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Quenched invariance principles for random walks on percolation clusters

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We consider a supercritical Bernoulli percolation model in $\mathbb{Z}^d$, $d \geq 2$, and study the simple symmetric random walk on the infinite percolation cluster. The aim of this paper is to prove the almost sure (quenched) invariance principle for this random walk.

Keywords: percolation; random walk; invariance principle; infinite cluster

1. Introduction

Let $d \geq 2$. In this paper, we study the simple random walk on the infinite component of supercritical Bernoulli bond percolation on the lattice $\mathbb{Z}^d$. Percolation is a classical construction from statistical mechanics to select random sub-graphs of a fixed graph. Applied to the d-dimensional lattice, the construction is well known: one successively considers the different edges of the grid and decides to keep or delete a given edge by tossing a fixed coin. (See below for a more formal definition.) Call p the probability that an edge is kept. The shape of the resulting (random) sub-graph thus obtained dramatically depends on the percolation parameter p: below some critical value, p_c, with probability 1, all the connected components of the percolation graph are finite but, when $p > p_c$, the percolation graph almost surely has a unique infinite connected component called the *infinite cluster* and denoted by $\mathcal{C}(\omega)$. By construction, $\mathcal{C}(\omega)$ is a random infinite connected sub-graph of the grid $\mathbb{Z}^d$. The exact value of p_c is unknown except in dimension 2, where it equals 0.5.

In statistical mechanics, percolation plays the prominent role of a toy model for disordered environments, as the title of de Gennes (1976) indicates. Since its first rigorous formulation by J. M. Hammersley in 1956, percolation gave rise to a rich mathematical theory, much of which focused on the geometric properties of the percolation graph. We refer to Kesten (1982) and Grimmett (1999). In the supercritical regime $p > p_c$, one would expect the geometry of the infinite cluster to be close to the geometry of the full grid. Indeed, by construction, the law of $\mathcal{C}(\omega)$ is invariant under translations of $\mathbb{Z}^d$. Note, however, that, except in the trivial case $p=1$, the geometry of $\mathcal{C}(\omega)$ will undergo fluctuations; for instance, big holes will appear somewhere as well as

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long linear parts. As a matter of fact, with probability 1, $\mathcal{C}(\omega)$ contains somewhere a translate of any finite sub-graph of the grid.

Closely connected to a description of the geometry of $\mathcal{C}(\omega)$ are questions related to potential theory on $\mathcal{C}(\omega)$. In probabilistic terms, one wonders to what extent the simple random walk on $\mathcal{C}(\omega)$ behaves similarly to the simple random walk on $\mathbb{Z}^d$. Grimmett *et al.* (1993) proved that the random walk is recurrent in dimension 2 and transient in larger dimensions. In this paper, we will discuss the extension of Donsker's invariance principle to the simple random walk on $\mathcal{C}(\omega)$, namely can we prove that, after proper rescaling, the law of the random walk converges to the law of a Brownian motion in $\mathbb{R}^d$? In order to give a more precise meaning to this question, one should distinguish weak (also called *annealed* or *averaged*) and strong (also called *individual*, *almost sure* or *quenched*) forms of the invariance principle. The annealed form of the invariance principle, stating the convergence of the law of the walk towards Brownian motion when one averages with respect to the law of environment, was proved by De Masi *et al.* (1989) as a special case of a general theorem on random walks among random conductances. The annealed invariance principle does not give a complete description of the behaviour of the random walk for a given realization of the percolation. For instance, after being averaged over the percolation randomness, the law of the random walk clearly inherits all the symmetries of the grid $\mathbb{Z}^d$. On the other hand, owing to the fluctuations caused by randomness in the percolation process, a given realization of the infinite cluster has no symmetry. Nevertheless, on a large scale, one would expect symmetries to be restored, and that should reflect on the isotropy of the limiting behaviour of the random walk. Different recent approaches have been proposed in order to prove the quenched invariance principle: Sidoravicius & Sznitman (2004) deduced the quenched invariance principle from the annealed one through variance estimates in dimensions higher than 4. Here, we shall establish the quenched invariance principle in any dimension using the construction of a corrector and the notion of *two-scale convergence*. Independently of our work and at the same time, Berger & Biskup (in press) recently published the same statement. Although they also rely on the construction of a corrector, their method to prove the sub-linear growth of the corrector is quite different from ours.

We now turn to a more precise description of the model and the statement of our result. Consider supercritical Bernoulli bond percolation in $\mathbb{Z}^d$, $d\geq 2$. For $x, y\in\mathbb{Z}^d$, we write $x\sim y$ if x and y are neighbours in the grid $\mathbb{Z}^d$, and let $\mathbb{E}^d$ be the set of non-oriented nearest neighbour pairs (x, y). We identify a sub-graph of $\mathbb{Z}^d$ with a functional $\omega:\mathbb{E}_d\to\{0, 1\}$, writing $\omega(x, y)=1$ if the edge (x, y) is present in ω and $\omega(x, y)=0$ otherwise. Thus, $\Omega=\{0,1\}^{\mathbb{E}_d}$ might be identified with the set of sub-graphs of $\mathbb{Z}^d$. Edges pertaining to ω are then called *open*. Connected components of such a sub-graph will be called *clusters*, and the cluster of ω containing a point $x\in\mathbb{Z}^d$ is denoted by $\mathcal{C}_x(\omega)$.

Now define Q to be the probability measure on $\{0,1\}^{\mathbb{E}_d}$ under which the random variables $(\omega(e), e\in\mathbb{E}_d)$ are Bernoulli independent variables with common parameter (p) and let

$$p_c = \sup\{p; Q[0\in\mathcal{C}(\omega)] = 0\}$$

be the critical probability. It is known that $p_c\in]0,1[$ (see Grimmett 1999). Throughout the paper, we choose a parameter p such that

$$p > p_c. \tag{1.1}$$

Then, Q almost surely (Q.a.s.), the graph ω has a unique infinite cluster denoted by $\mathcal{C}(\omega)$.

We are interested in the behaviour of the simple symmetric random walk on $\mathcal{C}_0(\omega)$: let $D(\mathbb{R}_+,\mathbb{Z}^d)$ be the space of càd-làg $\mathbb{Z}^d$-valued functions on $\mathbb{R}_+$ and $X(t)$, $t\in\mathbb{R}_+$, be the coordinate maps from $D(\mathbb{R}_+, \mathbb{Z}^d)$ to $\mathbb{Z}^{\mathrm{d}}$. $D(\mathbb{R}_+, \mathbb{Z}^d)$ is endowed with the Skorohod topology. For a given sub-graph $\omega\in\{0,1\}^{\mathbb{E}_d}$, and for $x\in\mathbb{Z}^d$, let P_x^ω be the probability measure on $D(\mathbb{R}_+, \mathbb{Z}^d)$ under which the coordinate process is the Markov chain starting at $X(0)=x$ and with generator

$$\mathcal{L}^\omega f(x) = \frac{1}{n^\omega(x)}\sum_{y\sim x}\omega(x, y)(f(y)-f(x)), \tag{1.2}$$

where $n^\omega(x)$ is the number of neighbours of x in the cluster $\mathcal{C}_x(\omega)$.

The behaviour of $X(t)$ under P_x^ω can be described as follows: starting from point x, the random walker waits for an exponential time of parameter 1 and then chooses, uniformly at random, one of its neighbours in $\mathcal{C}_x(\omega)$, say y, and moves to it. This procedure is then iterated with independent hoping times. The walker clearly never leaves the cluster of ω it started from. Since edges are not oriented, the measures with weights $n^\omega(x)$ on the possibly different clusters of ω are reversible.

Let Q_0 be the conditional measure $Q_0(.)=Q(.|0\in\mathcal{C}(\omega))$, and let $Q_0.P_x^\omega$ be the so-called *annealed* semi-direct product measure law defined by

$$Q_0.P_x^\omega[F(\omega, X(.))] = \int P_x^\omega[F(\omega, X(.))]\mathrm{d}Q_0(\omega).$$

Note that $X(t)$ is not Markovian anymore under $Q_0.P_x^\omega$. As already alluded to at the beginning of this introduction, it was proved by De Masi *et al.* (1989) that, under $Q_0.P_0^\omega$, the process $(X^\varepsilon(t)=\varepsilon X(t/\varepsilon^2), t\in\mathbb{R}_+)$ satisfies an invariance principle as ε tends to 0, i.e. it converges in law to a non-degenerate Brownian motion. The proof is based on the *point of view of the particle*. It relies on the fact that the law of the environment ω viewed from the current position of the Markov chain is reversible when considered under the annealed measure. We shall prove theorem 1.1.

Theorem 1.1. *Q almost surely on the event $0\in\mathcal{C}(\omega)$, under P_0^ω, the process $(X^\varepsilon(t)=\varepsilon X(t/\varepsilon^2)$, $t\in\mathbb{R}_+)$ converges in law as ε tends to 0 to a Brownian motion with covariance matrix $\sigma^2 Id$, where σ^2 is positive and does not depend on ω.*

Our strategy of proof follows the classical pattern introduced by Kozlov (1985) for averaging random walks with random conductances. The method of Kozlov was successfully used under ellipticity assumptions that are clearly not satisfied here. We refer in particular to the first part of Sidoravicius & Sznitman (2004), in which random walks in elliptic environments are considered. The main idea is to modify the process $X(t)$ by the addition of a *corrector* in such a way that the sum is a martingale under P_0^ω and to use a martingale invariance principle. Then one has to prove that, in the rescaled limit, the corrector can be neglected, or equivalently that the corrector has sub-linear growth. For this second step, in a classical elliptic set-up, one would invoke the Poincaré inequality and the compact embedding of H^1 into L^2. For percolation models, a weaker but still suitable form of the Poincaré inequality was proved by Mathieu & Remy (2004) (see also Barlow 2004). However, another difficulty arises: our reference measure is the counting measure on the cluster at the origin. When rescaled, it does converge to the Lebesgue measure on $\mathbb{R}^d$, but for a fixed ε it is of course singular. Thus, rather than using classical functional analysis tools, one has to turn to L^2 techniques in varying spaces or to two-scale convergence arguments as they have been recently developed for the theory of homogenization of singular random structures by Jikov & Piatnitski (2006). An elementary self-contained construction of the corrector is given in §2*b*. We also provide an approach to two-scale convergence avoiding explicit reference to the results of Jikov & Piatnitski (2006). For background material on homogenization theory in both periodic and random environments, we refer to Jikov *et al.* (1994), in which percolation models are considered in ch. 9.

Note on the constants. Throughout the paper, β and c will denote positive constants depending only on d and p, whose values might change from place to place.

2. Proof of the theorem

Let $|x| = \max|x_i|$. We use the notation $x \cdot y$ for the scalar product of the two vectors $x, y \in \mathbb{R}^d$. We also use the notation $Q_0(.) = Q(.|0 \in \mathcal{C}(\omega))$.

(a) Tightness

We start by recalling the Gaussian upper bound obtained by Barlow (2004) for walks on percolation clusters. A corresponding lower bound also holds, but we will not need it here. Note that Barlow's bound is used in the proof of tightness only. Remember that $p > p_c$, so that, Q.a.s., the percolation sub-graph ω contains a unique infinite cluster denoted by $\mathcal{C}(\omega)$.

Statement from Barlow (2004): Q.a.s., for any $x \in \mathcal{C}(\omega)$ there exists a random variable S_x such that, whenever x and y belong to $\mathcal{C}(\omega)$, if $t \geq |x-y|$ and $t \geq S_x$ then

$$P_x^\omega[X(t) = y] \leq ct^{-d/2}\exp\left(-\frac{|y-x|^2}{ct}\right). \tag{2.1}$$

Moreover,

$$Q[x \in \mathcal{C}(\omega), S_x \geq t] \leq c\exp(-ct^{\epsilon(d)}) \quad \text{with} \quad \epsilon(d) > 0. \tag{2.2}$$

In case $t \leq |x-y|$, then the upper bound on $P_x^\omega[X(t) = y]$ is of the form

$$P_x^\omega[X(t) = y] \leq c\exp\left(-\frac{|y-x|}{c}\right). \tag{2.3}$$

Indeed, if t is much smaller than $|x-y|$, say $2t \leq |x-y|$, then (2.3) is an easy estimate on the tail on the Poisson distribution. In case $t \leq |x-y| \leq 2t$, then (2.3) follows from the Carne–Varopoulos bound (see appendix C in Mathieu & Remy (2004)).

Also, observe that the same estimates on the tail on the Poisson distribution imply that for $2t \leq |x-y|$ we have

$$P_x^\omega[\exists s \leq t; \quad X(s) = y] \leq c\exp\left(-\frac{|y-x|}{c}\right). \tag{2.4}$$

Let us assume that $t \geq S_x$. Combining (2.1) and (2.3), it is now an easy exercise to conclude that

$$E_x^\omega[|X(t)-x|^2] \leq c(t+1). \tag{2.5}$$

Lemma 2.1. *Q almost surely on the event $0 \in \mathcal{C}(\omega)$, under P_0^ω, the sequence of processes $(X^\varepsilon(t) = \varepsilon X(t/\varepsilon^2), t \in \mathbb{R}_+)$ is tight in the Skorohod topology.*

Proof. It is sufficient to check that Q.a.s. on the event $0 \in \mathcal{C}(\omega)$, for any $T > 0$ one has

$$\limsup_{\delta \to 0} \limsup_{\varepsilon \to 0} \sup_\tau E_0^\omega[|X^\varepsilon(\tau+\delta) - X^\varepsilon(\tau)|^2] = 0,$$

where τ is any stopping time in the filtration generated by X^ε that is bounded by T (see Ethier & Kurtz 1986, p. 138).

We have

$$E_0^\omega[|X^\varepsilon(\tau+\delta) - X^\varepsilon(\tau)|^2] = \varepsilon^2 E_0^\omega\left[\left|X\left(\frac{\tau+\delta}{\varepsilon^2}\right) - X\left(\frac{\tau}{\varepsilon^2}\right)\right|^2\right]$$

$$\le \varepsilon^2 E_0^\omega\left[\left|X\left(\frac{\tau+\delta}{\varepsilon^2}\right) - X\left(\frac{\tau}{\varepsilon^2}\right)\right|^2; \left|X\left(\frac{\tau}{\varepsilon^2}\right)\right| \le 2\frac{T}{\varepsilon^2}\right]$$

$$+ \varepsilon^2 E_0^\omega\left[\left|X\left(\frac{\tau+\delta}{\varepsilon^2}\right) - X\left(\frac{\tau}{\varepsilon^2}\right)\right|^2; \left|X\left(\frac{\tau}{\varepsilon^2}\right)\right| \ge 2\frac{T}{\varepsilon^2}; \left|X\left(\frac{\tau+\delta}{\varepsilon^2}\right) - X\left(\frac{\tau}{\varepsilon^2}\right)\right| \ge 2\frac{\delta}{\varepsilon^2}\right]$$

$$+ \varepsilon^2 E_0^\omega\left[\left|X\left(\frac{\tau+\delta}{\varepsilon^2}\right) - X\left(\frac{\tau}{\varepsilon^2}\right)\right|^2; \left|X\left(\frac{\tau}{\varepsilon^2}\right)\right| \ge 2\frac{T}{\varepsilon^2}; \left|X\left(\frac{\tau+\delta}{\varepsilon^2}\right) - X\left(\frac{\tau}{\varepsilon^2}\right)\right| \le 2\frac{\delta}{\varepsilon^2}\right].$$

The third term in this last inequality is bounded by $2\delta P_0^\omega[|X(\tau/\varepsilon^2)| \ge 2(T/\varepsilon^2)]$ and, as follows from the exponential bound (2.4), it tends to 0 as ε tends to 0. (Remember that τ is bounded by T.) We use the Markov property at time τ/ε^2 to bound the second term by $\varepsilon^2 \sup_y E_y^\omega[|X(\delta/\varepsilon^2) - y|^2; |X(\delta/\varepsilon^2) - y| \ge 2\delta/\varepsilon^2]$ and we deduce from (2.3) that it converges to 0 as ε tends to 0.

We also use the strong Markov property to estimate the first term by

$$\varepsilon^2 E_0^\omega\left[\left|X\left(\frac{\tau+\delta}{\varepsilon^2}\right) - X\left(\frac{\tau}{\varepsilon^2}\right)\right|^2; \left|X\left(\frac{\tau}{\varepsilon^2}\right)\right| \le 2\frac{T}{\varepsilon^2}\right] \le \varepsilon^2 \sup_{y \in \mathcal{C}_0(\omega); |y| \le 2T/\varepsilon^2} E_y^\omega\left[\left|X\left(\frac{\delta}{\varepsilon^2}\right) - y\right|^2\right].$$

Since we are conditioning on the event $\mathcal{C}_0(\omega) = \mathcal{C}(\omega)$, one may replace the condition $y \in \mathcal{C}_0(\omega)$ by the condition $y \in \mathcal{C}(\omega)$. From (2.5), it then follows that $\varepsilon^2 E_y^\omega[|X(\delta/\varepsilon^2) - y|^2]$ is bounded by $c\delta + c\varepsilon^2$ provided that $\delta/\varepsilon^2 \ge \sup_{y \in \mathcal{C}(\omega);\ |y| \le T/\varepsilon^2} S_y$.

To check this last condition, we observe that

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \sup_{y \in \mathcal{C}(\omega);\ |y| \le T/\varepsilon^2} S_y \le \limsup_{n \to \infty} \left(\frac{1}{n+1}\right)^2 \sup_{y \in \mathcal{C}(\omega); |y| \le Tn^2} S_y.$$

From (2.2), we get

$$Q[\sup_{y \in \mathcal{C}(\omega);\ |y| \le Tn^2} S_y > \delta n^2] \le cTn^2 \exp(-c(\delta n^2)^{\epsilon(d)}).$$

It then follows from the Borel–Cantelli lemma that Q.a.s. on the event $0 \in \mathcal{C}(\omega)$

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \sup_{y \in \mathcal{C}(\omega); |y| \le T/\varepsilon^2} S_y \le \limsup_{n \to \infty} \left(\frac{1}{n+1}\right)^2 \sup_{y \in \mathcal{C}(\omega);\ |y| \le Tn^2} S_y = 0,$$

and we conclude that Q.a.s. on the event $0\in\mathcal{C}(\omega)$, we have

$$\limsup_{\varepsilon\to 0}\sup_{\tau} E_0^{\omega}[|X^{\varepsilon}(\tau+\delta)-X^{\varepsilon}(\tau)|^2]\le c\delta.$$

■

(b) Construction of the corrector

In this section, we prove the existence of a corrector to the process X, i.e. we construct a random field $\chi(\omega, x)$ such that the process $M(t)=X(t)+\chi(\omega, X(t))$ is a martingale under P_0^{ω} for Q almost all ω s.t. $0\in\mathcal{C}(\omega)$. Then, we argue that the martingale M satisfies an invariance principle by checking the conditions of theorem 5.1 part a in Helland (1982).

Random fields. We recall that $\Omega=\{0, 1\}^{\mathbb{E}_d}$ is the set of sub-graphs of $\mathbb{Z}^d$. We shall denote with $\mathcal{B}$ the set of neighbours of the origin in $\mathbb{Z}^d$. With some abuse of notation, we write $\omega(b)$ instead of $\omega(0, b)$ when $b\in\mathcal{B}$. We use the notation $x.\omega$ to denote the natural action of $\mathbb{Z}^d$ on Ω by translations. Ω is equipped with the product sigma field.

We endow $\Omega\times\mathcal{B}$ with the measure M defined by

$$\int u\mathrm{d}M = Q\left[\sum_{b\in\mathcal{B}}\omega(b)u(\omega, b)\mathbf{1}_{0\in\mathcal{C}(\omega)}\right].$$

Note that, if two random fields u and v coincide in $L^2(\Omega\times\mathcal{B},M)$, then Q.a.s. on the event $0\in\mathcal{C}(\omega)$ it holds $u(\omega, b)=v(\omega, b)$ for any $b\in\mathcal{B}$ such that $\omega(b)=1$.

We are now going to introduce two subspaces of $L^2(\Omega\times\mathcal{B},M)$, called $L^2_{\rm pot}$ and $L^2_{\rm sol}$. To this end, we first define local functions and their gradients.

A function $u:\Omega\to\mathbb{R}$ is said to be *local* if it only depends on a finite number of coordinates. We associate to u its *gradient*: $\nabla^{(\omega)}u\ :\ \Omega\times\mathcal{B}\to\mathbb{R}$ defined by

$$\nabla^{(\omega)}u(\omega, b) = u(b.\omega)-u(\omega).$$

Definition 2.2. The closure in $L^2(\Omega\times\mathcal{B}, M)$ of the set of gradients of local fields is called $L^2_{\rm pot}$. The orthogonal complement of $L^2_{\rm pot}$ in $L^2(\Omega\times\mathcal{B}, M)$ is called $L^2_{\rm sol}$. Note that 'pot' stands for potential and 'sol' stands for solenoidal.

From the definition of potential vector fields, it follows that they possess so-called *co-cycle* property.

Lemma 2.3. *Fields in $L^2_{\rm pot}$ satisfy a co-cycle relation: on the event $0\in\mathcal{C}(\omega)$, for any $u\in L^2_{\rm pot}$ and any closed path in $\mathcal{C}(\omega)$ of the form $\gamma=(x_0, x_1, \dots, x_k)$ with $x_i\sim x_{i+1}$, $\omega(x_i, x_{i+1})=1$ and $x_0=x_k=0$, then $\sum_{i=1}^k u(x_{i-1}.\omega, x_i-x_{i-1})=0$.*

Let us write down explicitly what it means for a square integrable field v to be in $L^2_{\rm sol}$: let u be a local function on Ω. Then

$$\begin{aligned}Q\left[\sum_{b\in\mathcal{B}}\omega(b)v(\omega, b)\nabla^{(\omega)}u(\omega, b)\mathbf{1}_{0\in\mathcal{C}(\omega)}\right] &= Q\left[\sum_{b\in\mathcal{B}}v(\omega, b)\nabla^{(\omega)}u(\omega, b)\mathbf{1}_{0\in\mathcal{C}(\omega),b\in\mathcal{C}(\omega)}\right]\\ &= Q\left[\sum_{b\in\mathcal{B}}v(\omega, b)(u(b.\omega)-u(\omega))\mathbf{1}_{0\in\mathcal{C}(\omega),b\in\mathcal{C}(\omega)}\right].\end{aligned}$$

Using the translation invariance of Q and the symmetry of the set $\mathcal{B}$, we then get

$$\begin{aligned} Q\left[\sum_{b\in\mathcal{B}} v(\omega,b)u(b.\omega)\mathbf{1}_{0\in\mathcal{C}(\omega),b\in\mathcal{C}(\omega)}\right] &= Q\left[\sum_{b\in\mathcal{B}} v((-b).b.\omega,b)u(b.\omega)\mathbf{1}_{0\in\mathcal{C}(b.\omega),-b\in\mathcal{C}(b.\omega)}\right] \\ &= Q\left[\sum_{b\in\mathcal{B}} v((-b).\omega,b)u(\omega)\mathbf{1}_{0\in\mathcal{C}(\omega),-b\in\mathcal{C}(\omega)}\right] \\ &= Q\left[\sum_{b\in\mathcal{B}} v(b.\omega,-b)u(\omega)\mathbf{1}_{0\in\mathcal{C}(\omega),b\in\mathcal{C}(\omega)}\right] \\ &= Q\left[\sum_{b\in\mathcal{B}} \omega(b)v(b.\omega,-b)u(\omega)\mathbf{1}_{0\in\mathcal{C}(\omega)}\right], \end{aligned}$$

so that

$$\begin{aligned} &Q\left[\sum_{b\in\mathcal{B}} v(\omega,b)(u(b.\omega)-u(\omega))\mathbf{1}_{0\in\mathcal{C}(\omega),b\in\mathcal{C}(\omega)}\right] \\ &\quad = Q\left[\sum_{b\in\mathcal{B}} \omega(b)u(\omega)(v(b.\omega,-b)-v(\omega,b))\mathbf{1}_{0\in\mathcal{C}(\omega)}\right]. \end{aligned}$$

Thus, we have proved the following integration by parts formula:

$$\int v\nabla^{(\omega)}u\mathrm{d}M = -Q\left[n^{\omega}(0)u(\nabla^{(\omega)*}v)\mathbf{1}_{0\in\mathcal{C}(\omega)}\right], \tag{2.6}$$

where

$$\nabla^{(\omega)*}v(\omega) = \frac{1}{n^{\omega}(0)}\sum_{b\in\mathcal{B}}\omega(b)(v(\omega,b)-v(b.\omega,-b)). \tag{2.7}$$

Relation (2.6) holds for a square integrable random field v and any local function u.

As a consequence, taking v to be a constant, note that $\int\nabla^{(\omega)}u\,\mathrm{d}M=0$ for any local u. By extension, we will also have $\int u\,\mathrm{d}M=0$ for any $u\in L^2_{\text{pot}}$.

A square integrable random field v is in L^2_{sol} if it satisfies $\nabla^{(\omega)*}v=0$ Q.a.s. on the set $0\in\mathcal{C}(\omega)$.

Definition of the corrector. Let $b\in\mathcal{B}$. Define the random field $\hat{b}(\omega,e)=\mathbf{1}_{e=b}-\mathbf{1}_{e=-b}$. Let G_b be the unique solution in L^2_{pot} satisfying the equation

$$\hat{b}+G_b(\omega,e)\in L^2_{\text{sol}}. \tag{2.8}$$

(G_b is simply the projection of $-\hat{b}$ on L^2_{pot}.)

We define the corrector $\chi:\cup_{\omega\in\Omega}(\omega,\mathcal{C}_0(\omega))\to\mathbb{R}^d$ by the equation

$$\chi(\omega,x+e)\cdot b-\chi(\omega,x)\cdot b = G_b(x.\omega,e), \tag{2.9}$$

for any $x\in\mathbb{Z}^d$, b, $e\in\mathcal{B}$. (In this equation, $\chi(.)\cdot b$ stands for the usual scalar product of the two $\mathbb{R}^d$ vectors $\chi(.)$ and b. Note that there is no ambiguity because $G_b=-G_{-b}$, as can be seen directly from equation (2.8).) Observe that, unlike G_b, the corrector χ is not a homogeneous field.

The solution to (2.8) being unique in L^2_{pot}, the value of $G_b(\omega,e)$ is uniquely determined whenever $0\in\mathcal{C}(\omega)$ and $e\in\mathcal{B}$ satisfy $\omega(e)=1$. Therefore, $G_b(x.\omega,e)$ is well defined Q.a.s. on the set $0\in\mathcal{C}(\omega)$ for any x and e s.t. x and $x+e$ belong to

$\mathcal{C}(\omega)$. Thus, if x belongs to $\mathcal{C}_0(\omega)$, then the value of $\chi(\omega, x) - \chi(\omega, 0)$ can be computed by integrating (2.9) along a path in $\mathcal{C}_0(\omega)$ from the origin to x. That this value does not depend on the choice of the path is an immediate consequence of the co-cycle relation satisfied by G_b. We conclude that $\chi(\omega, x)$ is uniquely determined by equation (2.9) up to an additive constant (which might depend on ω). We summarize these properties in the following statement.

Lemma 2.4. *The corrector* $\chi = \chi(\omega, x)$ *is uniquely defined by (2.9) up to an additive (random) constant.*

The martingale property. Let $X(t)$ be a random walk in $\mathbb{Z}^d$ with generator (1.2), $X(0) = 0$. Consider the random process

$$\mathcal{M}(t) = X(t) + \chi(\omega, X(t)).$$

Note that since the process $X(t)$, starting from the origin, never leaves $\mathcal{C}(\omega)$ $\chi(\omega, X(t))$ is well defined on the event $\{\omega : 0 \in \mathcal{C}(\omega)\}$.

Proposition 2.5. *The process* $\mathcal{M}$ *is a martingale under* P_0^ω *for* Q *almost all* ω *s.t.* $0 \in \mathcal{C}(\omega)$.

Proof. We choose ω s.t. $0 \in \mathcal{C}(\omega)$. Since $G_b \in L^2_{\text{pot}}$, the co-cycle relation (see lemma 2.3) implies that $G_b(\omega, e) + G_b(e.\omega, -e) = 0$ for any $e \in \mathcal{B}$ s.t. $\omega(e) = 1$. Comparing the expression (1.2) of L^ω with the definition (2.7) of $\nabla^{(\omega)*}$, we then see that $\mathcal{L}^\omega \chi(\omega, x) \cdot b = 1/2 \nabla^{(\omega)*} G_b(x.\omega)$ for any $x \in \mathcal{C}(\omega)$.

Let $\phi(x) = x + \chi(\omega, x)$. Noting that $\hat{b}(\omega, e) = e \cdot b$ and $\hat{b}(e.\omega, -e) = -b \cdot e$, we see that $\nabla^{(\omega)*} \hat{b}(\omega) = 2/n^\omega(0) \sum_{e \in \mathcal{B}} \omega(e) b \cdot e$. Therefore,

$$\begin{aligned} \mathcal{L}^\omega \phi(\omega, x) \cdot b &= \frac{1}{n^\omega(x)} \sum_{e \in \mathcal{B}} \omega(x, x+e) e \cdot b + \mathcal{L}^\omega \chi(\omega, x) \\ &= \frac{1}{2} \nabla^{(\omega)*} \hat{b}(x.\omega) + \frac{1}{2} \nabla^{(\omega)*} G_b(x.\omega) = 0. \end{aligned}$$

This last equality holds for any $x \in \mathcal{C}(\omega)$. We have proved the martingale property.

The invariance principle. Let $\mathcal{M}(t) = X(t) + \chi(\omega, X(t))$.

In order to prove the convergence of the rescaled martingales $\mathcal{M}^\varepsilon(t) = \varepsilon \mathcal{M}(t/\varepsilon^2)$ towards a Brownian motion, we will use theorem 5.1 part a from Helland (1982). For the reader's convenience, we provide here the formulation of this theorem.

Theorem 2.6 (Helland 1982). *Let* m^ε *be a family of martingales with associated quadratic variation processes* $\langle m^\varepsilon \rangle$ *satisfying the following two conditions*:

(*i*) *for any* $t > 0$, *as* ε *tends to* 0, *then* $\langle \mathrm{m}^\varepsilon \rangle(t)$ *converges in probability towards* $\sigma^2 t$, *and*

(*ii*) *for any* $t > 0$ *and for any* $\eta > 0$, *as* ε *tends to* 0,

$$\sum_{0 \le s \le t} (m^\varepsilon(s) - m^\varepsilon(s-))^2 \mathbf{1}_{|m^\varepsilon(s) - m^\varepsilon(s-)| \ge \eta} \to 0,$$

in probability, then, as ε tends to 0, the sequence of processes $m^{\varepsilon}(.)$ converges in law in the Skorohod topology to a Brownian motion with variance σ^2.

More precisely, for any $b \in \mathcal{B}$, we check that, for any $t>0$, as ε tends to 0, $(1/t)\langle \mathcal{M}^{\varepsilon} \cdot b\rangle(t)$ almost surely converges to some constant, and

$$E_0^{\omega}\left[\sum_{0\le s\le t}(\mathcal{M}^{\varepsilon}(s)\cdot b-\mathcal{M}^{\varepsilon}(s-)\cdot b)^2 \mathbf{1}_{|\mathcal{M}^{\varepsilon}(s)\cdot b-\mathcal{M}^{\varepsilon}(s-)\cdot b|\ge \eta}\right]\to 0.$$

(See (2.11) and (2.12).) Both conditions follow from the computation of the bracket of the martingale $\mathcal{M}$ and the ergodic theorem applied to the Markov process $(X(t-).\omega)$ that represents the evolution of the environment ω as seen from the moving particle.

We start computing the bracket of the martingale $\mathcal{M}$ using the representation of the Markov chain X by Poisson processes: to each pair of neighbouring points $x, y \in \mathcal{C}_0(\omega)$, such that $\omega(x, y)=1$, attach a Poisson process of rate $1/n^{\omega}(x)$, say $N_t^{x,y}$, all of them being independent. Let X be the càd-làg solution of the equation $X(0)=0$,

$$\mathrm{d}X(t) = \sum_{y\sim X(t-)} \omega(X(t-), y)(y-X(t-))\mathrm{d}N_t^{X(t-),y}.$$

Then the law of the random process $(X(t), t\ge 0)$ is P_0^{ω}.

Let ω be such that $0\in\mathcal{C}(\omega)$. Let $\mathcal{M}(t)=X(t)+\chi(\omega, X(t))$. From the previous paragraph, we already know that $\mathcal{M}$ is a martingale. Its bracket can be computed using Itô's formula. We fix a direction $b\in\mathcal{B}$. Then

$$\begin{aligned}
&\mathrm{d}\langle \mathcal{M}\cdot b\rangle(t)\\
&=\frac{1}{n^{\omega}(X(t-))}\sum_{y\sim Y(t-)}\omega(X(t-), y)(y\cdot b+\chi(\omega, y)\cdot b - X(t-)\cdot b-\chi(\omega, X(t-))\cdot b)^2 \mathrm{d}t\\
&=\frac{1}{n^{X(t-).\omega}(0)}\sum_{e\in\mathcal{B}} X(t-).\omega(e)(e\cdot b+G_b(X(t-).\omega, e))^2\mathrm{d}t. \qquad (2.10)
\end{aligned}$$

Let $\tilde{Q}_0$ be the probability measure

$$\tilde{Q}_0(A) = \frac{\int_A n^{\omega}(0)\mathrm{d}Q_0(\omega)}{\int n^{\omega}(0)\mathrm{d}Q_0(\omega)}.$$

The random process $X(t-).\omega$ is Markovian under P_0^{ω}. The measure $\tilde{Q}_0$ is reversible, invariant and ergodic with respect to $X(t-).\omega$ (see lemma 4.9 in De Masi *et al.* (1989)). Observe that $\tilde{Q}_0$ is obviously absolutely continuous with respect to Q_0. As a consequence, by Birkhoff's ergodic theorem we get, Q.a.s. on the set $0\in\mathcal{C}(\omega)$,

$$\frac{\langle \mathcal{M}\cdot b\rangle(t)}{t}\underset{t\to\infty}{\rightarrow}\tilde{Q}_0\left(\frac{1}{n^{\omega}(0)}\sum_{e\in\mathcal{B}}\omega(e)(e\cdot b+G_b(\omega, e))^2\right).$$

Now let $\mathcal{M}^{\varepsilon}(t)=\varepsilon\mathcal{M}(t/\varepsilon^2)$. We have proved that, for any $t>0$, as ε tends to 0,

$$\langle \mathcal{M}^{\varepsilon}\cdot b\rangle(t)\to t\tilde{Q}_0\left(\frac{1}{n^{\omega}(0)}\sum_{e\in\mathcal{B}}\omega(e)(e\cdot b+G_b(\omega, e))^2\right). \qquad (2.11)$$

For any function $f: \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ that vanishes on the diagonal, the process

$$\sum_{0 \le s \le t} f(X(s), X(s-)) - \int_0^t \mathrm{d}s \frac{1}{n^{X(s-).\omega}(0)} \sum_{e \in \mathcal{B}} X(s-).\omega(e) f(X(s-)+e, X(s-))$$

is a local martingale. Applying this to $f(x, y) = (b\cdot(x+\chi(\omega,x)) - b\cdot(y+\chi(\omega, y)))^2 \mathbf{1}_{|b\cdot(x+\chi(\omega,x)) - b\cdot(y+\chi(\omega, y))| \ge \eta}$ for some direction b and some $\eta > 0$, we get that

$$\sum_{0 \le s \le t} (\mathcal{M}(s)\cdot b - \mathcal{M}(s-)\cdot b)^2 \mathbf{1}_{|\mathcal{M}(s)\cdot b - \mathcal{M}(s-)\cdot b| \ge \eta}$$

$$- \int_0^t \mathrm{d}s \frac{1}{n^{X(s-).\omega}(0)} \sum_{e \in \mathcal{B}} X(s-).\omega(e)(e\cdot b + G_b(X(s-).\omega, e))^2 \mathbf{1}_{|e\cdot b + G_b(X(s-).\omega, e)| \ge \eta}$$

is a martingale. Taking expectations and using the ergodic theorem for the process $X(s-).\omega$, we get, on the set $0 \in \mathcal{C}(\omega)$,

$$E_0^\omega \left[\frac{1}{t} \sum_{0 \le s \le t} (\mathcal{M}(s)\cdot b - \mathcal{M}(s-)\cdot b)^2 \mathbf{1}_{|\mathcal{M}(s)\cdot b - \mathcal{M}(s-)\cdot b| \ge \eta} \right]$$
$$= \frac{1}{t} \int_0^t \mathrm{d}s E_0^\omega \left[\frac{1}{n^{X(s-).\omega}(0)} \sum_{e \in \mathcal{B}} X(s-).\omega(e)(e\cdot b + G_b(X(s-).\omega, e))^2 \mathbf{1}_{|e\cdot b + G_b(X(s-).\omega, e)| \ge \eta} \right]$$
$$\to \tilde{Q}_0 \left(\frac{1}{n^\omega(0)} \sum_{e \in \mathcal{B}} \omega(e)(e\cdot b + G_b(\omega, e))^2 \mathbf{1}_{|e\cdot b + G_b(\omega, e)| \ge \eta} \right) < \infty.$$

Then, for any $t > 0$,

$$E_0^\omega \left[\sum_{0 \le s \le t} (\mathcal{M}^\varepsilon(s)\cdot b - \mathcal{M}^\varepsilon(s-)\cdot b)^2 \mathbf{1}_{|\mathcal{M}^\varepsilon(s)\cdot b - \mathcal{M}^\varepsilon(s-)\cdot b| \ge \eta} \right]$$

$$= \varepsilon^2 E_0^\omega \left[\sum_{0 \le s \le t/\varepsilon^2} (\mathcal{M}(s)\cdot b - \mathcal{M}(s-)\cdot b)^2 \mathbf{1}_{|\mathcal{M}(s)\cdot b - \mathcal{M}(s-)\cdot b| \ge \eta/\varepsilon} \right] \to 0. \tag{2.12}$$

From the martingale convergence theorem, theorem 5.1 part a in Helland (1982), we then deduce that, Q.a.s. on the set $0 \in \mathcal{C}(\omega)$, the law of the process $\varepsilon X(./\varepsilon^2) + \varepsilon\chi(\omega, X(./\varepsilon^2))$ under P_0^ω converges to the law of a Brownian motion with a deterministic covariance matrix $A = \sigma^2 Id$ with

$$\sigma^2 = \tilde{Q}_0 \left(\frac{1}{n^\omega(0)} \sum_{e \in \mathcal{B}} \omega(e)(e\cdot e_1 + G_{e_1}(\omega, e))^2 \right), \tag{2.13}$$

and e_1 being the first coordinate vector. Thus, we proved the following statement.

Lemma 2.7. *Q_0.a.s., the family $\mathcal{M}^\varepsilon(t) = \varepsilon X(./\varepsilon^2) + \varepsilon\chi(\omega, X(./\varepsilon^2))$ converges in law in the Skorohod topology to a Brownian motion with covariance matrix $\sigma^2 Id$, where σ^2 is defined in (2.13).*

(c) *Convergence of the corrector*

We now check that the contribution of the corrector is negligible in the limit. Let us recall that, by lemmas 2.1 and 2.7, the process $\varepsilon\chi(\omega, X(t/\varepsilon^2))$ is tight in the Skorohod topology on any time interval. Therefore, it suffices to prove that, for all t, $\varepsilon\chi(\omega, X(t/\varepsilon^2))$ converges to 0 in P_0^ω probability, Q.a.s. on the set $0\in\mathcal{C}(\omega)$. In view of (2.1), it is sufficient to show that

$$\lim_{\varepsilon\to 0}\varepsilon^d \sum_{y\in\mathcal{C}_0(\omega);|y|\le 1/\varepsilon} |\varepsilon\chi(\omega,\, y)|^2 = 0 \quad Q_0\text{.a.s.}$$

We first note that the tightness of the family $\varepsilon X(t/\varepsilon^2)$ implies the relation

$$\lim_{t\to 0}\lim_{\varepsilon\to 0} P_0^\omega\left[\left|\varepsilon X\left(\frac{t}{\varepsilon^2}\right)\right| \ge K\right] = 0 \quad Q_0\text{.a.s., and for any } K>0. \tag{2.14}$$

Below, we use the Poincaré inequality to prove that there exist some constants $a_\varepsilon(\omega)$ such that

$$\lim_{\varepsilon\to 0}\varepsilon^d \sum_{y\in\mathcal{C}_O(\omega);\; |y|\le 1/\varepsilon} |\varepsilon\chi(\omega,\, y) - a_\varepsilon|^2 = 0 \quad Q_0\text{.a.s.} \tag{2.15}$$

As a consequence of (2.1), (2.15) yields

$$\lim_{t\to 0}\lim_{\varepsilon\to 0} P_0^\omega\left[\left|\varepsilon\chi\left(\omega, X\left(\frac{t}{\varepsilon^2}\right)\right) - a_\varepsilon\right| \ge K\right] = 0 \quad Q_0\text{.a.s.,} \quad \text{and} \quad \text{for} \quad \text{any} \quad \mathrm{K}>0.$$

Indeed,

$$\begin{aligned}
&\lim_{t\to 0}\lim_{\varepsilon\to 0} P_0^\omega\left\{\left|\varepsilon\chi\left(\omega, X\left(\frac{t}{\varepsilon^2}\right)\right) - a_\varepsilon\right| \ge K\right\} \\
&\quad = \lim_{t\to 0}\lim_{\varepsilon\to 0} P_0^\omega\left\{\left|\varepsilon\chi\left(\omega, X\left(\frac{t}{\varepsilon^2}\right)\right) - a_\varepsilon\right| \ge K, \left|X\left(\frac{t}{\varepsilon^2}\right)\right| \le \frac{1}{\varepsilon}\right\} \\
&\quad\quad + \lim_{t\to 0}\lim_{\varepsilon\to 0} P_0^\omega\left\{\left|\varepsilon\chi\left(\omega, X\left(\frac{t}{\varepsilon^2}\right)\right) - a_\varepsilon\right| \ge K, \left|X\left(\frac{t}{\varepsilon^2}\right)\right| > \frac{1}{\varepsilon}\right\}.
\end{aligned}$$

By (2.1) and (2.15), the first limit on the right-hand side is equal to zero. The second one is zero owing to (2.14).

But the invariance principle for the process $\varepsilon X(t/\varepsilon^2) + \varepsilon\chi(\omega, X(t/\varepsilon^2))$ implies that

$$\lim_{t\to 0}\lim_{\varepsilon\to 0} P_0^\omega\left[|\varepsilon X\left(\frac{t}{\varepsilon^2}\right) + \varepsilon\chi\left(\omega, X\left(\frac{t}{\varepsilon^2}\right)\right)| \ge K\right] = 0 \quad Q_0\text{.a.s.,}$$

and for any $\mathrm{K}>0$.

Thus, taking the difference, we see that a_ε tends to 0 and

$$\lim_{\varepsilon\to 0}\varepsilon^d \sum_{y\in\mathcal{C}_0(\omega);\; |y|\le 1/\varepsilon} |\varepsilon\chi(\omega,\, y)|^2 = 0 \quad Q_0\text{.a.s.}$$

It remains to justify (2.15).

Poincaré inequalities. Let us recall that the function $G_b = G_b(\omega, e)$, $\omega\in\Omega$, $e\in\mathcal{B}$, is defined as a unique solution to problem (2.8) and $\chi(\omega, x)$ satisfies (2.9). Since G_b is square integrable, the spatial ergodic theorem (see Krengel 1985, p. 205) implies that $\varepsilon^d \sum_{e\in\mathcal{B}} \sum_{x\in\mathcal{C}_0(\omega);\ |x|\le 1/\varepsilon} x.\omega(e)(G_b(x.\omega, e))^2$ has a Q.a.s. finite limit. Therefore,

$$\limsup_{\varepsilon} \varepsilon^d \sum_{e\in\mathcal{B}} \sum_{x\in\mathcal{C}_0(\omega);\ |x|\le 1/(1-a)\varepsilon} x.\omega(e)(G_b(x.\omega, e))^2 < \infty, \tag{2.16}$$

Q_0.a.s. and for any constant $0<a<1$.

We quote from Mathieu & Remy (2004), theorem 1.3. For some $\varepsilon>0$, define $\mathcal{C}_0^\varepsilon$ to be the connected component of the intersection of $\mathcal{C}_0(\omega)$ with the box $[-1/\varepsilon, 1/\varepsilon]^d$ that contains the origin. There exists a constant β such that, Q_0.a.s. for small enough ε, for any function $u:\mathcal{C}_0^\varepsilon \to \mathbb{R}$ one has

$$\frac{1}{\#\mathcal{C}_0^\varepsilon} \sum_{x,\, y\in\mathcal{C}_0^\varepsilon} (u(x)-u(y))^2 \le \beta\varepsilon^{-2} \sum_{x\sim y\in\mathcal{C}_0^\varepsilon} \omega(x,\ y)(u(x)-u(y))^2.$$

Since $\#\mathcal{C}_0^\varepsilon$ is of order ε^{-d} for small enough ε and since $\mathcal{C}_0(\omega)\cap[-1/\varepsilon, 1/\varepsilon]^d \subset \mathcal{C}_0^{(1-a)\varepsilon}$ for some constant a, we therefore have a constant β such that, Q_0.a.s. for small enough ε, for any function $u:\mathcal{C}_0(\omega)\to\mathbb{R}$

$$\varepsilon^d \sum_{x,\, y\in\mathcal{C}_0(\omega);\ |x|,|y|\le 1/\varepsilon} (u(x)-u(y))^2 \le \beta\varepsilon^{-2} \sum_{x\sim y\in\mathcal{C}_0^{(1-a)\varepsilon}} \omega(x,\ y)(u(x)-u(y))^2.$$

We use this last inequality for the functions $u(x)=\chi(\omega, x)\cdot b$ to get

$$\begin{aligned} &\varepsilon^d \sum_{x,\, y\in\mathcal{C}_0(\omega);\ |x|,|y|\le 1/\varepsilon} |\chi(\omega, x)-\chi(\omega,\ y)|^2 \\ &\quad \le \beta\varepsilon^{-2} \sum_{b\in\mathcal{B}} \sum_{e\in\mathcal{B}} \sum_{x\in\mathcal{C}_0(\omega);\ |x|\le 1/(1-a)\varepsilon} x.\omega(e)(G_b(x.\omega, e))^2. \end{aligned}$$

By (2.16), we therefore get

$$\limsup_{\varepsilon} \varepsilon^{2d} \sum_{x,\, y\in\mathcal{C}_0(\omega);\ |x|,|y|\le 1/\varepsilon} |\varepsilon\chi(\omega, x)-\varepsilon\chi(\omega,\ y)|^2 < \infty,$$

Q_0.a.s., and

$$\limsup_{\varepsilon} \varepsilon^{2d} \sum_{x,\, y\in\mathcal{C}_0(\omega);\ |x|,|y|\le 1/\varepsilon} n^\omega(x)n^\omega(y)|\varepsilon\chi(\omega, x)-\varepsilon\chi(\omega,\ y)|^2 < \infty.$$

Indeed, observe that the presence of the bounded factors $n^\omega(x)$ and $n^\omega(y)$ is harmless. This last inequality is equivalent to

$$\limsup_{\varepsilon} \varepsilon^d \sum_{x\in\mathcal{C}_0(\omega);\ |x|\le 1/\varepsilon} n^\omega(x)|\varepsilon\chi(\omega, x)-a_\varepsilon|^2 < \infty, \tag{2.17}$$

Q_0.a.s., where

$$a_\varepsilon = \sum_{x\in\mathcal{C}_0(\omega);\ |x|\le 1/\varepsilon} n^\omega(x)\varepsilon\chi(\omega, x)$$

is the mean value of $\varepsilon\chi(\omega, x)$ on the set $\{x\in\mathcal{C}_0(\omega);\ |x|\le 1/\varepsilon\}$ with respect to the measure with weight $n^\omega(x)$.

Two-scale convergence. We first introduce some notation. Let $\Gamma=]-1,1[^d$. For $\omega\in\Omega$ and $\varepsilon>0$, we define the measures

$$\mu_\omega=\sum_{z\in\mathcal{C}(\omega)} n^\omega(z)\delta_z,\quad \mu_\omega^\varepsilon=\varepsilon^d\sum_{z\in\mathcal{C}(\omega)} n^\omega(z)\delta_{\varepsilon z}.$$

Given a direction $e\in\mathcal{B}$, the gradient of a function $\phi:\mathbb{R}^d\to\mathbb{R}$ is

$$\nabla_e^\varepsilon\phi(z)=\frac{1}{\varepsilon}(\phi(z+\varepsilon e)-\phi(z)).$$

Let us now choose $b_0\in\mathcal{B}$ and let

$$\psi^\varepsilon(\omega,z)=\left(\varepsilon\chi\left(\omega,\frac{1}{\varepsilon}z\right)-a_\varepsilon\right)\cdot b_0.$$

Thus ψ^ε is well defined for $z\in\varepsilon\mathcal{C}_0(\omega)$. From the definition of χ, we have

$$\nabla_e^\varepsilon\psi^\varepsilon(\omega,z)=G_{b_0}\left(\frac{1}{\varepsilon}z.\omega,e\right),$$

for $z\in\varepsilon\mathcal{C}_0(\omega)$.

Remember that, for $z'\in\mathbb{Z}^d$, the expression $z'.\omega$ denotes the graph obtained by translating ω by z'. In particular, for $z\in\varepsilon\mathbb{Z}^d$, $(1/\varepsilon)z.\omega(e)$ is either 0 or 1, depending on whether the edge $(z,z+e)$ belongs to ω or not. We sometimes prefer the notation $(z/\varepsilon).\omega(e)$ in order to avoid possible confusion.

In our new notation, (2.16) and (2.17) now read

$$C_1(\omega)=\sup_{e\in\mathcal{B}}\sup_\varepsilon\int_\Gamma\left(\frac{z}{\varepsilon}\right).\omega(e)(\nabla_e^\varepsilon\psi^\varepsilon(\omega,z))^2\mathrm{d}\mu_\omega^\varepsilon(z)<\infty, \tag{2.18}$$

and

$$C_2(\omega)=\sup_\varepsilon\int_\Gamma(\psi^\varepsilon(\omega,z))^2\mathrm{d}\mu_\omega^\varepsilon(z)<\infty, \tag{2.19}$$

for Q_0 almost any ω. For further reference, let us call Ω_1 the set of ω in Ω such that $0\in\mathcal{C}(\omega)$, $C_1(\omega)<\infty$ and $C_2(\omega)<\infty$ and observe that $Q_0(\Omega_1)=1$.

Define the measure

$$\mathcal{P}(A)=Q\left[\mathbf{1}_A(\omega)n^\omega(0)\mathbf{1}_{0\in\mathcal{C}(\omega)}\right]. \tag{2.20}$$

According to the ergodic theorem, for any smooth function $\phi\in C^\infty(\Gamma)$ and any $u\in L^1(\Omega,\mathcal{P})$ we have

$$\int_\Gamma\phi(z)u\left(\frac{1}{\varepsilon}z.\omega\right)\mathrm{d}\mu_\omega^\varepsilon(z)\to\left(\int_\Gamma\phi(z)\mathrm{d}z\right)\left(\int_\Omega u(\omega')\mathrm{d}\mathcal{P}(\omega')\right), \tag{2.21}$$

Q_0.a.s.

We endow Ω with its natural (product) topology to turn it into a compact space. We will use the notation $C(\Omega)$ for continuous real-valued functions defined on Ω. Using standard separability arguments, we see that (2.21) holds simultaneously for any $\phi\in C^\infty(\Gamma)$ and $u\in C(\Omega)$ on a set of full Q_0 measure. More precisely, let Ω_2 be the set of ω in Ω such that $0\in\mathcal{C}(\omega)$, and for any functions $\phi\in C^\infty(\Gamma)$ and $u\in C(\Omega)$ one has

$$\int_\Gamma\phi(z)u\left(\frac{1}{\varepsilon}z.\omega\right)\mathrm{d}\mu_\omega^\varepsilon(z)\to\left(\int_\Gamma\phi(z)\mathrm{d}z\right)\left(\int_\Omega u(\omega')\mathrm{d}\mathcal{P}(\omega')\right), \tag{2.22}$$

and, for any $e \in \mathcal{B}$,

$$\int_\Gamma \phi(z) \frac{1}{\varepsilon} z.\omega(e) u\left(\frac{1}{\varepsilon} z.\omega\right) G_{b_0}\left(\frac{1}{\varepsilon} z.\omega, e\right) \mathrm{d}\mu_\omega^\varepsilon(z) \to \left(\int_\Gamma \phi(z) \mathrm{d}z\right)\left(\int_\Omega \omega'(e) u(\omega') G_{b_0}(\omega', e) \mathrm{d}\mathcal{P}(\omega')\right). \tag{2.23}$$

Then $Q_0(\Omega_2)=1$. Finally, let $\Omega_0=\Omega_1 \cap \Omega_2$. In the sequel, α will denote an element of Ω_0.

Our next goal is to introduce a version of two-scale convergence adapted to the model studied. To this end, consider the family of linear functionals

$$L^{\varepsilon,\alpha}(u, \phi) = \int_\Gamma \phi(z) \psi^\varepsilon(\alpha, z) u\left(\frac{1}{\varepsilon} z.\alpha\right) \mathrm{d}\mu_\alpha^\varepsilon(z).$$

Using the Cauchy–Schwartz inequality, we get

$$(L^{\varepsilon,\alpha}(u, \phi))^2 \le \int_\Gamma (\psi^\varepsilon(\alpha, z))^2 \mathrm{d}\mu_\alpha^\varepsilon(z) \int_\Gamma \phi(z)^2 u\left(\frac{1}{\varepsilon} z.\alpha\right)^2 \mathrm{d}\mu_\alpha^\varepsilon(z).$$

From (2.19) and (2.22), we deduce that for $\phi \in C^\infty(\Gamma)$ and $u \in C(\Omega)$

$$\limsup_\varepsilon \, (L^{\varepsilon,\alpha}(u, \phi))^2 \le C_2(\alpha) \int_\Gamma \phi(z)^2 \mathrm{d}z \int_\Omega u(\omega)^2 \mathrm{d}\mathcal{P}(\omega).$$

Therefore, applying the diagonal procedure, we conclude that, up to extracting a sub-sequence, we can assume that, for any smooth ϕ and any continuous $u \in C(\Omega)$, $L^{\varepsilon,\alpha}(u, \phi)$ has a limit of say $L^\alpha(u, \phi)$, where L^α is a linear functional satisfying

$$(L^\alpha(u, \phi))^2 \le C_2(\alpha) \int_\Gamma \phi(z)^2 \mathrm{d}z \int_\Omega u(\omega)^2 \mathrm{d}\mathcal{P}(\omega).$$

Thus, L^α can be extended as a continuous linear functional on $L^2(\Omega \times \Gamma, \mathrm{d}\mathcal{P} \times \mathrm{d}x)$ and, by Riesz's theorem, there exists a function $v^\alpha \in L^2(\Omega \times \Gamma, \mathrm{d}\mathcal{P} \times \mathrm{d}x)$ such that

$$L^\alpha(u, \phi) = \int_\Gamma \phi(z) \mathrm{d}z \int_\Omega u(\omega) v^\alpha(\omega, z) \mathrm{d}\mathcal{P}(\omega).$$

Let us summarize the preceding discussion: we have proved that, up to extracting a sub-sequence, for $\phi \in C^\infty(\Gamma)$ and $u \in C(\Omega)$,

$$\int_\Gamma \phi(z) \psi^\varepsilon(\alpha, z) u\left(\frac{1}{\varepsilon} z.\alpha\right) \mathrm{d}\mu_\alpha^\varepsilon(z) \to \int_\Gamma \phi(z) \mathrm{d}z \int_\Omega u(\omega) v^\alpha(\omega, z) \mathrm{d}\mathcal{P}(\omega). \tag{2.24}$$

We will prove lemma 2.8.

Lemma 2.8. *For any $\alpha \in \Omega_0$, $v^\alpha(\omega, z)=0$ for Lebesgue almost any $z \in \Gamma$ and $\mathcal{P}$ almost any ω.*

As a consequence of this lemma, we have that for Q_0 almost any α, for any function $\phi \in C^\infty(\Gamma)$,

$$\int_\Gamma \phi(z) \psi^\varepsilon(\alpha, z) \mathrm{d}\mu_\alpha^\varepsilon(z) \to 0.$$

Since we also have uniform bounds on the L^2 norm of ψ^{ε} (see (2.19)), we deduce that, for any rectangle $A\subset\Gamma$,

$$\int_A \psi^{\varepsilon}(\alpha, z)\mathrm{d}\mu_{\alpha}^{\varepsilon}(z)\to 0.$$

We conclude that, for any rectangle $A\subset[-1,1]^d$, Q_0.a.s.

$$\varepsilon^d \sum_{x\in\mathcal{C}(\omega);\ \varepsilon x\in A} n^{\omega}(x)(\varepsilon\chi(\omega, x)-a_{\varepsilon})\to 0. \tag{2.25}$$

Remark 2.9. The content of this part of the paper, including the proof of lemma 2.8 below, should be compared with the results of Jikov & Piatnitski (2006). The convergence in (2.24) is known as 'two-scale convergence'. The only difference between our setting and that of Jikov & Piatnitski (2006) is the discrete nature of the grid; continuous diffusions are considered in Jikov & Piatnitski (2006).

It is also possible to directly apply the results of Jikov & Piatnitski (2006) to justify lemma 2.8. We refer the interested reader to the first version of the present paper on the arXiv e-print archive for details. Here, we preferred to give a more self-contained approach, but most of the arguments are mere copies of the proofs in Jikov & Piatnitski (2006) with some minor simplifications owing to the fact that, for instance, the Palm measure $\mathcal{P}$ is explicit and absolutely continuous w.r.t. Q.

Proof of lemma 2.8. The proof is in three steps. Throughout the following proof, ϕ is always assumed to be in $C_o^{\infty}(\Gamma)$, the space of smooth functions with compact support in Γ.

Step 1. We check the integration by parts formula

$$\int_{\Gamma}\phi(z)\nabla^{(\omega)*}u\left(\frac{1}{\varepsilon}z.\alpha\right)\mathrm{d}\mu_{\alpha}^{\varepsilon}(z) = -\varepsilon\int_{\Gamma}\frac{1}{n^{\alpha}\left(\frac{1}{\varepsilon}z\right)} \times\sum_{e\in\mathcal{B}} u\left(\frac{1}{\varepsilon}z.\alpha, e\right)\left(\frac{z}{\varepsilon}\right).\alpha(e)\nabla_e^{\varepsilon}\phi(z)\mathrm{d}\mu_{\alpha}^{\varepsilon}(z), \tag{2.26}$$

where u is any function defined on $\Omega\times\mathcal{B}$ and ε is small enough (depending on the support of ϕ),

$$\begin{aligned}\int_{\Gamma}\phi(z)\nabla^{(\omega)*}u\left(\frac{1}{\varepsilon}z.\alpha\right)\mathrm{d}\mu_{\alpha}^{\varepsilon}(z) &= \varepsilon^d\sum_{x\in\mathcal{C}(\alpha)}\phi(\varepsilon x)\nabla^{(\omega)*}u(x.\alpha)n^{\alpha}(x)\\ &= \varepsilon^d\sum_{x\in\mathcal{C}(\alpha)}\phi(\varepsilon x)\sum_{e\in\mathcal{B}}x.\alpha(e)(u(x.\alpha, e)\\ &\quad -u(x.e.\alpha,-e))n^{\alpha}(x).\end{aligned}$$

But

$$\begin{aligned}&\sum_{x\in\mathcal{C}(\alpha)}\phi(\varepsilon x)\sum_{e\in\mathcal{B}}x.\alpha(e)u(x.e.\alpha,-e)n^{\alpha}(x)\\ &\quad=\sum_{x'\in\mathcal{C}(\alpha)}\sum_{e'\in\mathcal{B}}x'.\alpha(e')u(x'.\alpha, e')\phi(\varepsilon x'+\varepsilon e')n^{\alpha}(x'),\end{aligned}$$

with the change of variables $x'=x+e$ and $e'=-e$. Combining the last two equalities, one gets (2.26). Observe that boundary terms vanish because ϕ has compact support and ε is small enough.

Step 2. We prove that $v^\alpha(\omega, z)$ does not depend on ω, i.e. that Q_0.a.s.

$$v^\alpha(\omega, z) = \frac{\int v^\alpha(\omega', z)\mathrm{d}\mathcal{P}(\omega')}{\int \mathrm{d}\mathcal{P}(\omega')} = v^\alpha(z). \tag{2.27}$$

Indeed, let u be continuous on $\Omega\times\mathcal{B}$ and $\phi\in C_o^\infty(\Gamma)$ and use (2.24) and the integration by parts formula (2.26) to get

$$\int_\Gamma \phi(z)\mathrm{d}z\int_\Omega v^\alpha(\omega, z)\nabla^{(\omega)*}u(\omega)\mathrm{d}\mathcal{P}(\omega) = \lim_\varepsilon \int_\Gamma \phi(z)\psi^\varepsilon(\alpha, z)\nabla^{(\omega)*}u\left(\frac{1}{\varepsilon}z.\alpha\right)\mathrm{d}\mu_\alpha^\varepsilon(z)$$

$$= \lim_\varepsilon -\varepsilon\int_\Gamma \frac{1}{n^\alpha\left(\frac{1}{\varepsilon}z\right)}\sum_{e\in\mathcal{B}} u\left(\frac{1}{\varepsilon}z.\alpha, e\right)\left(\frac{z}{\varepsilon}\right).\alpha(e)\nabla_e^\varepsilon(\psi^\varepsilon(\alpha, .)\phi)(z)\mathrm{d}\mu_\alpha^\varepsilon(z).$$

Since u is continuous, it is bounded. Note that $(z/\varepsilon).\alpha(e)\le n^\alpha((1/\varepsilon)z)$. Besides,

$$\limsup_\varepsilon \int_\Gamma \left(\frac{z}{\varepsilon}\right).\alpha(e)(\nabla_e^\varepsilon(\psi^\varepsilon(\alpha, .)\phi)(z))^2\mathrm{d}\mu_\alpha^\varepsilon(z)\le 2C_1(\alpha)\|\phi\|_\infty^2 + 2C_2(\alpha)\|\nabla\phi\|_\infty^2 < \infty.$$

We conclude that, as ε tends to 0, the expression

$$\int_\Gamma \frac{1}{n^\alpha\left(\frac{1}{\varepsilon}z\right)} u\left(\frac{1}{\varepsilon}z.\alpha, e\right)\left(\frac{z}{\varepsilon}\right).\alpha(e)\nabla_e^\varepsilon(\psi^\varepsilon(\alpha, .)\phi)(z)\mathrm{d}\mu_\alpha^\varepsilon(z),$$

remains bounded and therefore

$$\lim_\varepsilon -\varepsilon\int_\Gamma \frac{1}{n^\alpha\left(\frac{1}{\varepsilon}z\right)}\sum_{e\in\mathcal{B}} u\left(\frac{1}{\varepsilon}z.\alpha, e\right)\left(\frac{z}{\varepsilon}\right).\alpha(e)\nabla_e^\varepsilon(\psi^\varepsilon(\alpha, .)\phi)(z)\mathrm{d}\mu_\alpha^\varepsilon(z) = 0,$$

and

$$\int_\Gamma \phi(z)\mathrm{d}z\int_\Omega v^\alpha(\omega, z)\nabla^{(\omega)*}u(\omega)\mathrm{d}\mathcal{P}(\omega) = 0.$$

By (2.6), we also have

$$\int_\Omega v^\alpha(\omega, z)\nabla^{(\omega)*}u(\omega)\mathrm{d}\mathcal{P}(\omega) = \int_\Omega v^\alpha(\omega, z)\nabla^{(\omega)*}u(\omega)n^\omega(0)1_{0\in\mathcal{C}(\omega)}\mathrm{d}Q(\omega)$$

$$= -\int u\nabla^{(\omega)}v^\alpha(., z)\mathrm{d}M.$$

Thus, we have proved that

$$\int_\Gamma \phi(z)\mathrm{d}z\int u\nabla^{(\omega)}v^\alpha(., z)\mathrm{d}M = 0,$$

for any $\phi\in C_o^\infty(\Gamma)$ and continuous u. We deduce that Q_0.a.s. for any $b\in\mathcal{B}$ such that $\omega(b)=1$ and for Lebesgue almost any z then $v^\alpha(b.\omega, z)=v^\alpha(\omega, z)$. Integrating this equality on a path between 0 and $x\in\mathcal{C}_0(\omega)$, we then get that Q_0.a.s. for any

$x \in \mathcal{C}(\omega)$ and for Lebesgue almost any z then $v^\alpha(\omega, z) = v^\alpha(x.\omega, z)$. Therefore, since μ_ω^ε charges only $\mathcal{C}(\omega)$, the ergodic theorem yields

$$v^\alpha(\omega, z) = \frac{\int_\Gamma v^\alpha\left(\frac{1}{\varepsilon} z'.\omega, z\right) \mathrm{d}\mu_\omega^\varepsilon(z')}{\int_\Gamma \mathrm{d}\mu_\omega^\varepsilon(z')} \underset{\varepsilon \to 0}{\to} \frac{\int v^\alpha(\omega', z) \mathrm{d}\mathcal{P}(\omega')}{\int \mathrm{d}\mathcal{P}(\omega')} = v^\alpha(z),$$

Q_0.a.s in ω and for Lebesgue almost any $z \in \Gamma$.

Step 3. We now prove that $v^\alpha(z)$ does not depend on z. To this end, we first prove that, for any smooth $\phi \in C_o^\infty(\Gamma)$ and any continuous $u \in L^2_{\mathrm{sol}}$, we have

$$\sum_{e \in \mathcal{B}} \left(\int_\Gamma \mathrm{d}z\, v^\alpha(z) \nabla\phi(z) \cdot e \right) \left(\int_\Omega \tilde{u}(\omega, e) \mathrm{d}\mathcal{P}(\omega) \right) = 0, \tag{2.28}$$

where

$$\tilde{u}(\omega, e) = \frac{\omega(e)}{n^\omega(0)} u(\omega, e). \tag{2.29}$$

We have

$$\begin{aligned}
&\left(\int_\Gamma \mathrm{d}z\, v^\alpha(z) \nabla\phi(z) \cdot e \right) \left(\int_\Omega \tilde{u}(\omega, e) \mathrm{d}\mathcal{P}(\omega) \right) \\
&= \lim_\varepsilon \int_\Gamma (\nabla\phi(z) \cdot e) \psi^\varepsilon(\alpha, z) \tilde{u}\left(\frac{1}{\varepsilon} z.\alpha, e \right) \mathrm{d}\mu_\alpha^\varepsilon(z) \\
&= \lim_\varepsilon \int_\Gamma \nabla_e^\varepsilon \phi(z) \psi^\varepsilon(\alpha, z) \tilde{u}\left(\frac{1}{\varepsilon} z.\alpha, e \right) \mathrm{d}\mu_\alpha^\varepsilon(z) \\
&= \lim_\varepsilon \int_\Gamma \nabla_e^\varepsilon (\phi \psi^\varepsilon(\alpha, .))(z) \tilde{u}\left(\frac{1}{\varepsilon} z.\alpha, e \right) \mathrm{d}\mu_\alpha^\varepsilon(z) \\
&\quad - \lim_\varepsilon \int_\Gamma \phi(z) \nabla_e^\varepsilon (\psi^\varepsilon(\alpha, .))(z) \tilde{u}\left(\frac{1}{\varepsilon} z.\alpha, e \right) \mathrm{d}\mu_\alpha^\varepsilon(z),
\end{aligned} \tag{2.30}$$

where we used (2.24) in the first equality and the regularity of ϕ for the second and third equalities. Using integration by parts and the definition of $\tilde{u}$, we get

$$\begin{aligned}
&\sum_{e \in \mathcal{B}} \int_\Gamma \nabla_e^\varepsilon (\phi \psi^\varepsilon(\alpha, .))(z) \tilde{u}\left(\frac{1}{\varepsilon} z.\alpha, e \right) \mathrm{d}\mu_\alpha^\varepsilon(z) \\
&= \sum_{e \in \mathcal{B}} \int_\Gamma \nabla_e^\varepsilon (\phi \psi^\varepsilon(\alpha, .))(z) \frac{\left(\frac{z}{\varepsilon}\right).\alpha(e)}{n^\alpha\left(\frac{1}{\varepsilon} z\right)} u\left(\frac{1}{\varepsilon} z.\alpha, e \right) \mathrm{d}\mu_\alpha^\varepsilon(z) \\
&= -\frac{1}{\varepsilon} \int_\Gamma \phi(z) \psi^\varepsilon(\alpha, z) \nabla^{(\omega)*} u\left(\frac{1}{\varepsilon} z.\alpha \right) \mathrm{d}\mu_\alpha^\varepsilon(z) = 0,
\end{aligned}$$

since $u \in L^2_{\mathrm{sol}}$ and therefore $\nabla^{(\omega)*} u = 0$.

We now turn to the second term in (2.30). Remember that $\nabla_e^\varepsilon(\psi^\varepsilon(\alpha,.))(z) = G_{b_0}\left(\frac{1}{\varepsilon}z.\alpha, e\right)$. Thus, as an application of (2.23),

$$\begin{aligned}
&\lim_{\varepsilon}\int_\Gamma \phi(z)\nabla_e^\varepsilon(\psi^\varepsilon(\alpha,.))(z)\tilde{u}\left(\frac{1}{\varepsilon}z.\alpha, e\right)\mathrm{d}\mu_\alpha^\varepsilon(z)\\
&\quad = \lim_{\varepsilon}\int_\Gamma \phi(z)\nabla_e^\varepsilon(\psi^\varepsilon(\alpha,.))(z)\frac{\left(\frac{z}{\varepsilon}\right).\alpha(e)}{n^\alpha\left(\frac{1}{\varepsilon}z\right)}u\left(\frac{1}{\varepsilon}z.\alpha, e\right)\mathrm{d}\mu_\alpha^\varepsilon(z)\\
&\quad = \left(\int_\Gamma \phi(z)\mathrm{d}z\right)\left(\int_\Omega G_{b_0}(\omega, e)\tilde{u}(\omega, e)\mathrm{d}\mathcal{P}(\omega)\right).
\end{aligned}$$

Replacing $\tilde{u}$ and $\mathcal{P}$ by their definitions in (2.29) and (2.20), respectively, we also have

$$\begin{aligned}
\sum_{e\in\mathcal{B}}\int_\Omega G_{b_0}(\omega, e)\tilde{u}(\omega, e)\mathrm{d}\mathcal{P}(\omega) &= \sum_{e\in\mathcal{B}}\int_\Omega G_{b_0}(\omega, e)\omega(e)u(\omega, e)\mathbf{1}_{0\in\mathcal{C}(\omega)}\mathrm{d}Q(\omega)\\
&= \int G_{b_0}u\mathrm{d}M = 0,
\end{aligned}$$

since $G_{b_0}\in L^2_{\text{pot}}$ and $u\in L^2_{\text{sol}}$. We conclude that (2.28) holds.

Equation (2.28) was proved for any continuous $u\in L^2_{\text{sol}}$. By density, it also holds for any $u\in L^2_{\text{sol}}$.

It remains to check the following fact.

Lemma 2.9. *For any direction $e\in\mathcal{B}$, there exists $u\in L^2_{\text{sol}}$ such that $\int_\Omega\tilde{u}(\omega, e)\mathrm{d}\mathcal{P}(\omega)\neq 0$ with $\tilde{u}$ defined in (2.29).*

Proof. Indeed, first of all note that, by definition of $\tilde{u}$,

$$\int_\Omega \tilde{u}(\omega, e)\mathrm{d}\mathcal{P}(\omega) = \int_{0\in\mathcal{C}(\omega)}\omega(e)u(\omega, e)\mathrm{d}Q(\omega).$$

Define the random field $\tilde{e}$ by $\tilde{e}(\omega, b) = 1_{b=e}$ (e is kept fixed.) Let G be the orthogonal projection of $-\tilde{e}$ on L^2_{pot} and let $u = G + \tilde{e}\in L^2_{\text{sol}}$. We write that u and $u-\tilde{e} = G$ are orthogonal,

$$\int_{0\in\mathcal{C}(\omega)}\omega(e)u(\omega, e)\mathrm{d}Q(\omega) = \int u\tilde{e}\mathrm{d}M = \int u^2\mathrm{d}M\neq 0,$$

because $\tilde{e}\notin L^2_{\text{sol}}$ and $u\neq 0$. ■

Conclusion of the proof of lemma 2.8. We deduce from (2.28) and lemma 2.9 that

$$\int_\Gamma \mathrm{d}z\, v^\alpha(z)\nabla\phi(z)\cdot e = 0,$$

for any smooth ϕ and any direction e. Therefore, v^α is Lebesgue almost surely constant.

The mean of ψ^ε w.r.t. the measure μ_ω^ε on Γ vanishes; remember, this is the way we chose a_ε. Therefore, v^α also has a vanishing mean w.r.t. the Lebesgue measure on Γ. And since, by steps 2 and 3, v^α is almost surely constant, we must have that Q_0.a.s. and for Lebesgue almost any z, $v^\alpha(\omega,z)=0$.

Scaling and strong L^2 convergence of χ. To conclude the proof of the theorem, we still have to prove the strong L^2 convergence in (2.15). It will be a consequence of the weak convergence (2.25) and of a scaling argument.

We choose a parameter $\delta>0$. We chop the box $[-1,1]^d$ into smaller boxes of side length of order δ: for $z\in\delta\mathbb{Z}^d$ s.t. $|z|\leq 1$, let B_z (resp. C_z) be the box of centre z and side length $M\delta$ (resp. side length δ). M is a constant whose value will be chosen later. For $\varepsilon>0$, we use the notation $B_z(\varepsilon)=((1/\varepsilon)B_z)\cap\mathbb{Z}^d$ and $C_z(\varepsilon)=((1/\varepsilon)C_z)\cap\mathbb{Z}^d$.

The following version of the Poincaré inequality was proved by Barlow (2004), see definition 1.7, theorem 2.18, lemma 2.13 and proposition 2.17 in that paper: there exist constants $M>1$ and β such that Q_0.a.s. for any $\delta>0$, for small enough ε, for any $z\in\delta\mathbb{Z}^d$ s.t. $|z|\leq 1$ and for any function $u:\mathbb{Z}^d\to\mathbb{R}$, one has

$$\frac{1}{\#C_z(\varepsilon)}\sum_{x,\,y\in\mathcal{C}(\omega)\cap C_z(\varepsilon)}(u(x)-u(y))^2\leq\beta\delta^2\varepsilon^{-2}\sum_{x\sim y\in\mathcal{C}(\omega)\cap B_z(\varepsilon)}\omega(x,\,y)(u(x)-u(y))^2.$$

We use this inequality for the function $\varepsilon\chi$ to get

$$\begin{aligned}&\frac{1}{\#C_z(\varepsilon)}\sum_{x,\,y\in\mathcal{C}(\omega)\cap C_z(\varepsilon)}|\varepsilon\chi(\omega,x)-\varepsilon\chi(\omega,\,y)|^2\\&\quad\leq\beta\delta^2\sum_{x\in\mathcal{C}(\omega)\cap B_z(\varepsilon)}\sum_{b\in\mathcal{B}}\sum_{e\in\mathcal{B}}\omega(x,x+e)(G_b(x.\omega,e))^2.\end{aligned}$$

This last inequality is equivalent to

$$\begin{aligned}&\frac{1}{\#C_z(\varepsilon)}\sum_{x,\,y\in\mathcal{C}(\omega)\cap C_z(\varepsilon)}n^{\omega}(x)n^{\omega}(y)|\varepsilon\chi(\omega,x)-\varepsilon\chi(\omega,\,y)|^2\\&\quad\leq\beta\delta^2\sum_{x\in\mathcal{C}(\omega)\cap B_z(\varepsilon)}\sum_{b\in\mathcal{B}}\sum_{e\in\mathcal{B}}\omega(x,x+e)(G_b(x.\omega,e))^2.\end{aligned}$$

Denoting with $a_\varepsilon(z)$ the mean value of $\varepsilon\chi(\omega,.)$ on the set $\mathcal{C}(\omega)\cap\mathcal{C}_z(\varepsilon)$ and with respect to the measure with weights $n^{\omega}(x)$, we get that, for all z,

$$\begin{aligned}&\sum_{x\in\mathcal{C}(\omega)\cap C_z(\varepsilon)}n^{\omega}(x)|\varepsilon\chi(\omega,x)-a_\varepsilon(z)|^2\\&\quad\leq\beta\delta^2\sum_{x\in\mathcal{C}(\omega)\cap B_z(\varepsilon)}\sum_{b\in\mathcal{B}}\sum_{e\in\mathcal{B}}\omega(x,x+e)(G_b(x.\omega,e))^2,\end{aligned}$$

and summing over all values of z,

$$\begin{aligned}&\sum_z\sum_{x\in\mathcal{C}(\omega)\cap C_z(\varepsilon)}n^{\omega}(x)|\varepsilon\chi(\omega,x)-a_\varepsilon(z)|^2\\&\quad\leq\beta\delta^2\sum_{x\in\mathcal{C}(\omega);|x|\leq 1/\varepsilon}\sum_{b\in\mathcal{B}}\sum_{e\in\mathcal{B}}\omega(x,x+e)(G_b(x.\omega,e))^2.\end{aligned}$$

(Remember that the value of β is allowed to change from line to line.) Multiplying by ε^d and applying the spatial ergodic theorem as before, we get

$$\limsup_{\varepsilon}\sum_z\varepsilon^d\sum_{x\in\mathcal{C}(\omega)\cap C_z(\varepsilon)}n^{\omega}(x)|\varepsilon\chi(\omega,x)-a_\varepsilon(z)|^2\leq\beta\delta^2\sum_{b\in\mathcal{B}}\int(G_b)^2\,\mathrm{d}M.$$

On the other hand, it follows from (2.25) that, for any z, $a_\varepsilon(z) - a_\varepsilon$ converges to 0. Therefore, we must also have

$$\limsup_{\varepsilon} \varepsilon^d \sum_z \sum_{x\in\mathcal{C}(\omega)\cap C_z(\varepsilon)} n^\omega(x)(\varepsilon\chi(\omega,x) - a_\varepsilon)^2 \le \beta\delta^2 \sum_{b\in\mathcal{B}} \int (G_b)^2 \,\mathrm{d}M,$$

and

$$\limsup_{\varepsilon} \varepsilon^d \sum_{x\in\mathcal{C}(\omega);\ |x|\le 1/\varepsilon} n^\omega(x)(\varepsilon\chi(\omega,x) - a_\varepsilon)^2 \le \beta\delta^2 \sum_{b\in\mathcal{B}} \int (G_b)^2 \,\mathrm{d}M,$$

and, since this holds for any $\delta > 0$, we deduce that

$$\varepsilon^d \sum_{x\in\mathcal{C}(\omega);\ |x|\le 1/\varepsilon} n^\omega(x)(\varepsilon\chi(\omega,x) - a_\varepsilon)^2 \to 0,$$

Q_0.a.s.

Since $n^\omega(x) \ge 1$ for $x\in\mathcal{C}(\omega)$, we conclude that (2.15) holds.

Conclusion of the proof of the theorem. As pointed out at the beginning of §2*c*, (2.15) implies that, for any $t>0$, the random variables $\varepsilon\chi(\omega, X(t/\varepsilon^2))$ converge to 0 in P_0^ω probability, Q.a.s. on the set $0\in\mathcal{C}(\omega)$. Therefore, the asymptotics of the finite-dimensional marginals of the two processes $(X^\varepsilon(t),\ t\in\mathbb{R}_+)$ and $(M^\varepsilon(t) = X^\varepsilon(t) + \varepsilon\chi(\omega, X(t/\varepsilon^2)),\ t\in\mathbb{R}_+)$ coincide, and, since we already proved that M^ε satisfies the invariance principle (see the conclusion of §2*b*), we deduce that the processes $(X^\varepsilon(t), t\in\mathbb{R}_+)$ converge in the sense of finite-dimensional distributions towards a Brownian motion with deterministic covariance matrix A. Combined with the tightness result of lemma 2.1, it gives the convergence of X^ε in the Skorohod topology.

That A is diagonal was proved by De Masi *et al.* (1989), theorem 4.7, 3. One can argue that A is positive as a consequence of the Gaussian lower bounds obtained by Barlow (2004), but the original proof was given by Grimmett & Marstrand (1990). ■

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