Nonlinearity **19** (2006) 2973–2988 doi:10.1088/0951-7715/19/12/014

Homogenization of quasilinear elliptic equations with non-uniformly bounded coefficients

Brahim Amaziane[1], **Leonid Pankratov**[1,2] **and Andrey Piatnitski**[3]

[1] Laboratoire de Mathématiques Appliquées, CNRS-UMR5142, Université de Pau, Av. de l'Université, 64000 Pau, France
[2] Department of Mathematics, B. Verkin Institut for Low Temperature Physics and Engineering, 47, av. Lenin, 61103, Kharkov, Ukraine
[3] Narvik University College, P O Box 385, Narvik, 8505, Norway and Lebedev Physical Institute RAS, Leninski prospect 53, Moscow, 119991, Russia

E-mail: brahim.amaziane@univ-pau.fr, leonid.pankratov@univ-pau.fr and andrey@sci.lebedev.ru

Received 30 March 2006, in final form 18 October 2006
Published 13 November 2006
Online at stacks.iop.org/Non/19/2973

Recommended by C Le Bris

Abstract

The goal of the paper is to study the asymptotic behaviour of solutions to a high contrast quasilinear equation of the form

$$-\mathrm{div}\,(|\nabla u^\varepsilon|^{p-2}\nabla u^\varepsilon) + G^\varepsilon(x)|u^\varepsilon|^{p-2}u^\varepsilon = f(x) \qquad \text{in} \quad \Omega,$$

where $\Omega \subset \mathbb{R}^n$ with $n \geqslant 2$, $1 < p \leqslant n$, and the coefficient $G^\varepsilon(x)$ is assumed to blow up as $\varepsilon \to 0$ on a set of N_ε isolated inclusions of asymptotically small measure. Here $N_\varepsilon \longrightarrow +\infty$ as $\varepsilon \to 0$. It is shown that the asymptotic behaviour, as $\varepsilon \to 0$, of the solution u^ε is described in terms of a homogenized quasilinear equation of the form

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) + B(x)|u|^{p-2}u = f(x) \qquad \text{in} \quad \Omega,$$

where the coefficient $B(x)$ is calculated as a local energy characteristic of the microstructure associated with the potential $G^\varepsilon(x)$ in the original problem. This result is then illustrated with a periodic example and a nonperiodic one.

Mathematics Subject Classification: 35B40, 35J60, 74Q05, 76M50

1. Introduction

In this paper we study the asymptotic behaviour of solutions to a high contrast quasilinear equation of the form

$$-\,\mathrm{div}\,(|\nabla u^\varepsilon|^{p-2}\nabla u^\varepsilon) + G^\varepsilon(x)|u^\varepsilon|^{p-2}u^\varepsilon = f(x) \tag{1.1}$$

with a small positive parameter ε. This equation is defined in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ ($n \geqslant 2$, $1 < p \leqslant n$), the homogeneous Neumann condition being imposed at the boundary $\partial\Omega$. We assume that the coefficient $G^\varepsilon(x)$ tends to infinity as $\varepsilon \to 0$ on a set consisting of N_ε isolated inclusions of asymptotically small measure. Here $N_\varepsilon \longrightarrow +\infty$ as $\varepsilon \to 0$.

Equation (1.1) with appropriate boundary conditions can describe, for example, the combustion in a medium with a gradient nonlinearity (see, e.g. [15]) and also non-Newtonian flows in porous media (see, e.g. [7]). Let us also mention that the studied homogenization problem is closely related to that for nonlinear Dirichlet problems (see, e.g. [2,5,9,11,16] and references therein).

A number of homogenization problems for equations with non-uniformly bounded coefficients have already been studied in the existing literature. We will not attempt a literature review here, but merely mention a few references, for instance [4,6,8,11,13] which deal with the case of differential operators whose coefficient tends to infinity on a set of asymptotically small measure. Linear equations of the form (1.1) have been considered in [11].

In the present paper we deal with a quasilinear elliptic problem in a domain with non-uniformly bounded coefficients. Following the approach introduced in [11], instead of a classical periodicity assumption, we impose certain conditions on the so-called local energetic characteristics associated with the boundary value problem (1.1). It will be shown that the asymptotic behaviour, as $\varepsilon \to 0$, of the solution u^ε is described by a homogenized quasilinear equation of the form

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) + B(x)|u|^{p-2}u = f(x) \qquad \text{in} \quad \Omega,$$

where the coefficient $B(x)$ is calculated as a local energy characteristic associated with the potential $G^\varepsilon(x)$. The proof of the main result is based on the variational homogenization techniques which are widely used nowadays in homogenization theory (see, e.g. [3,11,14,17] and references therein). Let us also mention that another nonperiodic homogenization approach was proposed recently in [12] for nonlinear monotone operators.

The paper is organized as follows. In section 2 we state the problem and formulate the main result. This result is proved in section 4; it relies on auxiliary results from section 3. Two examples of periodic and nonperiodic structures are considered in section 5.

Finally, we note that throughout the paper, C (sometimes subscripted) will denote a generic positive constant, independent of ε and may take different values for different occurences.

2. Statement of the problem and the main result

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geqslant 2$) with sufficiently smooth boundary. Let $\mathcal{F}^\varepsilon$ be an open subset in Ω ($\mathcal{F}^\varepsilon \subset \Omega$) consisting of small disjoint components $\mathcal{F}_i^\varepsilon$, i.e.

$$\mathcal{F}^\varepsilon = \bigcup_{i=1}^{N_\varepsilon} \mathcal{F}_i^\varepsilon,$$

where $N_\varepsilon \to +\infty$ as $\varepsilon \to 0$. We assume that the set $\mathcal{F}^\varepsilon$ is asymptotically distributed in a regular way in Ω, i.e. for any ball $V(y,r) = \{x \in \Omega \mid |x-y| < r\}$ of radius r centred at $y \in \Omega$ and $\varepsilon > 0$ small enough ($\varepsilon \leqslant \varepsilon_0(r)$), $V(y,r) \cap \mathcal{F}^\varepsilon \neq \emptyset$. It has a sufficiently smooth boundary $\partial\mathcal{F}^\varepsilon$ and

$$\mathrm{meas}\,\mathcal{F}^\varepsilon \longrightarrow 0 \qquad \text{as} \quad \varepsilon \to 0. \tag{2.1}$$

We set

$$\Omega^\varepsilon = \Omega \setminus \overline{\mathcal{F}^\varepsilon} \tag{2.2}$$

and consider the following variational problem:

$$\int_\Omega \left\{|\nabla u^\varepsilon|^p + G^\varepsilon(x)|u^\varepsilon|^p - p\, f(x)\, u^\varepsilon\right\} \mathrm{d}x \longrightarrow \inf, \qquad u^\varepsilon \in W^{1,p}(\Omega), \tag{2.3}$$

where $1 < p \leqslant n$, $f \in L^{p'}(\Omega)$ $(1/p + 1/p' = 1)$ and the function G^ε is given by

$$G^\varepsilon(x) = \begin{cases} g_i(\varepsilon) & \text{in} \quad \mathcal{F}_i^\varepsilon \ (i = 1, 2, ..., N_\varepsilon). \\ g_0(x) & \text{in} \quad \Omega^\varepsilon. \end{cases} \tag{2.4}$$

Here g_0 is a smooth strictly positive function in Ω and

$$\min_{i=1,..,N_\varepsilon} g_i(\varepsilon) \longrightarrow +\infty \qquad \text{as} \quad \varepsilon \to 0. \tag{2.5}$$

It is known (see, e.g. [10]) that, for any $\varepsilon > 0$, there exists a unique solution $u^\varepsilon \in W^{1,p}(\Omega)$ of the variational problem (2.3) and that u^ε solves the Neumann boundary value problem for the corresponding Euler equation:

$$-\mathrm{div}\,(|\nabla u^\varepsilon|^{p-2}\nabla u^\varepsilon) + G^\varepsilon(x)|u^\varepsilon|^{p-2}u^\varepsilon = f(x).$$

We study the asymptotic behaviour of u^ε as $\varepsilon \to 0$. The classical periodicity assumption is here replaced by an abstract one covering a variety of concrete behaviours such as the periodicity, the almost periodicity, and many more besides.

Let K_h^z be an open cube centred at $z \in \Omega$ with length equal to h $(0 < \varepsilon \ll h < 1)$. We set

$$\boldsymbol{I}_z^{\varepsilon,h}[\phi] = \int_{K_h^z} \left\{|\nabla\phi|^p + g_F^\varepsilon(x)|\phi|^p + h^{-p-\gamma}|\phi - 1|^p\right\} \mathrm{d}x, \tag{2.6}$$

where $0 < \gamma < p$;

$$g_F^\varepsilon(x) = \sum_{i=1}^{N_\varepsilon} g_i(\varepsilon)\mathbf{1}_i^\varepsilon(x) \tag{2.7}$$

with $\mathbf{1}_i^\varepsilon(x)$ being the characteristic functions of the sets $\mathcal{F}_i^\varepsilon$ $(i = 1, 2, ..., N_\varepsilon)$.

We introduce the local energy characteristics of the domain Ω associated with the variational problem (2.3). For $z \in \Omega$ we define

- the functional associated to the energy in Ω^ε:

$$b^{\varepsilon,h}(z) = \inf_{w^\varepsilon} \boldsymbol{I}_z^{\varepsilon,h}[w^\varepsilon], \tag{2.8}$$

  where the infimum is taken over $w^\varepsilon \in W^{1,p}(K_h^z)$;
- the functional associated to the p–capacity of the sets $\mathcal{F}^\varepsilon$ (for more details see, e.g. [2]):

$$a^{\varepsilon,h}(z) = \inf_{v^\varepsilon} \boldsymbol{I}_z^{\varepsilon,h}[v^\varepsilon], \tag{2.9}$$

  where the infimum is taken over $v^\varepsilon \in W(K_h^z, \mathcal{F}^\varepsilon)$, where

$$W(K_h^z, \mathcal{F}^\varepsilon) = \left\{v^\varepsilon \in W^{1,p}(K_h^z) \mid v^\varepsilon = 0 \text{ in } \mathcal{F}^\varepsilon\right\}. \tag{2.10}$$

Instead of the classical periodicity assumption on the microstructure of the disperse media, we impose the following conditions on the local energy characteristics of the domain Ω. Namely, we assume that

(C.1) there exists a constant C independent of the parameter γ such that

$$\overline{\lim_{h\to 0}}\ \overline{\lim_{\varepsilon\to 0}}\ h^{-n}a^{\varepsilon,h}(x) \leqslant C;$$

(C.2) for any $x \in \Omega$ there exist the limits

$$\lim_{h\to 0} \varliminf_{\varepsilon\to 0} h^{-n} b^{\varepsilon,h}(x) = \lim_{h\to 0} \varlimsup_{\varepsilon\to 0} h^{-n} b^{\varepsilon,h}(x) = b(x),$$

where $b \in C(\Omega)$.

Now we are in position to formulate the main result of the paper.

Theorem 2.1. *Let conditions* (C.1) and (C.2) *be satisfied. Then the solution* u^ε *of the variational problem (2.3) converges strongly in* $L^p(\Omega)$ *to a solution of the variational problem:*

$$J_{\text{hom}}[u] \equiv \int_\Omega \{|\nabla u|^p + B(x)|u|^p - pf(x)u\}\,\mathrm{d}x \longrightarrow \inf, \qquad u \in W^{1,p}(\Omega), \tag{2.11}$$

where

$$B(x) = g_0(x) + b(x). \tag{2.12}$$

Notice that the minimizer u of the variational problem (2.11) solves the Neumann boundary value problem for the quasilinear elliptic equation

$$-\text{div}\,(|\nabla u|^{p-2}\nabla u) + B(x)|u|^{p-2}u = f(x).$$

The proof of theorem 2.1 is given in section 4. The result is obtained by using the homogenization approach developed in [11] following the scheme developed in [1]. It also makes use of several auxiliary results given in section 3.

Remark 1. Let us notice that condition (C.1) means that the diameters of the inclusions $\mathcal{F}_i^\varepsilon$ are much smaller than the minimal distance between them (see, e.g. the relations (5.1) and (5.48) for the radii of the inclusions in the periodic and locally periodic examples in section 5). The functional $b^{\varepsilon,h}(z)$ given by (2.8) is a mesoscopic characteristic of the absorption of the nonhomogeneous medium Ω. Condition (C.2) means that this absorption is finite and its density is given by the function $b(x)$.

Condition (C.1) is used to prove an auxiliary result given in lemma 3.2 which will be then applied in the proof of theorem 2.1 (see section 4.2). Condition (C.2) is used to construct a convenient approximation for the solution of the variational problem (2.3) in the domain Ω (see lemma 3.1). We make use of this approximation in section 4.1. It is also used in the proof of the inequality (4.19) in section 4.2. The function $b(x)$ appears then in the homogenized functional (2.11) of theorem 2.1.

3. Auxiliary results

In this section we construct a convenient approximation for the solution of the variational problem (2.3) in the domain Ω. To this end we introduce first the following notation.

Let $\{x^\alpha\}$ be a periodic grid in Ω with a period $h' = h - h^{1+\gamma/p}$ ($\varepsilon \ll h \ll 1$ and $0 < \gamma < p$). Let us cover the domain Ω by the cubes K_h^α of length $h > 0$ centred at the points x^α. We associate with this covering a partition of unity $\{\varphi_\alpha\} : 0 \leqslant \varphi_\alpha(x) \leqslant 1$; $\varphi_\alpha(x) = 0$ for $x \notin K_h^\alpha$; $\varphi_\alpha(x) = 1$ for $x \in K_h^\alpha \setminus \cup_{\beta\neq\alpha} K_h^\beta$; $\sum_\alpha \varphi_\alpha(x) = 1$ for $x \in \Omega$; $|\nabla\varphi_\alpha(x)| \leqslant Ch^{-1-\gamma/p}$.

Denote by $K_{h'}^\alpha$ and Π_h^α the cube of length h' centred at the point x^α and the set $K_h^\alpha \setminus K_{h'}^\alpha$, respectively.

Lemma 3.1. *Assume that conditions* (C.1) and (C.2) *of theorem 2.1 are satisfied. Then for each* $h > 0$ *there is* $\varepsilon_0 = \varepsilon_0(h)$ *such that for any* $\varepsilon \leqslant \varepsilon_0$ *there exist sets* $\mathcal{B}_h^\varepsilon$ *(*$\mathcal{F}^\varepsilon \subset \mathcal{B}_h^\varepsilon \subset \Omega$*) and functions* Y_h^ε *satisfying the following conditions:*

(i) $0 \leqslant Y_h^\varepsilon(x) \leqslant 1$ *in* Ω*;*

(ii) $Y_h^\varepsilon(x) = 1$ *in* $\Omega \setminus \mathcal{B}_h^\varepsilon$;
(iii) $\overline{\lim}_{\varepsilon\to 0}$ meas $\mathcal{B}_h^\varepsilon = O(h^\gamma)$ *as* $h \to 0$;
(iv) *for any function* $w \in C^1(\Omega)$, *we have*

$$\overline{\lim_{\varepsilon\to 0}} \int_\Omega \left\{|\nabla Y_h^\varepsilon|^p + G^\varepsilon(x)|Y_h^\varepsilon|^p\right\}|w|^p\,\mathrm{d}x \leqslant \int_\Omega B(x)|w|^p\,\mathrm{d}x + o(1) \quad (3.1)$$

as $h \to 0$.

Proof of lemma 3.1.. Let $w_h^{\varepsilon,\alpha}$ be a minimizer of the functional in (2.8) with $z = x^\alpha$. It follows from conditions (C.1) and (C.2) that, as $h \to 0$,

$$\overline{\lim_{\varepsilon\to 0}} \int_{K_h^\alpha} \left\{\left|\nabla w_h^{\varepsilon,\alpha}\right|^p + g_F^\varepsilon(x)|w_h^{\varepsilon,\alpha}|^p\right\}\,\mathrm{d}x = O(h^n), \quad (3.2)$$

$$\overline{\lim_{\varepsilon\to 0}} \int_{\Pi_h^\alpha} \left\{\left|\nabla w_h^{\varepsilon,\alpha}\right|^p + g_F^\varepsilon(x)|w_h^{\varepsilon,\alpha}|^p\right\}\,\mathrm{d}x = o(h^n), \quad (3.3)$$

$$\overline{\lim_{\varepsilon\to 0}} \int_{K_h^\alpha} |w_h^{\varepsilon,\alpha} - 1|^p\,\mathrm{d}x = O(h^{n+p+\gamma}), \quad (3.4)$$

$$\overline{\lim_{\varepsilon\to 0}} \int_{\Pi_h^\alpha} |w_h^{\varepsilon,\alpha} - 1|^p\,\mathrm{d}x = o(h^{n+p+\gamma}). \quad (3.5)$$

In addition, using condition (C.2), we obtain

$$\overline{\lim_{\varepsilon\to 0}} \int_{K_h^\alpha} \left\{\left|\nabla w_h^{\varepsilon,\alpha}\right|^p + G^\varepsilon(x)\,|w_h^{\varepsilon,\alpha}|^p\right\}\,\mathrm{d}x \leqslant h^n B(x^\alpha) + o(h^n) \qquad \text{as} \quad h \to 0. \quad (3.6)$$

Furthermore, since $w_h^{\varepsilon,\alpha}$ minimizes the functional in (2.8), we have $0 \leqslant w_h^{\varepsilon,\alpha}(x) \leqslant 1$ and

$$\overline{\lim_{\varepsilon\to 0}}\ \text{meas}\, B_h^{\varepsilon,\alpha} \leqslant C\,h^{n+\gamma}, \quad (3.7)$$

where $B_h^{\varepsilon,\alpha} = \{x \in K_h^\alpha \cap \Omega^\varepsilon : w_h^{\varepsilon,\alpha} \leqslant 1 - h\}$. Let us introduce the function

$$W_h^{\varepsilon,\alpha} = \begin{cases} 1, & \text{if} \quad w_h^{\varepsilon,\alpha} \geqslant 1 - h, \\ (1-h)^{-1} w_h^{\varepsilon,\alpha}, & \text{otherwise.} \end{cases} \quad (3.8)$$

It is clear that $\left|W_h^{\varepsilon,\alpha} - 1\right| \leqslant \left|w_h^{\varepsilon,\alpha} - 1\right|$. One can easily show that the function $W_h^{\varepsilon,\alpha}$ satisfies the estimates (3.3)–(3.6). We set

$$\mathcal{B}_h^\varepsilon = \bigcup_\alpha B_h^{\varepsilon,\alpha}, \qquad Y_h^\varepsilon(x) = \sum_\alpha W_h^{\varepsilon,\alpha}(x)\varphi_\alpha(x).$$

Then, using the properties of the functions $W_h^{\varepsilon,\alpha}$ and $\{\varphi_\alpha\}$ and taking into account the estimate (3.7), it is easy to show that the functions $Y_h^\varepsilon(x)$ and the sets $\mathcal{B}_h^\varepsilon$ satisfy conditions (i)–(iv) of lemma 3.1.

Lemma 3.1 is proved. □

In what follows we make use of the following notation. We denote by $W(\Omega, \mathcal{F}^\varepsilon)$ the class of functions from the space $W^{1,p}(\Omega)$ such that these functions equal zero on the set $\mathcal{F}^\varepsilon$.

Lemma 3.2. *Let* ω *be an arbitrary function from the space* $W^{1,p}(\Omega)$ *and let conditions* (C.1) and (C.2) *of theorem 2.1 be satisfied. Then there exists a sequence* $\{w^\varepsilon\} \subset W(\Omega, \mathcal{F}^\varepsilon)$ *which converges weakly in the space* $W^{1,p}(\Omega)$ *to* ω *and such that, for* ε *sufficiently small* ($\varepsilon \leqslant \varepsilon_0(\omega)$),

$$\|w^\varepsilon\|_{W^{1,p}(\Omega)} \leqslant C\|\omega\|_{W^{1,p}(\Omega)}. \quad (3.9)$$

Proof of lemma 3.2.. Since the class $C^2(\overline{\Omega})$ is dense in the space $W^{1,p}(\Omega)$, it is sufficient to prove the assertions of lemma 3.2 for an arbitrary $\omega \in C^2(\overline{\Omega})$ only.

Let us introduce the function W_h^ε defined by

$$W_h^\varepsilon(x) = \sum_\alpha v_h^{\varepsilon,\alpha}(x)\omega(x)\varphi_\alpha(x), \tag{3.10}$$

where $v_h^{\varepsilon,\alpha}(x)$ is the minimizer of the functional (2.9) with $z = x^\alpha$. It follows from condition (C.1) that the function $v_h^{\varepsilon,\alpha}$ satisfies the inequalities (3.2)–(3.5) from lemma 3.1. Then using these inequalities we show that, for h and ε sufficiently small ($h \leqslant h_0(\omega)$, $\varepsilon \leqslant \varepsilon_0(\omega)$),

$$\|W_h^\varepsilon\|_{W^{1,p}(\Omega)} \leqslant C\|\omega\|_{W^{1,p}(\Omega)}.$$

Let $\hat\varepsilon(h)$ be a decreasing function such that $\lim_{h\to 0}\hat\varepsilon(h) = 0$. We set

$$h(\varepsilon) = \frac{1}{j} \qquad \text{for} \quad \hat\varepsilon\left(\frac{1}{j+1}\right) \leqslant \varepsilon \leqslant \hat\varepsilon\left(\frac{1}{j}\right), \quad j = 1, 2, \ldots$$

and

$$w^\varepsilon = W_h^\varepsilon\Big|_{h=h(\varepsilon)}.$$

It is clear that the function w^ε satisfies the inequality (3.9).

It remains to show that the sequence $\{w^\varepsilon\}$ converges weakly in $W^{1,p}(\Omega)$ to the function ω. According to (3.9), this sequence is a weakly compact set in the space $W^{1,p}(\Omega)$. Then it is sufficient to prove that it converges weakly in $L^p(\Omega)$ to the function ω. Let ϕ be an arbitrary function from $L^{p'}(\Omega)$. We have

$$\int_\Omega w^\varepsilon\phi\,\mathrm{d}x = \int_\Omega \omega\phi\,\mathrm{d}x + \sum_\alpha \int_\Omega \omega(v_{h(\varepsilon)}^{\varepsilon,\alpha} - 1)\varphi_\alpha\phi\,\mathrm{d}x. \tag{3.11}$$

It follows from lemma 3.1 that the second integral in (3.11) vanishes as $\varepsilon \to 0$. Therefore, the sequence $\{w^\varepsilon\}$ converges weakly in $L^p(\Omega)$ to the function ω.

Lemma 3.2 is proved. □

4. Proof of theorem 2.1

We begin this section by obtaining *a priori* estimates for the minimizer of problem (2.3):

$$J^\varepsilon[u^\varepsilon] \longrightarrow \inf, \qquad u^\varepsilon \in W^{1,p}(\Omega), \tag{4.1}$$

where

$$J^\varepsilon[u^\varepsilon] = \int_\Omega \left\{|\nabla u^\varepsilon|^p + G^\varepsilon(x)|u^\varepsilon|^p - p\,f(x)\,u^\varepsilon\right\}\mathrm{d}x. \tag{4.2}$$

Since $J^\varepsilon[u^\varepsilon] \leqslant J^\varepsilon[0] = 0$, by the Young inequality we get

$$\int_\Omega \left\{|\nabla u^\varepsilon|^p + G^\varepsilon(x)|u^\varepsilon|^p\right\}\mathrm{d}x \leqslant C\|f\|_{L^{p'}(\Omega)}\|u^\varepsilon\|_{L^p(\Omega)}. \tag{4.3}$$

It is clear that $G^\varepsilon(x) > G_0 > 0$. Therefore, the inequality (4.3) implies the estimate

$$\|u^\varepsilon\|_{W^{1,p}(\Omega)} \leqslant C. \tag{4.4}$$

Hence, $\{u^\varepsilon\}$ is a weakly compact set in the space $W^{1,p}(\Omega)$ and one can extract a subsequence (still denoted by $\{u^\varepsilon\}$) weakly converging to a function $u \in W^{1,p}(\Omega)$.

We will show that u is a solution of the variational problem (2.11). The proof will be done in two steps.

4.1. Step 1. Upper bound

Let $w = w(x)$ be an arbitrary smooth function in Ω and let Y_h^ε be the function and $\mathcal{B}_h^\varepsilon$ the set defined in lemma 3.1. We set

$$\vartheta_h^\varepsilon(x) = Y_h^\varepsilon(x) w(x).$$

It is clear that $\vartheta_{Mh}^\varepsilon \in W^{1,p}(\Omega)$ and since u^ε minimize the functional J^ε, then

$$J^\varepsilon[u^\varepsilon] \leqslant J^\varepsilon[\vartheta_h^\varepsilon]. \tag{4.5}$$

Let us estimate the right-hand side of the inequality (4.5). We have

$$\begin{aligned} J^\varepsilon[\vartheta_h^\varepsilon] &= \int_\Omega \left\{ |\nabla(w\, Y_h^\varepsilon)|^p + G^\varepsilon(x) |w\, Y_h^\varepsilon|^p - p\, f(x)\, w\, Y_h^\varepsilon \right\} \mathrm{d}x \\ &= \int_{\Omega\setminus\mathcal{B}_h^\varepsilon} |\nabla w|^p \,\mathrm{d}x + \int_\Omega \left\{ |\nabla Y_h^\varepsilon|^p + G^\varepsilon(x)|Y_h^\varepsilon|^p \right\} |w|^p \,\mathrm{d}x - \int_\Omega p\, f(x)\, w \,\mathrm{d}x \\ &\quad + \int_{\mathcal{B}_h^\varepsilon} \left\{ |\nabla w Y_h^\varepsilon + w \nabla Y_h^\varepsilon|^p - |w \nabla Y_h^\varepsilon|^p \right\} \mathrm{d}x - \int_\Omega p f w (Y_h^\varepsilon - 1) \,\mathrm{d}x. \end{aligned} \tag{4.6}$$

Now it follows from (4.6), and lemma 3.1 that

$$\lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}}\, J^\varepsilon[\vartheta_h^\varepsilon] \leqslant J_{\mathrm{hom}}[w], \tag{4.7}$$

where

$$J_{\mathrm{hom}}[w] = \int_\Omega \{ |\nabla w|^p + B(x)|w|^p - p f(x) w \} \,\mathrm{d}x, \quad B(x) = (g_0 + b)(x). \tag{4.8}$$

Finally, the inequalities (4.5) and (4.7) imply that

$$\overline{\lim_{\varepsilon\to 0}}\, J^\varepsilon[u^\varepsilon] \leqslant J_{\mathrm{hom}}[w]. \tag{4.9}$$

By density arguments, (4.9) holds for any function $w \in W^{1,p}(\Omega)$ as well.

4.2. Step 2. Lower bound

Let $\{u^\varepsilon\}$ be a sequence of solutions of the variational problem (4.1) which converges weakly in $W^{1,p}(\Omega)$ to a function $u \in W^{1,p}(\Omega)$. Let us show that

$$\underline{\lim_{\varepsilon\to 0}}\, J^\varepsilon[u^\varepsilon] \geqslant J_{\mathrm{hom}}[u]. \tag{4.10}$$

Let us approximate u by smooth functions, $u_\delta(x)$ $(\delta > 0)$, in Ω

$$\|u_\delta - u\|_{W^{1,p}(\Omega)} \leqslant \delta \tag{4.11}$$

and set $w_\delta = u_\delta - u$. Then according to lemma 3.2 there exists a sequence $\{w_\delta^\varepsilon \in W(\Omega, \mathcal{F}^\varepsilon)\}$ that converges weakly in $W^{1,p}(\Omega)$ to the function w_δ. We set

$$u_\delta^\varepsilon(x) = w_\delta^\varepsilon(x) + u^\varepsilon(x). \tag{4.12}$$

The function u_δ^ε equals u^ε on the set $\mathcal{F}^\varepsilon$, converges weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ to u_δ. Moreover, it satisfies the inequality

$$\|u_\delta^\varepsilon - u^\varepsilon\|_{W^{1,p}(\Omega)} \leqslant C \|u_\delta - u\|_{W^{1,p}(\Omega)}. \tag{4.13}$$

Then there exists a sequence $\{r^\varepsilon > 0\}$, $r^\varepsilon \to 0$, and sets $\mathcal{Q}^\varepsilon$ such that

$$\lim_{\varepsilon\to 0} \quad \text{meas} \quad \mathcal{Q}^\varepsilon = 0 \qquad \text{and} \quad |u_\delta^\varepsilon(x) - u_\delta(x)| \leqslant r^\varepsilon \qquad \text{in} \quad \Omega \setminus \mathcal{Q}^\varepsilon.$$

Let us define the functions

$$v_\delta^\varepsilon(x) = \begin{cases} u_\delta(x) + r^\varepsilon, & \text{if } u_\delta^\varepsilon(x) > u_\delta(x) + r^\varepsilon, \\ u_\delta^\varepsilon(x), & \text{if } |u_\delta^\varepsilon(x) - u_\delta(x)| \leqslant r^\varepsilon, \\ u_\delta(x) - r^\varepsilon, & \text{if } u_\delta^\varepsilon(x) < u_\delta(x) - r^\varepsilon. \end{cases} \tag{4.14}$$

Then the function v_δ^ε converges uniformly in Ω to u_δ as $\varepsilon \to 0$. We set

$$V_\delta^\varepsilon(x) = u_\delta^\varepsilon(x) - v_\delta^\varepsilon(x)$$

and

$$I^\varepsilon[u_\delta^\varepsilon] = \int_\Omega \left\{ |\nabla u_\delta^\varepsilon|^p + G^\varepsilon(x)|u_\delta^\varepsilon|^p \right\} \mathrm{d}x. \tag{4.15}$$

Notice that $u_\delta^\varepsilon(x) = v_\delta^\varepsilon(x)$ in $\Omega \setminus \mathcal{Q}^\varepsilon$.

Let us now represent the integral $I^\varepsilon[u_\delta^\varepsilon]$ as follows:

$$I^\varepsilon[u_\delta^\varepsilon] = \left(\int_\Omega |\nabla V_\delta^\varepsilon|^p \,\mathrm{d}x + \int_\Omega g_F^\varepsilon(x)|u_\delta^\varepsilon|^p \,\mathrm{d}x \right) + \left(\int_{\Omega\setminus\mathcal{Q}^\varepsilon} |\nabla u_\delta^\varepsilon|^p \,\mathrm{d}x + \int_{\Omega\setminus\mathcal{F}^\varepsilon} g_0(x)|u_\delta^\varepsilon|^p \,\mathrm{d}x \right)$$
$$+ \left(\int_{\mathcal{Q}^\varepsilon} |\nabla u_\delta^\varepsilon|^p \,\mathrm{d}x - \int_{\mathcal{Q}^\varepsilon} |\nabla V_\delta^\varepsilon|^p \,\mathrm{d}x \right) \equiv \theta_1^\varepsilon + \theta_2^\varepsilon + \theta_3^\varepsilon. \tag{4.16}$$

Consider the first term on the right-hand side of (4.16). First define $\Omega_\zeta \subset \Omega$:

$$\Omega_\zeta = \{x \in \Omega : |u_\delta(x)| > 2\zeta\} \qquad (\zeta \gg h > 0)$$

and cover Ω_ζ by cubes K_h^α of length h centred at x^α with nonintersecting interiors. For ε and h suffciently small, we have $|v_\delta^\varepsilon| > \zeta$ in K_h^α. As in [1], one can show that for $x \in \Omega \cap K_h^\alpha$ the following inequality holds true:

$$\left(1 + A_1 h^{\frac{p}{p-1}}\right) |\nabla V_\delta^\varepsilon|^p$$
$$\geqslant |v_\delta^\varepsilon|^p \left| \nabla \left(\frac{u_\delta^\varepsilon}{v_\delta^\varepsilon} \right) \right|^p - A_2 \left(1 + \frac{1}{h^p}\right) |u_\delta^\varepsilon - v_\delta|^p \frac{|\nabla v_\delta^\varepsilon|^p}{|v_\delta^\varepsilon|^p}, \tag{4.17}$$

where A_1, A_2 are positive constants independent of ε and δ.

Now we make use of the strong convergence of u_δ^ε in the space $L^p(\Omega)$ to the function u_δ, the definition and the properties of the function v_δ^ε (see (4.14)). For any $K_h^\alpha \subset \Omega_\zeta$, we obtain

$$\int_{K_h^\alpha} \left\{ |\nabla V_\delta^\varepsilon|^p + g_F^\varepsilon(x)|u_\delta^\varepsilon|^p \right\} \mathrm{d}x$$
$$\geqslant |u_\delta(x^\alpha)|^p \int_{K_h^\alpha} \left\{ \left| \nabla \left(\frac{u_\delta^\varepsilon}{v_\delta^\varepsilon} \right) \right|^p + g_F^\varepsilon(x) \left| \frac{u_\delta^\varepsilon}{v_\delta^\varepsilon} \right|^p \right\} \mathrm{d}x + o(h^3), \tag{4.18}$$

for ε small enough ($\varepsilon \leqslant \hat{\varepsilon}(h)$) and $h \to 0$. Condition (C.2) implies

$$\underline{\lim}_{\varepsilon\to 0} \int_{K_h^\alpha} \left\{ \left| \nabla \left(\frac{u_\delta^\varepsilon}{v_\delta^\varepsilon} \right) \right|^p + g_F^\varepsilon(x) \left| \frac{u_\delta^\varepsilon}{v_\delta^\varepsilon} \right|^p \right\} \mathrm{d}x \geqslant h^3 b(x^\alpha) + o(h^3) \tag{4.19}$$

as $h \to 0$. Now it follows from (4.18) and (4.19) that

$$\underline{\lim}_{\varepsilon\to 0} \theta_1^\varepsilon \geqslant \int_{\Omega_\zeta} b(x)|u_\delta|^p \,\mathrm{d}x. \tag{4.20}$$

Taking into account the definition of Ω_ζ and passing to the limit as $\zeta \to 0$ in (4.20) we get

$$\underline{\lim}_{\varepsilon=\varepsilon_j\to 0} \theta_1^\varepsilon \geqslant \int_\Omega b(x)|u_\delta|^p \,\mathrm{d}x. \tag{4.21}$$

In order to estimate θ_2^ε from below in (4.16) we argue as follows. Using the weak convergence of u_δ^ε in $W^{1,p}(\Omega)$ and strong convergence in $L^p(\Omega)$ to u_δ and (4.14) we get

$$\varliminf_{\varepsilon=\varepsilon_j\to 0} \theta_2^\varepsilon \geqslant \int_\Omega \left\{|\nabla u_\delta|^p + g_0(x)|u_\delta|^p\right\} \mathrm{d}x. \tag{4.22}$$

Finally consider the third term on the right-hand side of (4.16). Using (4.14) we have

$$|\theta_3^\varepsilon| \leqslant C_1 \int_{Q^\varepsilon} |\nabla u_\delta| \left\{|\nabla u_\delta^\varepsilon|^{p-1} + |\nabla u_\delta|^{p-1}\right\} \mathrm{d}x,$$

where C_1 is a constant independent of ε, δ. Since $u_\delta(x)$ is a smooth function in Ω, we finally get

$$|\theta_3^\varepsilon| \leqslant C \int_{Q^\varepsilon} \left\{1 + |\nabla u_\delta^\varepsilon|^{p-1}\right\} \mathrm{d}x. \tag{4.23}$$

Now it is easy to see that the definition of the function u_δ^ε, (4.4), (4.13), the estimate for the measure of Q^ε and Hölder's inequality yield

$$\varlimsup_{\varepsilon\to 0} |\theta_3^\varepsilon| = 0. \tag{4.24}$$

Thus, it follows from (4.21), (4.22), (4.24) and strong convergence of u_δ^ε in $L^p(\Omega)$ to u_δ that

$$\varliminf_{\varepsilon\to 0} J^\varepsilon[u_\delta^\varepsilon] \geqslant J_{\mathrm{hom}}[u_\delta]. \tag{4.25}$$

This inequality together with (4.11) and (4.13) yield (4.10).

Inequalities (4.9) and (4.10) mean that if a subsequence of solutions of problem (2.3) converges weakly in $W^{1,p}(\Omega)$ to a function $u = u(x)$, then u minimizes the functional J_{hom} in $W^{1,p}(\Omega)$, i.e. $u(x)$ is a solution of the variational problem (2.11). Since $b(x) \geqslant 0$, this problem has a unique solution and the whole sequence of solutions of problem (2.3) converges weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ to the function $u = u(x)$.

Theorem 2.1 is proved. □

5. Periodic and nonperiodic examples for $n = p = 3$

Theorem 2.1 of section 2 provides sufficient conditions for the existence of the homogenized problem (2.11). It is important to show that the 'intersection' of the conditions of theorem 2.1 is not empty. The goal of this section is to prove that for two examples all the conditions of theorem 2.1 are satisfied and to compute the coefficients of the homogenized problem (2.11) explicitly. Note that a periodic example for the case $n = 3$, $p = 2$ was already constructed in [11].

5.1. A periodic example

Let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary. Let $\mathcal{F}^\varepsilon$ be a union of balls $\mathcal{F}_i^\varepsilon$ periodically, with a period ε, distributed in the domain Ω. We assume that the radius of the ball which equals r_ε is defined by

$$r_\varepsilon = \exp\left(-\frac{1}{\sqrt{\ell}\varepsilon^{3/2}}\right), \tag{5.1}$$

where ℓ is a strictly positive constant. It is clear that meas $\mathcal{F}^\varepsilon \longrightarrow 0$ as $\varepsilon \to 0$.

Consider the variational problem:

$$\int_{\Omega} \left\{ |\nabla u^{\varepsilon}|^3 + G^{\varepsilon}(x)|u^{\varepsilon}|^3 - 3\, f(x)\, u^{\varepsilon} \right\} \mathrm{d}x \longrightarrow \inf, \qquad u^{\varepsilon} \in W^{1,3}(\Omega), \tag{5.2}$$

where $f \in L^{3/2}(\Omega)$ and the function G^{ε} is given by

$$G^{\varepsilon}(x) = \begin{cases} \boldsymbol{g}_{\varepsilon} & \text{in} \quad \mathcal{F}^{\varepsilon}, \\ g_0(x) & \text{in} \quad \Omega^{\varepsilon}, \end{cases} \tag{5.3}$$

with $\boldsymbol{g}_{\varepsilon} \longrightarrow +\infty$ as $\varepsilon \to 0$. Moreover, we assume that

$$\boldsymbol{g}_{\varepsilon} = \frac{1}{r_{\varepsilon}^3}. \tag{5.4}$$

The following result holds.

Theorem 5.1. *Let u^{ε} be the solution of the variational problem (5.2). Then u^{ε} converges strongly in $L^3(\Omega)$ to u the solution of the variational problem:*

$$J_{\mathrm{hom}}[u] = \int_{\Omega} \{|\nabla u|^3 + B(x)|u|^3 - 3f(x)u\}\,\mathrm{d}x \longrightarrow \inf, \qquad u \in W^{1,3}(\Omega), \tag{5.5}$$

where

$$B(x) = g_0(x) + 4\pi\ell. \tag{5.6}$$

Proof of theorem 5.1. Following the lines of section 2 we introduce K_h^z, an open cube centred at $z \in \Omega \subset \mathbb{R}^3$ with length equal to h $(0 < \varepsilon \ll h < 1)$ and we set

$$\boldsymbol{I}_z^{\varepsilon,h}[\phi] = \int_{K_h^z} \left\{ |\nabla\phi|^3 + g_F^{\varepsilon}(x)|\phi|^3 + h^{-3-\gamma}|\phi - 1|^3 \right\} \mathrm{d}x, \tag{5.7}$$

where

$$g_F^{\varepsilon}(x) = \boldsymbol{g}_{\varepsilon}\boldsymbol{I}_F^{\varepsilon}(x), \tag{5.8}$$

with $\boldsymbol{g}_{\varepsilon}$ defined in (5.4) and $\boldsymbol{I}_F^{\varepsilon}(x)$ being the characteristic function of the set $\mathcal{F}^{\varepsilon}$.

Then we introduce the local energy characteristics of the domain Ω associated with the variational problem (5.2). For $z \in \Omega$ we define

- the functional associated to the energy in Ω^{ε}:

$$b^{\varepsilon,h}(z) = \inf_{v^{\varepsilon}} \boldsymbol{I}_z^{\varepsilon,h}[v^{\varepsilon}], \tag{5.9}$$

  where the infimum is taken over $v^{\varepsilon} \in W^{1,3}(K_h^z)$;
- the functional associated to the 3-capacity of the sets $\mathcal{F}^{\varepsilon}$:

$$a^{\varepsilon,h}(z) = \inf_{w^{\varepsilon}} \boldsymbol{I}_z^{\varepsilon,h}[w^{\varepsilon}], \tag{5.10}$$

  where the infimum is taken over $w^{\varepsilon} \in W(K_h^z, \mathcal{F}^{\varepsilon})$, where

$$W(K_h^z, \mathcal{F}^{\varepsilon}) = \left\{ w^{\varepsilon} \in W^{1,3}(K_h^z) \mid w^{\varepsilon} = 0 \text{ in } \mathcal{F}^{\varepsilon} \right\}. \tag{5.11}$$

Our goal is to check the following conditions:

(C.1) there exists a constant C independent of the parameter γ such that

$$\overline{\lim_{h\to 0}}\, \overline{\lim_{\varepsilon\to 0}}\, h^{-3} a^{\varepsilon,h}(z) \leqslant C.$$

(C.2) for any $x \in \Omega$ there exist the limits

$$\lim_{h\to 0} \underline{\lim_{\varepsilon\to 0}}\, h^{-3} b^{\varepsilon,h}(z) = \lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}}\, h^{-3} b^{\varepsilon,h}(z) = 4\pi\ell.$$

5.1.1. Condition (C.1) We introduce the following function:

$$W^{\varepsilon}(x)=\begin{cases}0 & \text{in} \quad K_h^z\cap\mathcal{F}^{\varepsilon},\\ 1-\sum_i \dfrac{\mathsf{Ln}(|x-x^{i,\varepsilon}|)}{\mathsf{Ln}(r_\varepsilon)}\varphi\left(\dfrac{|x-x^{i,\varepsilon}|}{\alpha_\varepsilon}\right) & \text{in} \quad K_h^z\setminus\mathcal{F}^{\varepsilon},\end{cases} \tag{5.12}$$

where $x^{i,\varepsilon}$ are the centres of the balls $\mathcal{F}_i^{\varepsilon}$,

$$\mathsf{Ln}(\rho)=\ln\frac{1}{\rho}-\ln\frac{1}{\alpha_\varepsilon} \tag{5.13}$$

and $\varphi(t)$ is a smooth positive function defined by $\varphi\in C^2(\mathbb{R}_+)$ with $\varphi(t)=1$ for $t\leqslant 1/2$; $\varphi(t)=0$ for $t\geqslant 1$. The parameter α_ε is defined by

$$\alpha_\varepsilon=\exp\left(-\frac{1}{\sqrt{\ell}\varepsilon^{3/2-\kappa}}\right)\qquad\left(0<\kappa<\frac{3}{2}\right). \tag{5.14}$$

It is clear that $r_\varepsilon\ll\alpha_\varepsilon\ll\varepsilon$.

Now it follows from (5.10) that

$$a^{\varepsilon,h}(z)\leqslant\int_{K_h^z}\left\{|\nabla W^{\varepsilon}|^3+h^{-3-\gamma}|W^{\varepsilon}-1|^3\right\}\,\mathrm{d}x. \tag{5.15}$$

Consider the first integral on the right-hand side of (5.15). We have

$$\int_{K_h^z}|\nabla W^{\varepsilon}|^3\,\mathrm{d}x=4\pi\sum_{\mathcal{F}_i^{\varepsilon}\subset K_h^z}\left(\int_{r_\varepsilon}^{\alpha_\varepsilon/2}\left|\frac{\partial W^{\varepsilon}}{\partial\rho}\right|^3\rho^2\,\mathrm{d}\rho+\int_{\alpha_\varepsilon/2}^{\alpha_\varepsilon}\left|\frac{\partial W^{\varepsilon}}{\partial\rho}\right|^3\rho^2\,d\rho\right). \tag{5.16}$$

It is clear that $|\mathsf{Ln}(\rho)|\leqslant\ln 2$ for $\rho\in[\alpha_\varepsilon/2,\alpha_\varepsilon]$. Then it easily follows from (5.1) and (5.14) that

$$\int_{K_h^z}|\nabla W^{\varepsilon}|^3\,\mathrm{d}x$$
$$\leqslant C\frac{h^3}{\varepsilon^3}\left(\frac{1}{(\ln(1/r_\varepsilon)-\ln(1/\alpha_\varepsilon))^2}+\frac{1}{(\ln(1/r_\varepsilon)-\ln(1/\alpha_\varepsilon))^3}\right)\leqslant C_1h^3, \tag{5.17}$$

where C_1 is a constant independent of ε,γ.

Consider the second integral on the right-hand side of (5.15). We have

$$h^{-3-\gamma}\int_{K_h^z}|W^{\varepsilon}-1|^3\,\mathrm{d}x$$
$$=4\pi h^{-3-\gamma}\sum_{\mathcal{F}_i^{\varepsilon}\subset K_h^z}\int_{r_\varepsilon}^{\alpha_\varepsilon}\left|\frac{\mathsf{Ln}(\rho)}{\mathsf{Ln}(r_\varepsilon)}\varphi\left(\frac{\rho}{\alpha_\varepsilon}\right)\right|^3\rho^2\,\mathrm{d}\rho\leqslant C\frac{h^{-\gamma}}{\varepsilon^3}(\alpha_\varepsilon)^3. \tag{5.18}$$

Now it follows from (5.17) and (5.18) that

$$\overline{\lim_{h\to 0}}\;\overline{\lim_{\varepsilon\to 0}}\,h^{-3}a^{\varepsilon,h}(z)\leqslant C_1, \tag{5.19}$$

where C_1 is defined in (5.17) and it is independent of ε and the parameter γ.

Thus condition (C.1) is satisfied.

5.1.2. Condition (C.2). We will construct a function $V^{\varepsilon}(x)$ that is a 'good' approximation of v^{ε} the minimizer of the functional (5.9) and obtain then an asymptotic formula for the functional $b^{\varepsilon,h}(z)$.

Let $\mathcal{S}_0^R$ be a ball centred at the point zero and of radius $R > 0$. Consider the equation

$$-\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\left|\frac{\partial u}{\partial\rho}\right|\right)+u|u|=0 \qquad (\rho\geqslant 0). \tag{5.20}$$

One can show that, for any admissible positive boundary condition on $\partial\mathcal{S}_0^R$, there exists a unique positive non-decreasing solution $U(\rho)$ of the Dirichlet boundary value problem for the equation (5.20) such that $U(\rho)\in W^{1,3}(\mathcal{S}_0^R)$. Then the function $\mathsf{v}^\varepsilon(\rho)=U\left((\boldsymbol{g}_\varepsilon)^{1/3}\rho\right)$ is the solution of the following equation:

$$-\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial \mathsf{v}^\varepsilon}{\partial\rho}\left|\frac{\partial \mathsf{v}^\varepsilon}{\partial\rho}\right|\right)+\boldsymbol{g}_\varepsilon\mathsf{v}^\varepsilon|\mathsf{v}^\varepsilon|=0, \tag{5.21}$$

and it follows from (5.21) and the properties of the function $U(\rho)$ that

$$\frac{\partial\mathsf{v}^\varepsilon}{\partial\rho}(\rho)=\frac{(\boldsymbol{g}_\varepsilon)^{1/2}}{\rho}\left(\int_0^\rho|\mathsf{v}^\varepsilon(t)|^2\,t^2\,\mathrm{d}t\right)^{1/2}. \tag{5.22}$$

We define the function $V^\varepsilon(x)$ as follows:

$$V^\varepsilon(x)=\begin{cases}C_1^\varepsilon\mathsf{v}^\varepsilon(|x-x^{i,\varepsilon}|) & \text{in}\quad \mathcal{F}_h^\varepsilon,\\ 1-\sum_i C_2^\varepsilon\mathsf{Ln}(|x-x^{i,\varepsilon}|)\varphi\left(\dfrac{|x-x^{i,\varepsilon}|}{\alpha_\varepsilon}\right) & \text{in}\quad K_h^z\setminus\mathcal{F}_h^\varepsilon,\end{cases} \tag{5.23}$$

where

$$C_1^\varepsilon=\frac{1}{\mathsf{v}^\varepsilon(r_\varepsilon)+\mathsf{Ln}(r_\varepsilon)\,r_\varepsilon(\partial\mathsf{v}^\varepsilon/\partial\rho)(r_\varepsilon)},\qquad C_2^\varepsilon=C_1^\varepsilon\,r_\varepsilon\frac{\partial\mathsf{v}^\varepsilon}{\partial\rho}(r_\varepsilon) \tag{5.24}$$

and

$$\mathcal{F}_h^\varepsilon=\bigcup_{\mathcal{F}_i^\varepsilon\subset K_h^z}\mathcal{F}_i^\varepsilon \tag{5.25}$$

with $\mathcal{F}_i^\varepsilon$ such that $\mathcal{S}_i^\varepsilon\subset K_h^z$. Here $\mathcal{S}_i^\varepsilon$ is the ball centred at $x^{i,\varepsilon}$ and of radius α_ε.

From (5.4), it is easy to see that

$$r_\varepsilon\frac{\partial\mathsf{v}^\varepsilon}{\partial\rho}(r_\varepsilon)=\left(\int_0^1|U(t)|^2t^2\,\mathrm{d}t\right)^{1/2}=I_U=\mathrm{Const} \tag{5.26}$$

and

$$\mathsf{v}^\varepsilon(r_\varepsilon)=U(1). \tag{5.27}$$

Then we get the following asymptotic formulae for the constants $C_1^\varepsilon, C_2^\varepsilon$:

$$C_1^\varepsilon=\frac{1}{I_U\ln(1/r_\varepsilon)}(1+o(1)),\qquad C_2^\varepsilon=\frac{1}{\ln(1/r_\varepsilon)}(1+o(1))\qquad\text{as}\quad\varepsilon\to 0. \tag{5.28}$$

Now let $v_{\min}^\varepsilon=v_{\min}^\varepsilon(x)$ be the function that minimizes the functional (5.9). Let us represent this function in the form

$$v_{\min}^\varepsilon(x)=V^\varepsilon(x)+\zeta^\varepsilon(x), \tag{5.29}$$

where the function V^ε is defined in (5.23). Then

$$b^{\varepsilon,h}(z)=\int_{K_h^z}\left\{|\nabla v_{\min}^\varepsilon|^3+g_F^\varepsilon(x)|v_{\min}^\varepsilon|^3+h^{-3-\gamma}|v_{\min}^\varepsilon-1|^3\right\}\mathrm{d}x. \tag{5.30}$$

We will prove that the function ζ^ε gives a vanishing contribution (as $\varepsilon\to 0$ and $h\to 0$) in (5.30) and, therefore, the functional (5.9) may be computed by the function V^ε.

Since the function $v^\varepsilon_{\min} = V^\varepsilon + \zeta^\varepsilon$ minimizes the functional in (5.30), then we have

$$b^{\varepsilon,h}(z) \leqslant \int_{K_h^z} \left\{|\nabla V^\varepsilon|^3 + g_F^\varepsilon(x)|V^\varepsilon|^3 + h^{-3-\gamma}|V^\varepsilon - 1|^3\right\} \mathrm{d}x \equiv \boldsymbol{B}^{\varepsilon,h}(z). \tag{5.31}$$

Let us now estimate the functional (5.30) from below. To this end we make use of the following inequality:

$$|\xi_1 + \xi_2|^p \geqslant |\xi_1|^p + \delta_p|\xi_2|^p + p|\xi_1|^{p-2}(\xi_1, \xi_2), \tag{5.32}$$

where ξ_1, ξ_2 are arbitrary vectors from the space $\mathbb{R}^s$ $(s = 1, 2, ..)$, $0 < \delta_p \leqslant 1$ ($\delta_p = 1$ when $p = 2$). Now it follows from (5.30) and (5.32) that

$$\begin{aligned} b^{\varepsilon,h}(z) &\geqslant \boldsymbol{B}^{\varepsilon,h}(z) + \delta_3\boldsymbol{J}^{\varepsilon,h}(z) + 3\int_{K_h^z} \left\{(\nabla V^\varepsilon|\nabla V^\varepsilon|, \nabla\zeta^\varepsilon) + g_F^\varepsilon(x)|V^\varepsilon|V^\varepsilon\,\zeta^\varepsilon\right\} \mathrm{d}x \\ &\quad +3\,h^{-3-\gamma}\int_{K_h^z} |V^\varepsilon - 1|(V^\varepsilon - 1)\zeta^\varepsilon\,\mathrm{d}x, \end{aligned} \tag{5.33}$$

where

$$\boldsymbol{J}^{\varepsilon,h}(z) = \int_{K_h^z} \left\{|\nabla\zeta^\varepsilon|^3 + g_F^\varepsilon(x)|\zeta^\varepsilon|^3 + h^{-3-\gamma}|\zeta^\varepsilon|^3\right\} \mathrm{d}x. \tag{5.34}$$

Now the inequalities (5.31), (5.33) imply that

$$\begin{aligned} \boldsymbol{J}^{\varepsilon,h}(z) &\leqslant \frac{3}{\delta_3}\left|\int_{K_h^z} \left\{(\nabla V^\varepsilon|\nabla V^\varepsilon|, \nabla\zeta^\varepsilon) + g_F^\varepsilon(x)|V^\varepsilon|V^\varepsilon\,\zeta^\varepsilon + h^{-3-\gamma}|V^\varepsilon - 1|(V^\varepsilon - 1)\zeta^\varepsilon\right\} \mathrm{d}x\right| \\ &= \frac{3}{\delta_3}\left|\int_{K_h^z} \left\{\left(\Delta_3 V^\varepsilon + g_F^\varepsilon(x)|V^\varepsilon|V^\varepsilon\right) + h^{-3-\gamma}|V^\varepsilon - 1|(V^\varepsilon - 1)\right\}\zeta^\varepsilon\,\mathrm{d}x\right| \\ &\leqslant \frac{3}{\delta_3}\int_{K_h^z} \left\{|\ (\Delta_3 V^\varepsilon + g_F^\varepsilon(x)|V^\varepsilon|V^\varepsilon)\ | + h^{-3-\gamma}|V^\varepsilon - 1|^2\right\}|\zeta^\varepsilon|\,\mathrm{d}x. \end{aligned} \tag{5.35}$$

Let

$$\eta^\varepsilon(x) = -\Delta_3 V^\varepsilon + g_F^\varepsilon(x)|V^\varepsilon|V^\varepsilon. \tag{5.36}$$

Then the function η^ε equals zero everywhere in the cube K_h^z except the set $\mathcal{D}^\varepsilon = \cup_i\mathcal{D}_i^\varepsilon$, where

$$\mathcal{D}_i^\varepsilon = \left\{x \in \Omega \mid \alpha_\varepsilon/2 < |x - x^{i,\varepsilon}| < \alpha_\varepsilon\right\}$$

and

$$\eta^\varepsilon = -\Delta_3 V^\varepsilon = -\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\left|\frac{\partial V^\varepsilon}{\partial\rho}(\rho)\right|^2\right) \qquad \text{in} \quad \mathcal{D}_i^\varepsilon.$$

Moreover, since $|\mathrm{Ln}(\rho)| \leqslant \ln 2$ for $\rho \in [\alpha_\varepsilon/2, \alpha_\varepsilon]$, then the following estimate is valid:

$$|\eta^\varepsilon| \leqslant C\left(\ln\frac{1}{r_\varepsilon}\right)^{-2}\frac{1}{(\alpha_\varepsilon)^3} \qquad \text{in} \quad \mathcal{D}_i^\varepsilon. \tag{5.37}$$

Let $\mathcal{S}^\varepsilon$ be a union of the balls $\mathcal{S}_i^\varepsilon$ centred at $x^{i,\varepsilon}$ and of radius α_ε. It follows now from (5.35)–(5.37) and Hölder's inequality that

$$\begin{aligned} \boldsymbol{J}^{\varepsilon,h}(z) &\leqslant \frac{3}{\delta_3}\int_{\mathcal{D}^\varepsilon} \left\{|\eta^\varepsilon| + h^{-3-\gamma}|V^\varepsilon - 1|^2\right\}|\zeta^\varepsilon|\,\mathrm{d}x \\ &\leqslant Ch^2\left[\left(\ln\frac{1}{r_\varepsilon}\right)^{-2}\frac{1}{\varepsilon^2\alpha_\varepsilon} + h^{-3-\gamma}\frac{(\alpha_\varepsilon)^2}{\varepsilon^2}\right]\left(\int_{\mathcal{S}^\varepsilon} |\zeta^\varepsilon|^3\,\mathrm{d}x\right)^{1/3}. \end{aligned} \tag{5.38}$$

To estimate the second term in (5.38) we make use of the following inequality:

$$\int_{\mathcal{S}_0^\varepsilon} |v|^3\,\mathrm{d}x \leqslant C\left\{\frac{(\alpha_\varepsilon)^3}{\varepsilon^3}\int_{\mathcal{K}^\varepsilon}|v|^3\,\mathrm{d}x + (\alpha_\varepsilon)^3\ln^2\frac{1}{\alpha_\varepsilon}\int_{\mathcal{K}^\varepsilon}|\nabla v|^3\,\mathrm{d}x\right\}, \tag{5.39}$$

where v is an arbitrary function from $W^{1,3}(\mathcal{K}^\varepsilon)$, $\mathcal{K}^\varepsilon$ is a cube centred at the point zero and of length ε and $\mathcal{S}_0^\varepsilon$ is a ball centred at zero and of radius α_ε.

Then it follows now from (5.38), (5.39) that

$$\begin{aligned}\boldsymbol{J}^{\varepsilon,h}(z) \leqslant Ch^{3+\gamma/3}\Bigg(&\left[\ell^3 + h^{-9-3\gamma}\frac{(\alpha_\varepsilon)^9}{\varepsilon^9}\right]h^{-3-\gamma}\int_{K_h^z}|v|^3\,\mathrm{d}x\\ &+\left[h^{-3-\gamma}\frac{\ln^2(1/\alpha_\varepsilon)}{\ln^2(1/r_\varepsilon)}\ell^2 + h^{-12-4\gamma}\frac{(\alpha_\varepsilon)^9}{\varepsilon^6}\ln^2\frac{1}{\alpha_\varepsilon}\right]\int_{K_h^z}|\nabla v|^3\,\mathrm{d}x\Bigg)^{1/3}.\end{aligned}$$

From this inequality, for ε sufficiently small ($\varepsilon \leqslant \varepsilon_0(h)$), we get

$$\boldsymbol{J}^{\varepsilon,h}(z) \leqslant Ch^{3+\gamma/3}\left(\boldsymbol{J}^{\varepsilon,h}(z)\right)^{1/3}$$

and, therefore,

$$\overline{\lim_{\varepsilon\to 0}}\,\boldsymbol{J}^{\varepsilon,h}(z) = O(h^{9/2+\gamma/2}) = o(h^3). \tag{5.40}$$

Let us now calculate $\boldsymbol{B}^{\varepsilon,h}(z)$. We have

$$\boldsymbol{B}^{\varepsilon,h}(z) = \int_{K_h^z}\left\{|\nabla V^\varepsilon|^3 + g_F^\varepsilon(x)|V^\varepsilon|^3 + h^{-3-\gamma}|V^\varepsilon - 1|^3\right\}\mathrm{d}x. \tag{5.41}$$

Here

$$\int_{K_h^z}|\nabla V^\varepsilon|^3\,\mathrm{d}x = h^3\,4\pi\ell(1+o(1)) \qquad \text{as}\quad \varepsilon\to 0, \tag{5.42}$$

$$\int_{K_h^z} g_F^\varepsilon(x)|V^\varepsilon|^3\,\mathrm{d}x = \boldsymbol{g}_\varepsilon\int_{\mathcal{F}_h^\varepsilon}|V^\varepsilon|^3\,\mathrm{d}x = 4\pi\frac{h^3}{\varepsilon^3}\frac{1}{(I_U)^2\ln^3\dfrac{1}{r_\varepsilon}}(1+o(1)) \longrightarrow 0 \tag{5.43}$$

as $\varepsilon\to 0$

and

$$h^{-3-\gamma}\int_{K_h^z}|V^\varepsilon - 1|^3\,\mathrm{d}x \longrightarrow 0 \qquad \text{as}\quad \varepsilon\to 0. \tag{5.44}$$

Therefore,

$$\lim_{h\to 0}\lim_{\varepsilon\to 0} h^{-3}\boldsymbol{B}^{\varepsilon,h}(z) = 4\pi\ell. \tag{5.45}$$

Now let

$$\begin{aligned}\boldsymbol{j}^{\varepsilon,h}(z) = 3\int_{K_h^z}&\left\{(\nabla V^\varepsilon|\nabla V^\varepsilon|,\nabla\zeta^\varepsilon) + g_F^\varepsilon(x)|V^\varepsilon|V^\varepsilon\,\zeta^\varepsilon\right\}\mathrm{d}x\\ &+3\,h^{-3-\gamma}\int_{K_h^z}|V^\varepsilon - 1|(V^\varepsilon - 1)\zeta^\varepsilon\,\mathrm{d}x.\end{aligned}$$

Then it follows from Hölder's inequality and (5.45) that

$$\overline{\lim_{\varepsilon\to 0}}\,|\boldsymbol{j}^{\varepsilon,h}(z)| \leqslant 9\,\overline{\lim_{\varepsilon\to 0}}\left(\boldsymbol{B}^{\varepsilon,h}(z)\right)^{2/3}\left(\boldsymbol{J}^{\varepsilon,h}(z)\right)^{1/3} = o(h^3) \qquad \text{as}\quad h\to 0. \tag{5.46}$$

Finally, from the inequalities (5.31), (5.33), (5.46) and the equality (5.45) we get

$$\lim_{h\to 0}\lim_{\varepsilon\to 0} h^{-3}b^{\varepsilon,h}(z) = 4\pi\ell \tag{5.47}$$

and condition (C.2) is proved. This completes the proof of theorem 5.1.

Theorem 5.1 is proved. □

5.2. A nonperiodic example

As an application of the previous general result (cf theorem 2.1), we give now a nonperiodic example. More precisely, we will present a locally periodic example.

Let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary and $\{x^i\}$ be a periodic grid in Ω with a period ε. Let $\mathcal{F}^\varepsilon$ be a union of balls $\mathcal{F}_i^\varepsilon$ centred at the points x^i and of radii $r_\varepsilon^{(i)}$ defined by

$$r_\varepsilon^{(i)} = \exp\left(-\frac{1}{R(x^i)\varepsilon^{3/2}}\right), \tag{5.48}$$

where $R = R(x)$ is a strictly positive smooth function in Ω. As in the periodic case, it is clear that meas $\mathcal{F}^\varepsilon \longrightarrow 0$ as $\varepsilon \to 0$.

Consider the variational problem:

$$\int_\Omega \left\{|\nabla u^\varepsilon|^3 + G^\varepsilon(x)|u^\varepsilon|^3 - 3\, f(x)\, u^\varepsilon\right\} \mathrm{d}x \longrightarrow \inf, \qquad u^\varepsilon \in W^{1,3}(\Omega), \tag{5.49}$$

where $f \in L^{3/2}(\Omega)$ and the function G^ε is given by

$$G^\varepsilon(x) = \begin{cases} \dfrac{1}{(r_\varepsilon^{(i)})^3} & \text{in} \quad \mathcal{F}_i^\varepsilon; \ \ (i = 1, 2, \ldots, N_\varepsilon), \\ g_0(x) & \text{in} \quad \Omega^\varepsilon. \end{cases} \tag{5.50}$$

Following the lines of the proof of theorem 5.1 (with corresponding modifications) one can obtain the following result.

Theorem 5.2. *Let u^ε be the solution of the variational problem (5.49). Then u^ε converges strongly in $L^3(\Omega)$ to u the solution of the variational problem:*

$$J_{\mathrm{hom}}[u] = \int_\Omega \{|\nabla u|^3 + B(x)|u|^3 - 3f(x)u\}\, \mathrm{d}x \longrightarrow \inf, \qquad u \in W^{1,3}(\Omega), \tag{5.51}$$

where

$$B(x) = g_0(x) + 4\pi R^2(x). \tag{5.52}$$

Acknowledgments

The work of BA and LP was partially supported by the GdR MoMaS 2439 CNRS ANDRA BRGM CEA EDF whose support is gratefully acknowledged. This paper was completed when LP was visiting the Applied Mathematics Laboratory of the University of Pau. LP is grateful for the invitation and the hospitality provided. The authors are grateful to François Murat for his help.

References

[1] Amaziane B, Pankratov L and Piatnitski A 2006 Homogenization of a class of quasilinear elliptic equations in high-contrast fissured media, *Proc. R. Soc. Edinb.* at press

[2] Balzano M, Corbo Esposito A and Paderni G 1997 Nonlinear Dirichlet problems in randomly perforated domains, *Rend. Mat. Ser. VII* **17** 163–86

[3] Braides A and Defranceschi A 1998 *Homogenization of Multiple Integrals* (*Oxford Lecture Series in Mathematics and its Applications* vol 12) (Oxford: Clarendon)

[4] Briane M 2002 Homogenization of non-unifromly bounded operators: critical barrier for nonlocal effects *Arch. Ration. Mech. Anal.* **164** 73–101

[5] Dal Maso G and Murat F 1997 Asymptotic behaviour and correctors for Dirichlet problems in perforated domains with homogeneous monotone operators *Ann. Sc. Norm. Sup. Pisa* **7** 765–803

[6] Fenchenko V N and Khruslov E Ya 1981 Asymptotic of solution of differential equations with strongly oscillating matrix of coefficients which does not satisfy the condition of uniform boundedness *Dokl. Acad. Nauk. Ukr. SSR* **4** 23–27

[7] Hornung U 1997 *Homogenization and Porous Media* (New York: Springer)

[8] Khruslov E 1991 Homogenized models of composite media *Composite Media and Homogenization Theory* ed G Dal Maso and G F Dell'Antonio (Basle: Birkhäuser) 159–82

[9] Khruslov E and Pankratov L 2000 Homogenization of the Dirichlet variational problems in Orlicz–Sobolev spaces *Fields Inst. Commun.* **25** 345–66

[10] Ladyzhenskaya O A and Ural'tseva N N 1973 *Linear and Quasilinear Elliptic Equations* (New York: Academic)

[11] Marchenko V A and Khruslov E Ya 2006 *Homogenization of Partial Differential Equations* (Boston, MA: Birkhäuser)

[12] Nguetseng G and Nnang H 2003 Homogenization of nonlinear monotone operators beyond the periodic setting *Electron. J. Diff. Eqns* **36** 1–24

[13] Panfilov M 2000 *Macroscale Models of Flow Through Highly Heterogeneous Porous Media* (Dordrecht: Kluwer)

[14] Pankov A 1997 *G-Convergence and Homogenization of Nonlinear Partial Differential Operators* (Dordrecht: Kluwer)

[15] Samarskii A, Galaktionov V, Kurdumov S and Mikhailov A 1995 *Blow–up in Quasilinear Parabolic Equations* (Berlin: De Gruyter)

[16] Skrypnik I V 1990 *Methods of Investigation of Nonlinear Elliptic Boundary Value Problems* (Moscow: Nauka)

[17] Zhikov V V, Kozlov S M and Oleinik O A 1994 *Homogenization of Differential Operators and Integral Functionals* (Berlin: Springer)