Propagation of a solution of a hyperbolic equation with rapidly oscillating coefficients

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We consider the following Cauchy problem in the semispace R^{n+1}_+:

$$(1)\quad \begin{cases} \left(\frac{\partial^2}{\partial t^2}-\frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)\frac{\partial}{\partial x_j}-b_i\left(x, \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon}\right)\frac{\partial}{\partial x_i}-c\left(x, \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon}\right)\right)u^\varepsilon(x, t)=0, \\ u^\varepsilon(x, t)|_{t=0}=\varphi(x), \qquad \frac{\partial}{\partial t}u^\varepsilon(x, t)|_{t=0}=\psi(x). \end{cases}$$

Here the coefficients $a_{ij}(y, \tau)$ are smooth and periodic in R^{n+1} and satisfy the condition of uniform ellipticity. We omit the summation sign for repeated indices. We also assume that the lowest coefficients are smooth and bounded on any compact set. Finally, suppose that the initial functions $\varphi(x)$ and $\psi(x)$ have compact support. In this note we consider the behaviour of the support of a solution of (1) for small ε.

We denote by $Q^\varepsilon_{x_0}$ the domain of dependence [1] of (1) with vertex at the point $(x_0, 0)$, and by $\mathscr{P}^\varepsilon_{x_0}(\tau)$ the section of this set by the plane $\{t = \tau\}$.

Theorem 1. *There is a convex cone Q_{x_0} with rectilinear generators, whose form does not depend on x_0, such that for all $T > 0$*

$$\lim_{\varepsilon\to 0}\ \sup_{(x, t)\in \partial Q^\varepsilon_{x_0},\ t<T} \rho((x, t), \partial Q_{x_0})=0.$$

The convergence is uniform in x_0.

Remark. In view of the very complicated behaviour of the bicharacteristics of (1) for small ε (see [2]), we have not succeeded in obtaining any explicit expression for the limiting domain of dependence Q_{x_0}.

The situation is much better when the coefficients a_{ij} depend only on a single argument, for example y_1. Let us consider this case in more detail. We denote by $M\{f\}$ the mean of the periodic function f over the period. We define the matrix:

$$b_{ij}(y_1)=a_{ij}(y_1)-\frac{a_{1i}(y_1)\, a_{j1}(y_1)}{a_{11}(y_1)}\Bigg|^{n}_{ij=2}.$$

It can be proved that when a_{ij} is positive definite, so is b_{ij}. Let $H^\varepsilon(x, p, t, E)$ be the Hamiltonian of (1). We define the average Hamiltonian as follows:

$$(2)\qquad \overline{H}(x, p, t, E)=p_1-M\left\{\frac{a_{1l}(y)}{a_{11}(y)}\, p_l \pm \frac{\sqrt{E^2-b_{ij}(y)\, p_i p_j}}{\sqrt{a_{11}(y)}}\right\}.$$

In the space $(p_2, \ldots, p_n) \in R^{n-1}$ we construct a set $\mathscr{L}$ on which the average Hamiltonian is defined and takes real values:

$$\mathscr{L}=\{p'=(p_2, \ldots, p_n)\,|\, b_{ij}(y)\, p_i p_j<1\}.$$

The projections of the bicharacteristics of (2) on the (x, t)-space are rays. We draw rays from $(x_0, 0)$ corresponding to all possible p' of $\mathscr{L}$.

Lemma. *The set of rays thus constructed forms the boundary of a cone* Q'_{x_0}. *Its section by the plane* $\{t = 1\}$ *is given parametrically as follows:*

$$
(3) \qquad \begin{cases}
x_1 - x_{01} = \pm \left(M \left\{ \dfrac{1}{\sqrt{a_{11}(y)\,(1 - b_{ij}(y)\,p_i p_j)}} \right\} \right)^{-1} = \theta(p'), \\
x_2 - x_{02} = \left(M \left\{ \mp \dfrac{a_{12}(y)}{a_{11}(y)} + \dfrac{b_{2i}(y)\,p_i}{\sqrt{a_{11}(y)\,(1 - b_{ij}(y)\,p_i p_j)}} \right\} \right) \cdot \theta(p'), \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
x_n - x_{0n} = \left(M \left\{ \mp \dfrac{a_{1n}(y)}{a_{11}(y)} + \dfrac{b_{ni}(y)\,p_i}{\sqrt{a_{11}(y)\,(1 - b_{ij}(y)\,p_i p_j)}} \right\} \right) \cdot \theta(p').
\end{cases}
$$

The parameters $(p_2, \ldots, p_n)$ *range over the set* $\mathcal{L}$.

Theorem 2. *The set* Q'_{x_0} *coincides with the limiting domain of dependence* Q_{x_0}, *and the following bound holds uniformly in* x_0 *and* t *from* $(0, \infty)$:

$$\sup_{(x,\,t) \in \partial Q^{\varepsilon}_{x_0}} \rho((x, t), \partial Q_{x_0}) < c\varepsilon.$$

If in (1) there are no lowest coefficients and the initial functions φ and ψ are sufficiently smooth, then by [3], the solutions of (1) as $\varepsilon \to 0$ converge to a function that is the solution of an equation with constant coefficients. For such an equation the domain of dependence is a cone in whose section by the plane $\{t = 1\}$ there lies the ellipsoid

$$q^{ij}(x_i - x_{0i})(x_j - x_{0j}) \leqslant 1,$$

where q^{ij} is the inverse to the matrix of the coefficients q_{ij}. It is interesting to compare the limiting domain of dependence and the domain of dependence $Q^0_{x_0}$ of the average equation.

Exmaple. In R^3_+ we consider the equation

$$\frac{\partial^2}{\partial t^2} u^{\varepsilon}(x, t) - \frac{\partial}{\partial x_1} a_{11}\left(\frac{x_1}{\varepsilon}\right) \frac{\partial}{\partial x_1} u^{\varepsilon}(x, t) - \frac{\partial}{\partial x_2} a_{22}\left(\frac{x_1}{\varepsilon}\right) \frac{\partial}{\partial x_2} u^{\varepsilon}(x, t) = 0,$$

$$u^{\varepsilon}(x, t)|_{t=0} = \varphi(x), \qquad \frac{\partial}{\partial t} u^{\varepsilon}(x, t)|_{t=0} = \psi(x).$$

The section of the domain of dependence of the average equation by the plane $\{t = 1\}$ has the form

$$\mathcal{P}^0_{x_0}(1) = \{q^{11}(x_1 - x_{01})^2 + q^{22}(x_2 - x_{02})^2 \leqslant 1\},$$

$$q_{11} = M\left\{\frac{1}{a_{11}(y)}\right\}, \qquad q_{22} = (M\{a_{22}(y)\})^{-1}.$$

Comparing these formulae with (3) we can verify that the distance between the boundaries of the sections $\mathcal{P}_{x_0}(1)$ and $\mathcal{P}^0_{x_0}(1)$ are greater than zero if and only if both the coefficients $a_{11}(y_1)$ and $a_{22}(y_1)$ are not constant functions.

References

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