpositive self-adjoint operator in $L^2(\mathbb{R}^d)$, and therefore for any $m > 0$ the operator $(L^0 - m)^{-1}$ is bounded. Let $u_0(x)$ be a solution of the equation

$$\Theta\cdot\nabla\nabla u_0 - m u_0 = f, \quad \text{ i.e., } \; u_0 = (L^0 - m)^{-1} f, \tag{9}$$

with the same right-hand side f as in (7).

Our main results read as follows.

Theorem 1. *Let functions $a(x)$, $\lambda(x)$, and $\mu(x)$ satisfy conditions* (3)–(5). *Then for each $m > 0$ the family of resolvents $(L^\varepsilon - m)^{-1}$ converges strongly to the resolvent $(L^0 - m)^{-1}$, as $\varepsilon \to 0$, that is, for any $f \in L^2(\mathbb{R}^d)$ it holds that*

$$\|(L^\varepsilon - m)^{-1} f - (L^0 - m)^{-1} f\|_{L^2(\mathbb{R}^d)} \to 0, \quad \text{as } \varepsilon \to 0. \tag{10}$$

The analogous result holds in the space $C_0(\mathbb{R}^d)$ under natural additional assumptions on the operator L.

THEOREM 2. *Suppose the function $a(x)$ satisfies the following conditions:*

$$a(x) \in C(\mathbb{R}^d),\ a(x) = a(-x),\ a(x) \geq 0, \quad a(x) \leq \frac{K}{1 + |x|^{d+\delta}} \quad \text{with } \delta > 2. \tag{11}$$

Assume furthermore that $\lambda(x), \mu(x)$ are continuous and periodic functions satisfying bounds (5). *Then for each $m > 0$ the family of resolvents $(L^\varepsilon - m)^{-1}$ converges strongly in $C_0(\mathbb{R}^d)$ to the resolvent $(L^0 - m)^{-1}$, as $\varepsilon \to 0$, that is, for any $f \in C_0(\mathbb{R}^d)$ it holds that*

$$\|(L^\varepsilon - m)^{-1} f - (L^0 - m)^{-1} f\|_{C_0(\mathbb{R}^d)} \ \to 0, \qquad \text{as } \ \varepsilon \to 0. \tag{12}$$

Consider the Markov semigroup $T(t)$ generated by the operator L defined in (1) and the corresponding Markov jump process X. The family of semigroups $T^\varepsilon(t)$ is generated by the operators L^ε given by (6). It is obtained by diffusion scaling of the semigroup $T(t)$, and we denote the corresponding rescaled Markov jump processes by X_ε. We show that the processes X_ε converges in the path space to a Brownian motion $B^{(2\Theta)}$ with covariance matrix 2Θ, where matrix Θ is defined in (38).

THEOREM 3 (invariance principle). *Let X_ε be a Markov process corresponding to the semigroup $T^\varepsilon(t)$ with an initial distribution ν, and X_0 be a Markov process corresponding to the semigroup $T^0(t)$ with the same initial distribution. Then the Markov processes X_ε and X_0 have sample paths in $D_{\mathbb{R}^d}[0,\infty)$, and $X_\varepsilon \Rightarrow X_0$ in $D_{\mathbb{R}^d}[0,\infty)$.*

3. Correctors and auxiliary cell problem. In this section we introduce a number of auxiliary functions and quantities that will be used in the further analysis. We are going to approximate the solution u^ε of problem (7) using an ansatz constructed in terms of the solution u_0 of the limit problem (9). To this end we consider auxiliary periodic problems, whose solutions (so-called correctors) are used in the construction of this ansatz and define the coefficients of effective operator L^0 in (9). We first deal with functions from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

LEMMA 4. *Assume that $u \in \mathcal{S}(\mathbb{R}^d)$. Then there exist functions $\varkappa_1 \in (L^2(\mathbb{T}^d))^d$ and $\varkappa_2 \in (L^2(\mathbb{T}^d))^{d^2}$ (a vector function $\varkappa_1$ and a matrix function $\varkappa_2$) and a positive definite matrix Θ such that for the function w^ε defined by*

$$w^\varepsilon(x) = u(x) + \varepsilon \varkappa_1\left(\frac{x}{\varepsilon}\right) \cdot \nabla u(x) + \varepsilon^2 \varkappa_2\left(\frac{x}{\varepsilon}\right) \cdot \nabla\nabla u(x) \tag{13}$$

we have

$$L^\varepsilon w^\varepsilon = \Theta \cdot \nabla\nabla u + \phi_\varepsilon, \qquad \text{where } \ \lim_{\varepsilon \to 0} \|\phi_\varepsilon\|_{L^2(\mathbb{R}^d)} = 0. \tag{14}$$

Proof. After substituting w^ε defined in (13) for u in (6) we get

$$\begin{aligned}(L^\varepsilon w^\varepsilon)(x) = \frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big) \lambda\Big(\frac{x}{\varepsilon}\Big) \mu\Big(\frac{y}{\varepsilon}\Big) \Big\{ & u(y) + \varepsilon \varkappa_1\Big(\frac{y}{\varepsilon}\Big) \cdot \nabla u(y) + \varepsilon^2 \varkappa_2\Big(\frac{y}{\varepsilon}\Big) \\ & \cdot \nabla\nabla u(y) - u(x) - \varepsilon \varkappa_1\Big(\frac{x}{\varepsilon}\Big) \cdot \nabla u(x) \\ & - \varepsilon^2 \varkappa_2\Big(\frac{x}{\varepsilon}\Big) \cdot \nabla\nabla u(x) \Big\} dy.\end{aligned}$$

After change of variables $\frac{x-y}{\varepsilon} = z$ we get

(15)
$$
\begin{aligned}
&(L^\varepsilon w^\varepsilon)(x) \\
&\quad = \frac{1}{\varepsilon^2}\int_{\mathbb{R}} dz\; a(z)\lambda\Big(\frac{x}{\varepsilon}\Big)\mu\Big(\frac{x}{\varepsilon}-z\Big)\Big\{u(x-\varepsilon z)+\varepsilon\varkappa_1\Big(\frac{x}{\varepsilon}-z\Big)\cdot\nabla u(x-\varepsilon z) \\
&\qquad\qquad + \varepsilon^2\varkappa_2\Big(\frac{x}{\varepsilon}-z\Big)\cdot\nabla\nabla u(x-\varepsilon z) - u(x) \\
&\qquad\qquad - \varepsilon\varkappa_1\Big(\frac{x}{\varepsilon}\Big)\cdot\nabla u(x) - \varepsilon^2\varkappa_2\Big(\frac{x}{\varepsilon}\Big)\cdot\nabla\nabla u(x)\Big\}.
\end{aligned}
$$

Using the following identity based on the integral form of the remainder term in the Taylor expansion

$$
u(y) = u(x) + \int_0^1 \frac{\partial}{\partial t}\; u(x+(y-x)t)\; dt = u(x) + \int_0^1 \nabla u(x+(y-x)t)\cdot(y-x)\; dt,
$$

$$
u(y) = u(x) + \nabla u(x)\cdot(y-x) + \int_0^1 \nabla\nabla u(x+(y-x)t)(y-x)\cdot(y-x)(1-t)\; dt,
$$

which is valid for any $x, y \in \mathbb{R}^d$, we can rearrange (15) as follows:

$$
\begin{aligned}
&(L^\varepsilon w^\varepsilon)(x) \\
&\quad = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} dz\, a(z)\lambda\Big(\frac{x}{\varepsilon}\Big)\mu\Big(\frac{x}{\varepsilon}-z\Big)\Big\{(u(x) - \varepsilon z\cdot\nabla u(x) + \varepsilon^2\int_0^1 \nabla\nabla u(x-\varepsilon z t) \\
&\qquad \cdot z\otimes z\,(1-t)\,dt + \varepsilon\varkappa_1\Big(\frac{x}{\varepsilon}-z\Big) \\
&\qquad \cdot\Big(\nabla u(x) - \varepsilon\nabla\nabla u(x)\,z \\
&\qquad\quad + \varepsilon^2\int_0^1 \nabla\nabla\nabla u(x-\varepsilon z t) z\otimes z(1-t)\; dt\Big) \\
&\qquad + \varepsilon^2\varkappa_2\Big(\frac{x}{\varepsilon}-z\Big)\cdot\nabla\nabla u(x-\varepsilon z) - u(x) - \varepsilon\varkappa_1\Big(\frac{x}{\varepsilon}\Big) \\
&\qquad \cdot\nabla u(x) - \varepsilon^2\varkappa_2\Big(\frac{x}{\varepsilon}\Big)\cdot\nabla\nabla u(x)\Big\},
\end{aligned}
$$

where

$$
z\otimes z = \{z^i z^j\}\big|_{i,j=1}^d, \qquad \nabla\nabla u(\cdot)z = \frac{\partial^2 u}{\partial x^i\partial x^j}(\cdot)z^j
$$

and

$$
\nabla\nabla\nabla u(\cdot)z\otimes z = \frac{\partial^3 u}{\partial x^i\partial x^j\partial x^k}(\cdot)z^j z^k.
$$

Collecting power-like terms in the last relation we obtain

(16)
$$
\begin{aligned}
&(L^\varepsilon w^\varepsilon)(x) \\
&\quad = \frac{1}{\varepsilon}\lambda\Big(\frac{x}{\varepsilon}\Big)\nabla u(x)\cdot\int_{\mathbb{R}^d}\Big\{-z+\varkappa_1\Big(\frac{x}{\varepsilon}-z\Big)-\varkappa_1\Big(\frac{x}{\varepsilon}\Big)\Big\}a(z)\mu\Big(\frac{x}{\varepsilon}-z\Big)\,dz \\
&\qquad + \lambda\Big(\frac{x}{\varepsilon}\Big)\nabla\nabla u(x)\cdot\int_{\mathbb{R}^d}\Big\{\frac{1}{2}z\otimes z - z\otimes\varkappa_1\Big(\frac{x}{\varepsilon}-z\Big)+\varkappa_2\Big(\frac{x}{\varepsilon}-z\Big)-\varkappa_2\Big(\frac{x}{\varepsilon}\Big)\Big\} \\
&\qquad a(z)\mu\Big(\frac{x}{\varepsilon}-z\Big)\,dz + \phi_\varepsilon(x)
\end{aligned}
$$

with

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} dz\, a(z)\lambda\Big(\frac{x}{\varepsilon}\Big)\mu\Big(\frac{x}{\varepsilon}-z\Big)$$

$$\tag{17} \begin{aligned} &\times\Big\{\varepsilon^2\int_0^1 \nabla\nabla u(x-\varepsilon zt)\cdot z\otimes z\,(1-t)\;dt - \frac{\varepsilon^2}{2}\nabla\nabla u(x)\cdot z\otimes z \\ &\quad + \varepsilon^3\varkappa_1\Big(\frac{x}{\varepsilon}-z\Big)\cdot\int_0^1 \nabla\nabla\nabla u(x-\varepsilon zt)z\otimes z(1-t)\,dt \\ &\quad - \varepsilon^3\varkappa_2\Big(\frac{x}{\varepsilon}-z\Big)\cdot\int_0^1 \nabla\nabla\nabla u(x-\varepsilon zt)z\,dt\Big\}. \end{aligned}$$

Next we prove that $\|\phi_\varepsilon\|_{L^2(\mathbb{R}^d)}$ is vanishing as $\varepsilon\to 0$.

PROPOSITION 5. *Let $u\in\mathcal{S}(\mathbb{R}^d)$, and assume that λ,μ are periodic functions satisfying bounds* (5)*, and all the components of $\varkappa_1$ and $\varkappa_2$ are elements of $L^2(\mathbb{T}^d)$. Then*

$$\tag{18} \|\phi_\varepsilon(x)\|_{L^2(\mathbb{R}^d)} \to 0, \qquad \textit{as}\;\; \varepsilon\to 0.$$

Proof. The first term on the right-hand side in (17) (the term of order ε^0) reads

$$\tag{19} \begin{aligned} &\phi_\varepsilon^{(1)}(x) \\ &\quad = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} dz\, a(z)\lambda\Big(\frac{x}{\varepsilon}\Big)\mu\Big(\frac{x}{\varepsilon}-z\Big)\varepsilon^2\int_0^1\Big(\nabla\nabla u(x-\varepsilon zt)-\nabla\nabla u(x)\Big)\cdot z\otimes z(1-t)\,dt \\ &\quad = \int_{|z|\le R} dz\; a(z)\lambda\Big(\frac{x}{\varepsilon}\Big)\mu\Big(\frac{x}{\varepsilon}-z\Big)\int_0^1\Big(\nabla\nabla u(x-\varepsilon zt)-\nabla\nabla u(x)\Big)\cdot z\otimes z(1-t)\,dt \\ &\qquad + \int_{|z|>R} dz\; a(z)\lambda\Big(\frac{x}{\varepsilon}\Big)\mu\Big(\frac{x}{\varepsilon}-z\Big)\int_0^1\Big(\nabla\nabla u(x-\varepsilon zt)-\nabla\nabla u(x)\Big)\cdot z\otimes z(1-t)\;dt \\ &\quad := \phi_\varepsilon^{(1,\le R)}(x)+\phi_\varepsilon^{(1,>R)}(x). \end{aligned}$$

Then

$$\begin{aligned} &\|\phi_\varepsilon^{(1,\le R)}\|_{L^2(\mathbb{R}^d)} \\ &\quad\le \alpha_2^2\sup_{|z|\le R}\|\nabla\nabla u(x-\varepsilon zt)-\nabla\nabla u(x)\|_{L^2(\mathbb{R}^d)^{d^2}}\int_{\mathbb{R}^d}|z|^2\;a(z)\int_0^1(1-t)\,dt\,dz \\ &\quad = \frac{\alpha_2^2}{2}\sup_{|z|\le R}\|\nabla\nabla u(x-\varepsilon zt)-\nabla\nabla u(x)\|_{L^2(\mathbb{R}^d)^{d^2}}\int_{\mathbb{R}^d}|z|^2\;a(z)\;dz \end{aligned}$$

and

$$\|\phi_\varepsilon^{(1,>R)}\|_{L^2(\mathbb{R}^d)} \le 2\alpha_2^2\|\nabla\nabla u(x)\|_{L^2(\mathbb{R}^d)^{d^2}}\int_{|z|>R}|z|^2\;a(z)\;dz.$$

If we take $R=R(\varepsilon)=\frac{1}{\sqrt{\varepsilon}}$, then both

$$\|\phi_\varepsilon^{(1,\le R(\varepsilon))}\|_{L^2(\mathbb{R}^d)}\to 0 \quad\text{and}\quad \|\phi_\varepsilon^{(1,>R(\varepsilon))}\|_{L^2(\mathbb{R}^d)}\to 0, \qquad \text{as}\;\;\varepsilon\to 0.$$

This yields

$$\tag{20} \|\phi_\varepsilon^{(1)}\|_{L^2(\mathbb{R}^d)}\to 0, \qquad \text{as}\;\;\varepsilon\to 0.$$

For the second term on the right-hand side of (17)

$$\phi_\varepsilon^{(2)}(x) = \varepsilon \int_{\mathbb{R}^d} dz\, a(z)\lambda\Big(\frac{x}{\varepsilon}\Big)\mu\Big(\frac{x}{\varepsilon} - z\Big)\varkappa_1\Big(\frac{x}{\varepsilon} - z\Big) \cdot \int_0^1 \nabla\nabla\nabla u(x - \varepsilon z t) z \otimes z\,(1-t)\,dt$$

we have

$$\begin{aligned} &\|\phi_\varepsilon^{(2)}(x)\|_{L^2(\mathbb{R}^d)} \\ &\le \frac{\varepsilon}{2}\alpha_2^2 \sup_{z,q\in\mathbb{R}^d} \Big\| \varkappa_1\Big(\frac{x}{\varepsilon} - z\Big)\ \nabla\nabla\nabla u(x - \varepsilon z + q)\Big\|_{L^2(\mathbb{R}^d)^{d^2}} \int_{\mathbb{R}^d} |z|^2\, a(z)\, dz. \end{aligned} \tag{21}$$

We estimate now $\sup_{z,q\in\mathbb{R}^d} \|\varkappa_1(\frac{x}{\varepsilon} - z)\,\nabla\nabla\nabla u(x-\varepsilon z + q)\|_{L^2(\mathbb{R}^d)^{d^2}}$. Taking $y = x - \varepsilon z$ and considering the fact that the function $\varkappa_1$ is periodic we get

$$\sup_{q\in\mathbb{R}^d} \Big\| \varkappa_1\Big(\frac{y}{\varepsilon}\Big)\ \nabla\nabla\nabla u(y+q)\Big\|_{L^2(\mathbb{R}^d)^{d^2}} = \sup_{q\in\varepsilon\mathbb{T}^d} \Big\| \varkappa_1\Big(\frac{y}{\varepsilon}\Big)\ \nabla\nabla\nabla u(y+q)\Big\|_{L^2(\mathbb{R}^d)^{d^2}}.$$

Let us show that this quantity admits a uniform in ε upper bound. Indeed, denoting $I_k(\varepsilon) = \varepsilon k + \varepsilon\mathbb{T}^d,\ k\in\mathbb{Z}^d$ with $\mathbb{T} = [0,1]^d$, we have

$$\begin{aligned} &\sup_{q\in\varepsilon\mathbb{T}^d} \Big\| \varkappa_1\Big(\frac{y}{\varepsilon}\Big)\ \nabla\nabla\nabla u(y+q)\Big\|^2_{(L^2(\mathbb{R}^d))^{d^2}} \\ &\le \sup_{q\in\varepsilon\mathbb{T}^d} \sum_{i,j,l,m=1}^d \sum_{k\in\mathbb{Z}^d} \int_{I_k(\varepsilon)} \Big[\varkappa_1^i\Big(\frac{y}{\varepsilon}\Big)\Big]^2\ \big[\partial_{x^j}\partial_{x^l}\partial_{x^m} u(y+q)\big]^2\ dy \\ &\le \sum_{j,l,m=1}^d \sum_{k\in\mathbb{Z}^d} \max_{y\in I_k(\varepsilon),\ q\in\varepsilon\mathbb{T}^d} \big[\partial_{x^j}\partial_{x^l}\partial_{x^m} u(y+q)\big]^2 \int_{I_k(\varepsilon)} \varkappa_1^2\Big(\frac{y}{\varepsilon}\Big)\ dy \\ &= \|\varkappa_1\|^2_{(L^2(\mathbb{T}^d))^d}\ \varepsilon^d \sum_{j,l,m=1}^d \sum_{k\in\mathbb{Z}^d} \max_{y\in I_k(\varepsilon),\ q\in\varepsilon\mathbb{T}^d} \big[\partial_{x^j}\partial_{x^l}\partial_{x^m} u(y+q)\big]^2 \\ &\longrightarrow \|\varkappa_1\|^2_{(L^2(\mathbb{T}^d))^d} \sum_{j,l,m=1}^d \|\partial_{x^j}\partial_{x^l}\partial_{x^m} u\|^2_{L^2(\mathbb{R}^d)}, \end{aligned}$$

as $\varepsilon \to 0$. Here we have used the fact that for a functions $\psi \in \mathcal{S}(\mathbb{R}^d)$

$$\varepsilon^d \sum_{k\in\mathbb{Z}^d} \max_{y\in I_k(\varepsilon),\ q\in\varepsilon\mathbb{T}^d} \psi(y+q) \to \int_{\mathbb{R}^d} \psi(x)\ dx, \quad \varepsilon \to 0.$$

Thus from estimate (21) it follows that $\|\phi_\varepsilon^{(2)}\|_{L^2(\mathbb{R}^d)} \to 0$, as $\varepsilon \to 0$.

Similarly for the third term on the right-hand side of (17) we have

$$\begin{aligned} &\|\phi_\varepsilon^{(3)}(x)\|_{L^2(\mathbb{R}^d)} \\ &= \varepsilon \Big\| \int_{\mathbb{R}^d} dz\ a(z)\lambda\Big(\frac{x}{\varepsilon}\Big)\mu\Big(\frac{x}{\varepsilon} - z\Big)\varkappa_2\Big(\frac{x}{\varepsilon} - z\Big) \cdot \int_0^1 \nabla\nabla\nabla u(x - \varepsilon z t)\ z\ dt \Big\|_{L^2(\mathbb{R}^d)} \\ &\le \varepsilon c(d)\alpha_2^2 \Big(\int_{\mathbb{R}^d} |z|\ a(z)\ dz\Big) \sup_{q\in\mathbb{R}^d} \Big\| \varkappa_2\Big(\frac{y}{\varepsilon}\Big)\ \nabla\nabla\nabla u(y+q)\Big\|_{(L^2(\mathbb{R}^d))^d} \le \varepsilon C_3 \end{aligned} \tag{22}$$

for all sufficiently small ε, since as above we have for all i,j,l,m,n

$$\Big\| \varkappa_2^{in}\Big(\frac{y}{\varepsilon}\Big)\ \partial_{x^j}\partial_{x^l}\partial_{x^m} u(y+q)\Big\|^2_{L^2(\mathbb{R}^d)} \to \|\varkappa_2^{in}\|^2_{L^2(\mathbb{T}^d)} \|\partial_{x^j}\partial_{x^l}\partial_{x^m} u\|^2_{L^2(\mathbb{R}^d)},$$

as $\varepsilon \to 0$, uniformly in $q \in \mathbb{R}^d$. $\square$

Our next step of the proof deals with constructing the correctors $\varkappa_1$ and $\varkappa_2$. Denote $\xi = \frac{x}{\varepsilon}$ a variable on the period: $\xi \in \mathbb{T}^d = [0,1]^d$, and then $\lambda(\xi), \mu(\xi), \varkappa_1(\xi), \varkappa_2(\xi)$ are functions on $\mathbb{T}^d$ and (16) can be understood as equations for the functions $\varkappa_1(\xi), \varkappa_2(\xi)$, $\xi \in \mathbb{T}^d$ on the torus.

We collect all the terms of the order ε^{-1} in (16) and equate them to 0. This yields the following equation for the vector function $\varkappa_1(\xi) = \{\varkappa_1^i(\xi)\},\ \xi \in \mathbb{T}^d,\ i = 1, \dots, d$, as unknown function:

$$\int_{\mathbb{R}^d} \Big(- z^i + \varkappa_1^i(\xi - z) - \varkappa_1^i(\xi) \Big)\ a(z)\mu(\xi - z)\ dz = 0 \quad \forall\ i = 1, \dots, d. \tag{23}$$

Here $\varkappa_1(q),\ q \in \mathbb{R}^d$, is the periodic extension of $\varkappa_1(\xi),\ \xi \in \mathbb{T}^d$. Notice that (23) is a system of uncoupled equations. After change of variables $q = \xi - z \in \mathbb{R}^d$, (23) can be written in the vector form as

$$\int_{\mathbb{R}^d} a(\xi - q)\mu(q)(\varkappa_1(q) - \varkappa_1(\xi))\ dq = \int_{\mathbb{R}^d} a(\xi - q)(\xi - q)\mu(q)\ dq \tag{24}$$

or

$$A\varkappa_1 = h \tag{25}$$

with the operator A in $(L^2(\mathbb{T}^d))^d$ defined by

$$(A\bar\varphi)(\xi) = \int_{\mathbb{R}^d} a(\xi - q)\mu(q)(\bar\varphi(q) - \bar\varphi(\xi))\ dq = \int_{\mathbb{T}^d} \hat a(\xi - \eta)\mu(\eta)(\bar\varphi(\eta) - \bar\varphi(\xi))\ d\eta \tag{26}$$

and

$$\hat a(\eta) = \sum_{k \in \mathbb{Z}^d} a(\eta + k), \quad \eta \in \mathbb{T}^d. \tag{27}$$

Observe that the vector function

$$h(\xi) = \int_{\mathbb{R}^d} a(\xi - q)\mu(q)(\xi - q)\ dq \in (L^2(\mathbb{T}^d))^d, \tag{28}$$

because the function $h(\xi)$ is bounded for all $\xi \in \mathbb{T}^d$:

$$\left| \int_{\mathbb{R}^d} a(\xi - q)(\xi - q)\mu(q)\ dq \right| \le \alpha_2 \int_{\mathbb{R}^d} a(z)|z|\ dz < \infty.$$

In (25) the operator A applies componentwise. In what follows, abusing slightly the notation, we use the same notation A for the scalar operator in $L^2(\mathbb{T}^d)$ acting on each component in (25).

Let us denote

$$K\varphi(\xi) = \int_{\mathbb{R}^d} a(\xi - q)\mu(q)\varphi(q)\, dq, \qquad \varphi \in L^2(\mathbb{T}^d).$$

PROPOSITION 6. *The operator*

$$K\varphi(\xi) = \int_{\mathbb{R}^d} a(\xi - q)\mu(q)\varphi(q)\ dq = \int_{\mathbb{T}^d} \hat a(\xi - \eta)\mu(\eta)\varphi(\eta)\ d\eta, \quad \varphi \in L^2(\mathbb{T}^d), \tag{29}$$

is a compact operator in $L^2(\mathbb{T}^d)$.

Proof. First we prove that K is the bounded operator in $L^2(\mathbb{T}^d)$. The set of bounded functions $\mathbf{B}(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$ is dense in $L^2(\mathbb{T}^d)$. Let $\varphi \in \mathbf{B}(\mathbb{T}^d)$, and then the integral

$$\left| \int_{\mathbb{R}^d} a(\xi - q)\mu(q)\varphi(q)\ dq \right| \le \alpha_2\ a_1\ \max|\varphi(q)|$$

is bounded. Using Fubini's theorem and denoting $w(q) = \mu(q)\varphi(q)$ we get

$$\begin{aligned} \|K\varphi\|^2_{L^2(\mathbb{T}^d)} &= \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} a(q-\xi)w(q)\ dq\ \int_{\mathbb{R}^d} a(q'-\xi)w(q')\ dq'\ d\xi \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(z)a(z')\left(\int_{\mathbb{T}^d} w(\xi+z)w(\xi+z')d\xi\right) dz\ dz' \\ &\le \|w\|^2_{L^2(\mathbb{T}^d)}\left(\int_{\mathbb{R}^d} a(z)dz\right)^2 \le \alpha_2^2\|a\|^2_{L^1(\mathbb{R}^d)}\|\varphi\|^2_{L^2(\mathbb{T}^d)}. \end{aligned} \tag{30}$$

Consequently the operator K can be expanded on $L^2(\mathbb{T}^d)$, and we have

$$\|K\varphi\|_{L^2(\mathbb{T}^d)} \le \alpha_2\|a\|_{L^1(\mathbb{R}^d)}\|\varphi\|_{L^2(\mathbb{T}^d)}, \quad \varphi \in L^2(\mathbb{T}^d),$$

or

$$\|K\|_{\mathcal{L}(L^2(\mathbb{T}^d),L^2(\mathbb{T}^d))} \le \alpha_2\|a\|_{L^1(\mathbb{R}^d)}. \tag{31}$$

To prove the compactness of K we consider approximations of K by the following compact operators:

$$(K_N\varphi)(\xi) = \int_{\mathbb{R}^d} a_N(\xi-q)\mu(q)\varphi(q)\ dq \quad \text{ with }\ a_N(z) = a(z)\cdot\chi_{[-N,N]^d}(z).$$

Since $a - a_N \in L^1(\mathbb{R}^d)$, using (31) we get

$$\|K - K_N\|_{L^2(\mathbb{T}^d)} \le \alpha_2\|a - a_N\|_{L^1(\mathbb{R}^d)}.$$

Consequently, $\|K - K_N\|_{\mathcal{L}(L^2(\mathbb{T}^d),L^2(\mathbb{T}^d))} \to 0$, as $N \to \infty$, and K is a compact operator as the limit of the compact operators K_N. ∎

The operator

$$G\varphi(\xi) = \varphi(\xi)\ \int_{\mathbb{R}^d} a(\xi-q)\mu(q)\ dq = \varphi(\xi)\int_{\mathbb{T}^d}\hat{a}(\xi-\eta)\mu(\eta)\ d\eta, \quad \varphi \in L^2(\mathbb{T}^d), \tag{32}$$

is the operator of multiplication by the function $G(\xi) = \int_{\mathbb{R}^d} a(\xi-q)\mu(q)\ dq$. Observe that

$$0 < g_0 \le G(\xi) \le g_2 < \infty.$$

Thus, the operator A in (26) can be written as $A = K - G$, where G and K were defined in (32) and (29). Therefore $-A$ is the sum of a positive invertible operator G and a compact operator $-K$, and the Fredholm theorem applies to (25). It is easy to see that $Ker\ A^* = \{\mu(\xi)\}$, and then the solvability condition for (25) takes the form

$$\int_{\mathbb{T}^d} h(\xi)\mu(\xi)\ d\xi\ =\ 0. \tag{33}$$

The validity of condition (33) for the function h defined in (28) immediately follows from Proposition 7.

Proposition 7. *For any periodic functions* $\mu(y)$, $\lambda(y)$, $y \in \mathbb{R}^d$ *we have if* $a(x-y) = a(y-x)$, *and then*

$$\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} a(x-y)\mu(y)\lambda(x)\ dx\ dy = \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} a(x-y)\mu(x)\lambda(y)\ dx\ dy; \tag{34}$$

if $b(x-y) = -b(y-x)$, *then*

$$\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} b(x-y)\mu(y)\lambda(x)\ dx\ dy = -\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} b(x-y)\mu(x)\lambda(y)\ dx\ dy. \tag{35}$$

Proof. Using periodicity of μ and λ we get for any $z \in \mathbb{R}^d$

$$\int_{\mathbb{T}^d} \mu(z+x)\lambda(x)\ dx = \int_{\mathbb{T}^d} \mu(u)\lambda(u-z)\ du.$$

Consequently, we have

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} a(y-x)\mu(y)\lambda(x)\ dx\ dy &= \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} a(z)\mu(z+x)\lambda(x)\ dx\ dz \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} a(x-y)\mu(x)\lambda(y)\ dx\ dy.\end{aligned}$$

Recalling now the relation $a(x-y) = a(y-x)$ yields (34).

Similarly using that $b(x-y) = -b(y-x)$ we get

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} b(x-y)\mu(y)\lambda(x)\ dx\ dy &= -\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} b(y-x)\mu(y)\lambda(x)\ dx\ dy \\ &= -\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} b(z)\mu(z+x)\lambda(x)\ dx\ dz \\ &= -\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} b(x-y)\mu(x)\lambda(y)\ dx\ dy.\end{aligned}$$

Proposition 7 is now proved. ∎

Thus, the solution $\varkappa_1(\xi)$ of (25) exists and is unique up to a constant vector. In order to fix the choice of this vector we assume that the average of each component of $\varkappa_1(\xi)$ over the period is equal to 0.

At the next step we obtain the limit operator L^0. To this end we find the matrix function $\varkappa_2(\xi) = \{\varkappa_2^{ij}(\xi)\}$, $\varkappa_2^{ij} \in L^2(\mathbb{T}^d)$, such that the second term on the right-hand side of (16) takes the form

$$\sum_{i,j=1}^{d} \Theta^{ij}\ \frac{\partial^2 u(x)}{\partial x^i \partial x^j} \tag{36}$$

with a constant matrix $\Theta = \{\Theta^{ij}\}$. This leads to the following equation for the functions $\varkappa_2^{ij}(\xi)$ for any $i, j = 1, \dots, d$:

$$(A\varkappa_2^{ij})(\xi) = \frac{\Theta^{ij}}{\lambda(\xi)} - \int_{\mathbb{R}^d} a(z)\mu(\xi-z)\left(\frac{1}{2}z^i z^j - z^i \varkappa_1^j(\xi-z)\right)\ dz, \tag{37}$$

where the operator A is defined in (26). The matrix Θ is then determined from the following solvability condition for (37):

$$\Theta^{ij}\int_{\mathbb{T}^d}\frac{\mu(\xi)}{\lambda(\xi)}\,d\xi=\tilde\Theta^{ij}=\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\frac{1}{2}(\xi-q)^i(\xi-q)^j a(\xi-q)\mu(q)\mu(\xi)\;dq\;d\xi \tag{38}$$
$$-\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}a(\xi-q)\mu(q)\mu(\xi)(\xi-q)^i\varkappa_1^j(q)\;dq\;d\xi$$

for any i,j. Although the matrix Θ need not be symmetric, only its symmetric part matters in (36). Abusing slightly the notation we identify matrix Θ with its symmetric part.

Proposition 8. *The integrals on the right-hand side of* (38) *converge. Moreover, the symmetric part of the matrix* $\Theta=\{\Theta^{ij}\}$ *defined in* (38) *is positive definite.*

Proof. The first statement of the Proposition immediately follows from the existence of the second moment of the function $a(z)$. Since the integral $\int_{\mathbb{T}^d}\frac{\mu(\xi)}{\lambda(\xi)}\,d\xi$ equals a positive constant, it is sufficient to prove that the symmetric part of the right-hand side of (38) is positive definite. To this end we consider the following integrals, symmetric for all i,j,

$$I^{ij}=\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}a(\xi-q)\mu(q)\mu(\xi)\big((\xi-q)+(\varkappa_1(\xi)-\varkappa_1(q))\big)^i \tag{39}$$
$$\big((\xi-q)+(\varkappa_1(\xi)-\varkappa_1(q))\big)^j dq d\xi,$$

and prove that the symmetric part of the right-hand side of (38) is equal to I:

$$I^{ij}=\tilde\Theta^{ij}+\tilde\Theta^{ji}=\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}(\xi-q)^i(\xi-q)^j a(\xi-q)\mu(q)\mu(\xi)\;dq\;d\xi \tag{40}$$
$$-\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}a(\xi-q)\mu(q)\mu(\xi)(\xi-q)^i\varkappa_1^j(q)\;dq\;d\xi$$
$$-\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}a(\xi-q)\mu(q)\mu(\xi)(\xi-q)^j\varkappa_1^i(q)\;dq\;d\xi.$$

Using (35) we have

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}(\xi-q)^i a(\xi-q)\mu(q)\mu(\xi)\varkappa_1^j(\xi)\;dq\;d\xi$$
$$=-\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}(\xi-q)^i a(\xi-q)\mu(q)\mu(\xi)\varkappa_1^j(q)\;dqd\xi.$$

Consequently,

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}a(\xi-q)\mu(q)\mu(\xi)(\xi-q)^i(\varkappa_1(\xi)-\varkappa_1(q))^j\;dq\;d\xi \tag{41}$$
$$=-2\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}(\xi-q)^i a(\xi-q)\mu(q)\mu(\xi)\varkappa_1^j(q)\;dqd\xi.$$

Further, combining (24) on $\varkappa_1$ with (34)–(35), we get

$$\begin{aligned}
&\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} a(\xi-q)\mu(q)\mu(\xi)(\varkappa_1(\xi)-\varkappa_1(q))^i\varkappa_1^j(\xi)\,dq\,d\xi\\
&\quad=\int_{\mathbb{T}^d}\mu(\xi)\varkappa_1^j(\xi)\ \int_{\mathbb{R}^d} a(\xi-q)\mu(q)(\varkappa_1(\xi)-\varkappa_1(q))^i\,dq\,d\xi\\
&\quad=-\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} a(\xi-q)(\xi-q)^i\mu(q)\mu(\xi)\varkappa_1^j(\xi)\,dq\,d\xi\\
&\quad=\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} a(\xi-q)(\xi-q)^i\mu(q)\mu(\xi)\varkappa_1^j(q)\,dq\,d\xi
\end{aligned}$$

and

$$\begin{aligned}
&-\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} a(\xi-q)\mu(\xi)\mu(q)(\varkappa_1(\xi)-\varkappa_1(q))^i\varkappa_1^j(q)\,dq\,d\xi\\
&\quad=-\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} a(\xi-q)\mu(q)\mu(\xi)(\varkappa_1(q)-\varkappa_1(\xi))^i\varkappa_1^j(\xi)\,dq\,d\xi\\
&\quad=\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} a(\xi-q)\mu(q)\mu(\xi)(\varkappa_1(\xi)-\varkappa_1(q))^i\varkappa_1^j(\xi)\,dq\,d\xi\\
&\quad=\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} a(\xi-q)(\xi-q)^i\mu(q)\mu(\xi)\varkappa_1^j(q)\,dq\,d\xi.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} a(\xi-q)\mu(\xi)\mu(q)(\varkappa_1(\xi)-\varkappa_1(q))^i(\varkappa_1(\xi)-\varkappa_1(q))^j\,dq\,d\xi\\
&\quad=\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} a(\xi-q)\mu(\xi)\mu(q)\Big((\xi-q)^i\varkappa_1^j(q)+(\xi-q)^j\varkappa_1^i(q)\Big)\,dq\,d\xi,
\end{aligned}$$

which together with (39) and (41) implies (40).

The structure of (39) implies that $(Iv,v)\geq 0$ for all $v\in\mathbb{R}^d$, and moreover $(Iv,v)>0$ since $\varkappa_1(q)$ is the periodic function while q is the linear function, and consequently $[((\xi-q)+(\varkappa_1(\xi)-\varkappa_1(q)))\cdot v]^2$ cannot be identically 0 if $v\neq 0$. ◻

Thus equality (14) follows from (15)–(16), (23), and (37). Lemma 4 is proved. ◻

Under a natural additional assumption on the operator L the statement similar to that of Lemma 4 holds in the space $C_0(\mathbb{R}^d)$.

LEMMA 9. *Let conditions* (11) *be fulfilled, and assume that $\lambda(x),\mu(x)$ are continuous, periodic functions satisfying bounds* (5). *Then there exist periodic continuous functions $\varkappa_1$ and $\varkappa_2$ (a vector function $\varkappa_1$ and a matrix function $\varkappa_2$) and a positive definite matrix Θ such that for any function $u\in\mathcal{S}(\mathbb{R}^d)$ and the function w^ε defined by*

$$w^\varepsilon(x)=u(x)+\varepsilon\varkappa_1\left(\frac{x}{\varepsilon}\right)\cdot\nabla u(x)+\varepsilon^2\varkappa_2\left(\frac{x}{\varepsilon}\right)\cdot\nabla\nabla u(x)\tag{42}$$

we have

$$L^\varepsilon w^\varepsilon=\Theta\cdot\nabla\nabla u+\phi_\varepsilon,\qquad \textit{where}\ \lim_{\varepsilon\to 0}\|\phi_\varepsilon\|_{C_0(\mathbb{R}^d)}=0.\tag{43}$$

Proof. First we have to check that $w^\varepsilon \in C_0(\mathbb{R}^d)$. To this end it suffices to show that $\varkappa_1^i,\ \varkappa_2^{ij} \in C(\mathbb{T}^d)$, where $\varkappa_1,\ \varkappa_2$ are solutions of (25), (37), respectively. We recall that the equation on $\varkappa_1$ reads

$$\int_{\mathbb{T}^d} \hat{a}(\xi-\eta)\mu(\eta)(\varkappa_1(\eta)-\varkappa_1(\xi))\ d\eta = h(\xi), \tag{44}$$

where the function $\hat{a} \in C(\mathbb{T}^d)$ was defined by (27), and

$$h(\xi) = \int_{\mathbb{T}^d} \hat{b}(\xi-\eta)\mu(\eta)\ d\eta \in (C(\mathbb{T}^d))^d, \quad \hat{b}(\eta) = \sum_{k\in Z^d} a(\eta+k)\ k. \tag{45}$$

As was shown above, see (33), $\int_{\mathbb{T}^d} h(\xi)\mu(\xi)\ d\xi = 0$, and consequently the solvability condition holds, and there exists a solution $\varkappa_1 \in (L^2(\mathbb{T}^d))^d$ of (44). We will show now that $\varkappa_1^i \in C(\mathbb{T}^d)$ for all $i = 1,\dots,d$.

Let us rewrite (44) for each i as

$$(P-E)\varkappa_1^i = g^i, \quad g^i(\xi) = \frac{h^i(\xi)}{q(\xi)}, \quad q(\xi) = \int_{\mathbb{T}^d} \hat{a}(\xi-\eta)\mu(\eta)\ d\eta > 0 \tag{46}$$

with

$$(P\varphi)(\xi) = \int_{\mathbb{T}^d} p(\xi,\eta)\varphi(\eta)\ d\eta, \quad p(\xi,\eta) = \frac{\hat{a}(\xi-\eta)\mu(\eta)}{q(\xi)}, \quad \int_{\mathbb{T}^d} p(\xi,\eta)\ d\eta = 1\ \ \forall\ \xi. \tag{47}$$

Then P is a compact positive operator, such that P^n is a positivity improving operator in $C(\mathbb{T}^d)$ for some $n \in \mathbb{N}$ (i.e., if $\varphi \ge 0$, then $P^n\varphi > 0$), and by the Krein–Rutman theorem there exists the maximal eigenvalue $\lambda_0 = 1$ corresponding to the eigenfunction $\varphi_0(\eta) \equiv 1$, and other eigenvalues of P are less than 1 by the absolute value. Consequently, $C(\mathbb{T}^d) = \{1\} \oplus \mathcal{H}_1$ with

$$\mathcal{H}_1 = \left\{ \psi \in C(\mathbb{T}^d) :\ \int_{\mathbb{T}^d} \mu(\eta)q(\eta)\psi(\eta)\ d\eta = 0 \right\}.$$

One can easily check that $\mathcal{H}_1$ is an invariant subspace for P. Using Neumann decomposition for the operator $P_1 - E$ with $P_1 = P|_{\mathcal{H}_1}$ we can see that the operator $P - E$ is an invertable operator on $\mathcal{H}_1$ mapping $\mathcal{H}_1$ on itself. Thus,

$$\varkappa_1 = -(E-P_1)^{-1} g\ \in C(\mathbb{T}^d).$$

Similarly, we get $\varkappa_2^{ij} \in C(\mathbb{T}^d)$ for all $i,j = 1,\dots,d$, for the solutions of (37). Thus $w^\varepsilon \in C_0(\mathbb{R}^d)$.

The convergence

$$\|L^\varepsilon w^\varepsilon - L^0 u\|_{C_0(\mathbb{R}^d)} = \|\phi_\varepsilon\|_{C_0(\mathbb{R}^d)} \to\ 0 \quad \text{ as } \ \varepsilon \to 0$$

can now be justified in exactly the same way as in Lemma 4. ∎

4. Proof of Theorems 1 and 2.

Proof of Theorem 1. We consider first the case when $f \in \mathcal{S}(\mathbb{R}^d)$ and prove that a solution u^ε of problem (7) converges to a solution u_0 of problem (9) in the space $L^2(\mathbb{R}^d)$:

$$\|u^\varepsilon - u_0\|_{L^2(\mathbb{R}^d)}\ \to 0, \quad \text{ as } \ \varepsilon \to 0. \tag{48}$$

Let us consider a perturbation v^ε of the function u_0 defined by the ansatz (13):

$$v^\varepsilon(x) = u_0(x) + \varepsilon \varkappa_1\left(\frac{x}{\varepsilon}\right)\cdot\nabla u_0(x) + \varepsilon^2\varkappa_2\left(\frac{x}{\varepsilon}\right)\cdot\nabla\nabla u_0(x) \tag{49}$$

with periodic vector function $\varkappa_1(x) \in (L^2(\mathbb{T}^d))^d$ and periodic matrix function $\varkappa_2(x) \in (L^2(\mathbb{T}^d))^{d^2}$, that have been defined by (25) and (37). Since L^0 is an elliptic operator with constant coefficients and $f \in \mathcal{S}(\mathbb{R}^d)$, $u_0 \in \mathcal{S}(\mathbb{R}^d)$ and $v^\varepsilon \in L^2(\mathbb{R}^d)$ is correctly defined.

Lemma 10. *Let u^ε and v^ε be defined by* (7) *and* (9), (49), *respectively. If $\varkappa_1$ and $\varkappa_2$ are the solutions of* (25) *and* (37), *then for all $f \in \mathcal{S}(\mathbb{R}^d)$*

$$\|v^\varepsilon - u^\varepsilon\|_{L^2(\mathbb{R}^d)} \to 0$$

as $\varepsilon \to 0$.

Proof. We have from (14) that

$$L^\varepsilon v^\varepsilon = \Theta\ \nabla\nabla u_0 + \phi_\varepsilon,$$

where $\|\phi_\varepsilon\|_{L^2(\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$. Then

$$(L^\varepsilon - m)v^\varepsilon + m(v^\varepsilon - u_0) = \Theta\ \nabla\nabla u_0 - mu_0 + \phi_\varepsilon = f + \phi_\varepsilon.$$

Since $\|v^\varepsilon - u_0\|_{L^2(\mathbb{R}^d)} \to 0$ due to representation (49), we get

$$(L^\varepsilon - m)v^\varepsilon = f + \tilde\phi_\varepsilon \quad \text{with} \ \ \|\tilde\phi_\varepsilon\|_{L^2(\mathbb{R}^d)} \to 0. \tag{50}$$

For the operator $(L^\varepsilon - m)^{-1}$ we have

$$\|(L^\varepsilon - m)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))} \le C$$

with C being independent of ε. Then using (50) we obtain

$$u^\varepsilon = (L^\varepsilon - m)^{-1} f = (L^\varepsilon - m)^{-1}\Big((L^\varepsilon - m)v^\varepsilon - \tilde\phi_\varepsilon\Big) = v^\varepsilon - (L^\varepsilon - m)^{-1}\tilde\phi_\varepsilon$$

and

$$\|v^\varepsilon - u^\varepsilon\|_{L^2(\mathbb{R}^d)} = \|(L^\varepsilon - m)^{-1}\tilde\phi_\varepsilon\|_{L^2(\mathbb{R}^d)} \to 0.$$

This completes the proof of the lemma. ∎

Corollary 11.

$$\|u^\varepsilon - u_0\|_{L^2(\mathbb{R}^d)} \to 0 \quad \textit{as}\ \ \varepsilon \to 0,$$

i.e., (48) *holds for any $f \in \mathcal{S}(\mathbb{R}^d)$.*

We complete now the proof of Theorem 1. For any $f \in L^2(\mathbb{R}^d)$ there exists $f_\delta \in \mathcal{S}(\mathbb{R}^d)$ such that $\|f - f_\delta\|_{L^2(\mathbb{R}^d)} < \delta$. Since the operator $(L^\varepsilon - m)^{-1}$ is bounded uniformly in ε,

$$\|u^\varepsilon_\delta - u^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C_1\delta \tag{51}$$

and

$$\|u_{0,\delta} - u_0\|_{L^2(\mathbb{R}^d)} \le C_1\delta, \tag{52}$$

where

$$u^\varepsilon = (L^\varepsilon - m)^{-1} f, \;\; u_0 = (L^0 - m)^{-1} f, \;\; u^\varepsilon_\delta = (L^\varepsilon - m)^{-1} f_\delta, \;\; u_{0,\delta} = (L^0 - m)^{-1} f_\delta.$$

Since $\|u^\varepsilon_\delta - u_{0,\delta}\|_{L^2(\mathbb{R}^d)} \to 0$ by Corollary 11, (51)–(52) imply that

$$\overline{\lim_{\varepsilon\to 0}} \|u^\varepsilon - u_0\|_{L^2(\mathbb{R}^d)} \le 2C_1\delta$$

with an arbitrary small $\delta > 0$. This implies that $\|u^\varepsilon - u_0\|_{L^2(\mathbb{R}^d)} \to 0$, as $\varepsilon \to 0$. This completes the proof. ∎

Proof of Theorem 2. For any $f \in \mathcal{S}(\mathbb{R}^d)$, the convergence of u^ε to u_0 relies on the statement of Lemma 9. Since L^ε satisfy the positive maximum principle, the operators $(L^\varepsilon - m)^{-1}$ are bounded in $C_0(\mathbb{R}^d)$ for any $m > 0$ uniformly in ε. The remaining part of the proof of Theorem 2 follows the line of the proof of Theorem 1. ∎

5. Convergence of semigroups. In this section we consider the semigroups $T^\varepsilon(t)$ generated by the operators L^ε defined in (6) and show that the approximating sequence in the form of ansatz (13) or (42) can be used to prove the convergence of the semigroups.

5.1. Convergence of semigroups in $L^2(\mathbb{R}^d)$. Since for any $\varepsilon > 0$ the bounded operator defined in (6) is symmetric and nonpositive in $L^2(\mathbb{R}^d, \nu_\varepsilon)$, where $\nu(y) = \frac{\mu(y)}{\lambda(y)}$ and $\nu_\varepsilon(x) = \nu(\frac{x}{\varepsilon})$, by the Hille–Yosida theorem it is the generator of a strongly continuous contraction semigroup $T^\varepsilon(t)$ in $L^2(\mathbb{R}^d, \nu_\varepsilon)$. Denote $T^0(t)$ a strongly continuous contraction semigroup in $L^2(\mathbb{R}^d)$ generated by L^0.

Lemma 12. *For each $f \in L^2(\mathbb{R}^d)$ there holds $T^\varepsilon(t) f \to T^0(t) f, \; t \ge 0$. Moreover, this convergence is uniform on bounded time intervals.*

Proof. The space $S(\mathbb{R}^d)$ is a core for the operator L^0. By the approximation theorem [7, Chapter 1, Theorem 6.1] it is sufficient to show that for any $u \in S(\mathbb{R}^d)$ there exists $w^\varepsilon \in L^2(\mathbb{R}^d, \nu_\varepsilon)$ such that

$$\|w^\varepsilon - u\|_{L^2(\mathbb{R}^d, \nu_\varepsilon)} \to 0 \tag{53}$$

and

$$\|L^\varepsilon w^\varepsilon - L^0 u\|_{L^2(\mathbb{R}^d, \nu_\varepsilon)} \to 0 \tag{54}$$

as $\varepsilon \to 0$.

Notice that under our assumption (5) the bounds hold:

$$0 < \gamma_1 \le \nu_\varepsilon(x) \le \gamma_2 < \infty.$$

Therefore, the convergence (53)–(54) is equivalent to

$$\|w^\varepsilon - u\|_{L^2(\mathbb{R}^d)} \to 0, \quad \|L^\varepsilon w^\varepsilon - L^0 u\|_{L^2(\mathbb{R}^d)} \to 0. \tag{55}$$

For w^ε we take the function defined by (13):

$$w^\varepsilon(x) = u(x) + \varepsilon \varkappa_1\left(\frac{x}{\varepsilon}\right) \cdot \nabla u(x) + \varepsilon^2 \varkappa_2\left(\frac{x}{\varepsilon}\right) \cdot \nabla\nabla u(x).$$

Then the first convergence in (55) easily follows from the definition of w^ε, while the second one is a consequence of Lemma 4. Now the desired statements follow from [7, Theorem 6.1, Chapter 1]. ∎

COROLLARY 13. *The convergence of semigroups implies the convergence of solutions of the corresponding evolution equations.*

Let u^ε be the solution of the heat equation

$$\frac{\partial u^\varepsilon}{\partial t} = L^\varepsilon u^\varepsilon, \quad u^\varepsilon(x,0) = \phi(x), \quad \phi \in L^2(\mathbb{R}^d),$$

and u^0 be the solution of

$$\frac{\partial u^0}{\partial t} = L^0 u^0, \quad u^0(x,0) = \phi(x).$$

Then for each $t > 0$

$$\|u^\varepsilon(\cdot,t) - u^0(\cdot,t)\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as } \varepsilon \to 0.$$

5.2. Markov semigroup in $C_0(\mathbb{R}^d)$. Recall that in the space $C_0(\mathbb{R}^d)$ functions $a(x)$, $\lambda(x)$, and $\mu(x)$ satisfy conditions of Theorem 2.

LEMMA 14. *The semigroup $T(t)$ generated by the operator* (1) *is a Feller semigroup, i.e., it is a strongly continuous, positivity preserving, contraction and conservative semigroup in $C_0(\mathbb{R}^d)$.*

For each probability measure ν in $\mathbb{R}^d$ there exists a Markov jump process X corresponding to the semigroup $T(t)$ with the initial distribution ν and with a càdlàg modification, i.e., with sample paths in $D_{\mathbb{R}^d}[0,\infty)$ (right-continuous functions with finite left-hand limits).

Proof. Since L is bounded and satisfies the positive maximum principle, the first statement of Lemma follows from the Hille–Yosida theorem. In addition we can rewrite L as follows:

$$(Lf)(x) = \tilde{\lambda}(x) \int_{\mathbb{R}^d} (f(y) - f(x)) p(x,y)\, dy, \quad \int_{\mathbb{R}^d} p(x,y)\, dy = 1 \ \forall x \tag{56}$$

with

$$\tilde{\lambda}(x) = \lambda(x) q(x), \quad q(x) = \int_{\mathbb{R}^d} a(x-y)\mu(y) dy > 0, \quad p(x,y) = \frac{a(x-y)\mu(y)}{q(x)}.$$

This representation implies that L is a generator of jump Markov process with $T(t)1 = 1$.

The proof of the second statement follows from general results concerning Feller semigroups; see, e.g., [3, 7]. ∎

sample paths in $D_{\mathbb{R}^d}[0,\infty)$

Let us consider the family of strongly continuous contraction semigroups $T^\varepsilon(t)$ generated by the operators L^ε defined in (6) and the family of corresponding Markov processes X_ε. We denote by $T^0(t)$ the semigroup in $C_0(\mathbb{R}^d)$ generated by the operator L^0 given by (8). First we prove the result about convergence of the semigroups.

LEMMA 15. *For each $f \in C_0(\mathbb{R}^d)$ it holds that*

$$\lim_{\varepsilon \to 0} T^\varepsilon(t) f = T^0(t) f, \quad t \geq 0. \tag{57}$$

Moreover, this convergence is uniform on bounded time intervals.

Proof. Following the same reasoning as in the proof of Lemma 12 we again take $S(\mathbb{R}^d)$ as a core for L^0 in $C_0(\mathbb{R}^d)$, and for any $u \in S(\mathbb{R}^d)$ consider the approximation sequence w^ε given by (42). Then the convergence $w^\varepsilon \to u$ in the norm of the Banach space $C_0(\mathbb{R}^d)$ is due to representation (42). The convergence

$$\|L^\varepsilon w^\varepsilon - L^0 u\|_{C_0(\mathbb{R}^d)} = \|\phi_\varepsilon\|_{C_0(\mathbb{R}^d)} \to \ 0 \quad \text{ as } \ \varepsilon \to 0$$

follows from Lemma 9.

Thus we can apply the approximation theorem from [7] in the same way as in Lemma 12 and obtain convergence (57). ∎

We proceed with the proof of the main result of this section that states the invariance principle for the family of processes X_ε.

Proof of Theorem 3. The fact that X_ε has a modification in $D_{\mathbb{R}^d}[0,\infty)$ has been justified in Lemma 14. The limit process X_0 is a diffusion process that has continuous trajectories. The convergence in distributions $X_\varepsilon \ \Rightarrow \ X_0$ in the paths space $D_{\mathbb{R}^d}[0,\infty)$ follows from Lemma 15, which gives the convergence of finite-dimensional distributions, and [7, Theorem 2.5, Chapter 4].

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