GAKUTO International Series
Math. Sci. Appl., Vol.24 (2005)
Multi Scale Problems and Asymptotic Analysis, pp. 291-303

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Homogenization of a singular random one dimensional PDE

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Abstract: We study the homogenization problem for a $1D$ parabolic operator with a random large statistically homogeneous potential. It is shown that under proper normalization and mixing conditions this operator admits homogenization in law, the limits operator being a parabolic operator with random coefficients.

1. Introduction.

Our goal is to study the limits, as $\varepsilon \to 0$, of the solutions of the two linear parabolic one–dimensional PDEs

$$\begin{cases} \dfrac{\partial u_1^\varepsilon}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u_1^\varepsilon}{\partial x^2}(t,x) + \dfrac{1}{\sqrt{\varepsilon}} c\left(\dfrac{x}{\varepsilon}\right) u_1^\varepsilon(t,x), \quad t \geq 0,\, a < x < b; \\ u_1^\varepsilon(0,x) = g(x),\ a < x < b \quad u_1^\varepsilon(t,a) = g(a),\ u_1^\varepsilon(t,b) = g(b),\ t \geq 0, \end{cases} \tag{1}$$

and

$$\begin{cases} \dfrac{\partial u_2^\varepsilon}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u_2^\varepsilon}{\partial x^2}(t,x) + \dfrac{1}{\sqrt{\varepsilon}} c\left(\dfrac{x}{\varepsilon}\right) u_2^\varepsilon(t,x), \quad t \geq 0,\, x \in \mathbb{R}; \\ u_2^\varepsilon(0,x) = g(x), \quad x \in \mathbb{R}. \end{cases} \tag{2}$$

where $\{c(x),\ x \in \mathbb{R}\}$ is a zero–mean stationary mixing random field. We note that the highly oscillating factor $1/\sqrt{\varepsilon}$ is different from the correct factor $1/\varepsilon$ in the case of a periodic potential c, see [7] and [2]. Our results are stronger for the first equation, for which we have a uniqueness result for the limiting equation. In both cases, we obtain the convergence of the whole sequence to a well identified limit, using a Feynman–Kac representation.

The paper is organized as follows. Section 2 contains the assumptions and the statement of the main result. Section 3 contains weak convergence results, and section 4 combines those with some uniform integrability estimates, in order to prove the main result. Finally section 5 is concerned with the PDEs satisfied by the limits.

2. Set up and statement of the main result

We make the following assumptions:

(A.1) The initial condition g belongs to $C([a,b])$ (resp. to $L^2(\mathbb{R})\cap\mathcal{C}_b(\mathbb{R})$).

(A.2) The coefficient $\{c(x), x\in\mathbb{R}\}$ is a stationary centered and bounded random field defined on a probability space $(\Sigma,\mathcal{A},P)$, and we assume that

$$c_0\equiv\int_{-\infty}^{\infty}|Ec(0)c(x)|dx<\infty, \tag{3}$$

where E denotes expectation with respect to the probability measure P.

(A.3) Let

$$\mathcal{G}_x:=\sigma\{c(y),y\le x\},\quad \mathcal{G}^x:=\sigma\{c(y),y\ge x\}.$$

We assume that the random field c is ϕ-mixing in the following sense. Define, for $h>0$, $\phi(h)$ the mixing coefficient with respect to the algebras from above as

$$\phi(h):=\sup_{\{A\in\mathcal{G}_x,B\in\mathcal{G}^{x+h},P(A)>0\}}|P(B|A)-P(B)|.$$

We suppose that

$$\int_0^{\infty}\phi^{\frac12}(h)dh<\infty. \tag{4}$$

Let $\{B_t,\ t\ge 0\}$ denote a standard one–dimensional Brownian motion, defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, and let

$$X_t^x:=x+B_t,\quad ,x\in\mathbb{R},\ t\ge 0.$$

The Feynman-Kac formula for the solution of equation (1) reads

$$u_1^\varepsilon(t,x)=\mathbb{E}\left[g(X^x_{t\wedge\tau_x})\exp\left(\frac{1}{\sqrt{\varepsilon}}\int_0^{t\wedge\tau_x}c\left(\frac{X_s^x}{\varepsilon}\right)ds\right)\right], \tag{5}$$

where $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$, and τ_x denotes the first exit time from $[a,b]$ by the process $\{x+B_s,\ s\ge 0\}$, and the same formula for the solution of (2) reads

$$u_2^\varepsilon(t,x)=\mathbb{E}\left[g(X^x_t)\exp\left(\frac{1}{\sqrt{\varepsilon}}\int_0^{t}c\left(\frac{X_s^x}{\varepsilon}\right)ds\right)\right]. \tag{6}$$

Considering the assumption **(A.2)**, we define the finite quantity

$$\bar{c}^2=\int_{-\infty}^{\infty}E[c(0)c(x)]\,dx. \tag{7}$$

The main result of this paper is given by

Theorem 1.

$$u_1^\varepsilon(t,x)\to u_1(t,x):=\mathbb{E}\left[g(X^x_{t\wedge\tau_x})\exp\left(\bar{c}\int_{\mathbb{R}}L^{y-x}_{t\wedge\tau_x}W(dy)\right)\right]$$

$$u_2^\varepsilon(t,x) \to u_2(t,x) := \mathbb{E}\left[g(X_t^x)\exp\left(\bar{c}\int_{\mathbb{R}} L_t^{y-x} W(dy)\right)\right]$$

in P law, as $\varepsilon \to 0$, where W denotes a one dimensional standard Brownian motion defined on the probability space $(\Sigma, \mathcal{A}, P)$ and L_t^y is the local time at time t and point y of the process $\{X_t^0, t \geq 0\}$.

We introduce the notation

$$Y_t^{\varepsilon,x} := \frac{1}{\sqrt{\varepsilon}} \int_0^t c\left(\frac{X_s^x}{\varepsilon}\right) ds.$$

The first step in the proof of Theorem 1 is to establish the weak convergence of the process $\{Y_t^{\varepsilon,x}\}$ in the space $(\Sigma \times \Omega, \mathcal{A} \otimes \mathcal{F}, P \times \mathbb{P})$, which is done in the next section.

3. Weak convergence

The aim of this section is to prove the

Theorem 2. *For each $t > 0$,*

$$Y_t^{\varepsilon,x} \Rightarrow Y_t^x := \bar{c}\int_{\mathbb{R}} L_t^{y-x} W(dy),$$

weakly, as $\varepsilon \to 0$, where, as above, L_t^y is the local time at point y and time t of the Brownian motion $\{X_t^0,\ t \geq 0\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{W(y),\ y \in \mathbb{R}\}$ is a Wiener process defined on $(\Sigma, \mathcal{A}, P)$, so that (X, L) and W are independent.

We define

$$W_\varepsilon(x) = \frac{1}{\bar{c}\sqrt{\varepsilon}} \int_0^x c\left(\frac{y}{\varepsilon}\right) dy.$$

We first prove the

Proposition 3. *The sequence of random processes $\{W_\varepsilon\}$ converges weakly, as $\varepsilon \to 0$, in the space $\mathcal{C}(\mathbb{R})$, to a standard Wiener process $\{W\}$ defined on $(\Sigma, \mathcal{A}, P)$.*

Proof: Denote, for $x \geq 0$, $W_\varepsilon^1(x) = W_\varepsilon(x)$ and $W_\varepsilon^2(x) = W_\varepsilon(-x)$. According to the assumptions **(A.1), (A.2)** and the functional central limit theorem (see e.g. [1], pages 178, 179), it follows that

$$(W_\varepsilon^1, W_\varepsilon^2) \xrightarrow{\mathcal{D}} (W^1, W^2),$$

where $\{W^1(x), x \geq 0\}$ and $\{W^2(x), x \geq 0\}$ are mutually independent standard Brownian motions. Finally we denote by $\{W(x), x \in \mathbb{R}\}$ the process defined by

$$W(x) := W^1(x), \quad \text{for} \quad x \geq 0, \quad W(x) := W^2(-x), \quad \text{for} \quad x < 0.$$

□

We can now proceed with the

Proof of Theorem 2 : We deduce from Itô's formula that, if

$$\Phi_\varepsilon(x) := \int_0^x W_\varepsilon(y) dy,$$

$$\Phi_\varepsilon(X_t^x) = \Phi_\varepsilon(x) + \int_0^t W_\varepsilon(X_s^x)dX_s^x + \frac{1}{2}\int_0^t W_\varepsilon'(X_s^x)ds,$$

consequently

$$\begin{aligned} Y_t^{\varepsilon,x} &= \bar{c}\int_0^t W_\varepsilon'(X_s^x)ds \\ &= 2\bar{c}[\Phi_\varepsilon(X_t^x) - \Phi_\varepsilon(x) - \int_0^t W_\varepsilon(X_s^x)dX_s^x]. \end{aligned}$$

The mapping $f \to \int_0^{\cdot} f(y)dy$ is a continuous mapping from $C(\mathbb{R})$ into itself. Hence it follows from Proposition 3 that $(W_\varepsilon, \Phi_\varepsilon) \Rightarrow (W, \Phi)$ in $C(\mathbb{R})^2$ as $\varepsilon \to 0$, where $\Phi(x) = \int_0^x W(y)dy$, $x \in \mathbb{R}$. Moreover the mapping

$$f \to \int_0^t f(X_s^x)dX_s^x$$

is continuous from $C(\mathbb{R})$ into $L^0(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the topology of convergence in probability. Hence

$$Y_t^{\varepsilon,x} \to 2\bar{c}[\Phi(X_t^x) - \Phi(x) - \int_0^t W(X_s^x)dX_s^x]$$

in P law and $\mathbb{P}$ probability, hence also in $P \times \mathbb{P}$ law. The result now follows from the

Lemma 4. *The following relation holds a.s.*

$$\Phi(X_t^x) = \Phi(x) + \int_0^t W(X_s^x)dX_s^x + \frac{1}{2}\int_{\mathbb{R}} L_t^{y-x}W(dy).$$

Proof : Let

$$W_n(x) = (W * \rho_n)(x), \tag{8}$$

where $\rho_n(x) = n\rho(nx)$ and ρ is a smooth map from $\mathbb{R}$ into $\mathbb{R}_+$ with compact support, whose integral over $\mathbb{R}$ equals one, and Φ_n denote the indefinite integral of W_n. Then from Itô's formula

$$\begin{aligned} \Phi_n(X_t^x) &= \Phi_n(x) + \int_0^t W_n(X_s^x)dX_s^x + \frac{1}{2}\int_0^t W_n'(X_s^x)ds \\ &= \Phi_n(x) + \int_0^t W_n(X_s^x)dX_s^x + \frac{1}{2}\int_{\mathbb{R}} L_t^{y-x}W_n'(y)dy. \end{aligned}$$

The Lemma now follows by taking the limit as $n \to \infty$. In particular the last term in the above right hand side converges to

$$\int_{\mathbb{R}} L_t^{y-x}W(dy)$$

in $L^2(\Sigma, \mathcal{A}, P)$ for almost all $\omega \in \Omega$, where the last integral is interpreted as a Wiener integral for each fixed trajectory of $L_t^{\cdot-x}(\omega)$.

4. Convergence of the sequence u^ε

In order to deduce Theorem 1 from Theorem 2, we shall need some type of uniform integrability for

$$\left\{ \exp\left[\frac{1}{\sqrt{\varepsilon}} \int_0^t c\left(\frac{X_s^x}{\varepsilon}, \sigma\right) ds\right], \varepsilon > 0\right\}$$

under $\mathbb{P}$, for P almost all σ.

We first define the following $\mathbb{R}_+$–valued random variables, for $0 < \gamma < 1/2$:

$$\xi_\varepsilon^\gamma = \sup_{x\in\mathbb{R}} \frac{|W_\varepsilon(x)|}{(1+|x|)^{1-\gamma}}.$$

We have the

Lemma 5. *For any* $0 < \gamma < 1/2$ *and* $\varepsilon_0 > 0$*, the collection of random variables* $\{\xi_\varepsilon^\gamma,\ 0 < \varepsilon \le \varepsilon_0\}$ *is tight.*

Proof. Due to the symmetry it is sufficient to estimate $|W_\varepsilon(x)|$ for $x > 0$. We have

$$\begin{aligned} E(|W_\varepsilon(r)|^2) &= \varepsilon \int_0^{r/\varepsilon}\int_0^{r/\varepsilon} E(c(s)c(t))dsdt \\ &\le 2\varepsilon \int_0^{r/\varepsilon}\int_0^{\infty} |E(c(0)c(s))|dsdt \\ &\le 2rc_0. \end{aligned}$$

Denote

$$\eta_t = \int_0^\infty E(c(s+t)|\mathcal{G}_t)ds$$

Combining the estimate (2.23) in the case $p = \infty$ in Proposition 7.2.6. from [4] with our condition (4), we deduce that the stationary process $\{\eta_t,\ t \ge 0\}$ satisfies $|\eta_t| \le c_1$ a.s. for all $t > 0$, with a non-random constant c_1. Moreover,

$$\int_0^t c(r)dr - \eta_t$$

is a square integrable $\mathcal{G}_t$ martingale. Denote it by $\mathcal{N}_t$. Clearly

$$\begin{aligned} W_\varepsilon(t) &= \frac{\sqrt{\varepsilon}}{\bar{c}} \int_0^{t/\varepsilon} c(s)ds \\ &= \frac{\sqrt{\varepsilon}}{\bar{c}}\mathcal{N}_{t/\varepsilon} + \frac{\sqrt{\varepsilon}}{\bar{c}}\eta_{t/\varepsilon}, \end{aligned}$$

and thus we deduce from Doob's inequality

$$
\begin{aligned}
E\Big(\sup_{0\le t\le r}|W_\varepsilon(t)|^2\Big) &\le \frac{2}{\overline{c}^2}E(\sup_{0\le t\le r/\varepsilon}\left(\sqrt{\varepsilon}\mathcal{N}_t\right)^2)+2\frac{c_1^2\varepsilon}{\overline{c}^2}\\
&\le \frac{4}{\overline{c}^2}E((\sqrt{\varepsilon}\mathcal{N}_{r/\varepsilon})^2)+2\frac{c_1^2\varepsilon}{\overline{c}^2}\\
&\le 8E(|W_\varepsilon(r)|^2)+10\frac{c_1^2\varepsilon}{\overline{c}^2}\\
&\le C(\varepsilon+r),
\end{aligned}
$$

provided $C=(16c_0)\vee(10c_1^2/\overline{c}^2)$. Now for $j\ge 1$, $M>0$,

$$
\begin{aligned}
P\left(\sup_{2^{j-1}<r\le 2^j}\frac{|W_\varepsilon(r)|}{(1+r)^{1-\gamma}}\ge M\right) &\le P\left(\sup_{0\le r\le 2^j}|W_\varepsilon(r)|\ge(1+2^{j-1})^{1-\gamma}M\right)\\
&\le \frac{C(\varepsilon+2^j)}{M^2(1+2^{j-1})^{2-2\gamma}}\\
&\le (\varepsilon\vee 1)\frac{2C}{M^2}(1+2^{j-1})^{2\gamma-1}.
\end{aligned}
$$

Summing up over $j\ge 1$, we deduce that

$$
\begin{aligned}
P\left(\xi_\varepsilon^\gamma\ge M\right) &\le 2P\left(\sup_{r>0}\frac{|W_\varepsilon(r)|}{(1+r)^{1-\gamma}}\ge M\right)\\
&\le (\varepsilon\vee 1)\frac{4C}{M^2}\sum_{j=0}^{\infty}(1+2^j)^{2\gamma-1}\\
&\le (\varepsilon\vee 1)\frac{C'}{M^2}.
\end{aligned}
$$

The Lemma is established.

Remark 6. *We can in fact show that, as $\varepsilon\to 0$,*

$$\xi_\varepsilon^\gamma\Rightarrow\sup_{x\in\mathbb{R}}\frac{|W(x)|}{(1+|x|)^{1-\gamma}},$$

provided again $0<\gamma<1/2$, but we shall not use that result.

Before proving the next result, we state an elementary Lemma, which will be needed in the sequel, and whose proof relies on the following obvious inequality : whenever $x>0$, $0<p<2$, $cx^p\le\frac{2-p}{2}(4c)^{\frac{2}{2-p}}+\frac{x^2}{4}$.

Lemma 7. *Let Z be an $N(0,1)$ random variable, $c>0$ and $0<p<2$. Then*

$$\mathbb{E}\exp(c|Z|^p)\le\sqrt{2}\exp\left[\frac{2-p}{2}(4c)^{\frac{2}{2-p}}\right].$$

We next establish the

Proposition 8. *For any $0<\gamma<1/2$, there exists a continuous mapping $\Psi_\gamma:\mathbb{R}_+\to\mathbb{R}_+$ such that*

$$\mathbb{E}\left[\left(\exp\Big(\frac{1}{\sqrt{\varepsilon}}\int_0^t c\Big(\frac{X_s^x}{\varepsilon}\Big)ds\Big)\right)^2\right]\le \Psi_\gamma(\xi_\varepsilon^\gamma). \tag{9}$$

Proof. Since

$$\frac{1}{\sqrt{\varepsilon}}\int_0^t c\left(\frac{X_s^x}{\varepsilon}\right)ds = 2\overline{c}\int_x^{x+B_t} W_\varepsilon(y)dy - 2\overline{c}\int_0^t W_\varepsilon(x+B_s)dB_s,$$

we obtain

$$\begin{aligned}
\mathbb{E}\left(\exp\Big(\frac{1}{\sqrt{\varepsilon}}\int_0^t c\Big(\frac{X_s^x}{\varepsilon}\Big)ds\Big)^2\right) &= \mathbb{E}\exp\left(4\overline{c}\int_x^{x+B_t} W_\varepsilon(y)dy - 4\overline{c}\int_0^t W_\varepsilon(x+B_s)dB_s\right)\\
\le \left(\mathbb{E}\exp\Big(8\overline{c}\int_x^{x+B_t} W_\varepsilon(y)dy\Big)\right)^{1/2} &\left(\mathbb{E}\exp\Big(-8\overline{c}\int_0^t W_\varepsilon(x+B_s)dB_s\Big)\right)^{1/2}
\end{aligned} \tag{10}$$

Clearly, it suffices to estimate each factor on the r.h.s. of (10) separately.

$$\begin{aligned}
\mathbb{E}\exp\Big(8\overline{c}\int_x^{x+B_t} W_\varepsilon(y)dy\Big) &\le \mathbb{E}\exp\Big(8\overline{c}\int_x^{x+B_t} |W_\varepsilon(y)|dy\Big)\\
&\le \mathbb{E}\exp\Big(8\overline{c}\int_x^{x+B_t} \xi_\varepsilon^\gamma(1+|y|)^{1-\gamma}dy\Big)\\
&\le \mathbb{E}\exp\Big(\frac{\xi_\varepsilon^\gamma}{2-\gamma}(1+|y|)^{2-\gamma}\Big|_x^{x+B_t}\Big)\\
&\le \mathbb{E}\exp\Big(\frac{\xi_\varepsilon^\gamma}{2-\gamma}\big((1+|x|)^{2-\gamma}+|B_t|^{2-\gamma}\big)\Big)\\
&\le \sqrt{2}\exp\left[\frac{\xi_\varepsilon^\gamma}{2-\gamma}(1+|x|)^{2-\gamma}+\frac{\gamma}{2}\left(4\frac{\xi_\varepsilon^\gamma}{2-\gamma}t^{1-\gamma/2}\right)^{2/\gamma}\right]
\end{aligned}$$

The second factor on the r.h.s. of (10) can be estimated as follows

$$\begin{aligned}
&\mathbb{E}\exp\Big(-8\overline{c}\int_0^t W_\varepsilon(x+B_s)dB_s\Big)\\
&\le \left(\mathbb{E}\exp\Big(-16\overline{c}\int_0^t W_\varepsilon(x+B_s)dB_s - 128\overline{c}^2\int_0^t W_\varepsilon^2(x+B_s)ds\Big)\right)^{1/2}\\
&\quad\times\left(\mathbb{E}\exp\Big(128\overline{c}^2\int_0^t W_\varepsilon^2(x+B_s)ds\Big)\right)^{1/2}
\end{aligned}$$

The first term on the r.h.s. does not exceed 1. For the second one we have by Jensen's inequality

$$\begin{aligned}
\mathbb{E}\exp\Big(128\bar{c}^2\int_0^t W_\varepsilon^2(x+B_s)ds\Big) &\le t^{-1}\int_0^t \mathbb{E}\exp\left(128\bar{c}^2tW_\varepsilon^2(x+B_s)\right)ds \\
&\le t^{-1}\int_0^t \mathbb{E}\exp\left(128\bar{c}^2t(\xi_\varepsilon^\gamma)^2(1+|x+B_s|)^{2-2\gamma}\right)ds \\
&\le \exp\left[256\bar{c}^2t(\xi_\varepsilon^\gamma)^2(1+|x|)^{2-2\gamma}+\gamma\left(1024\bar{c}^2t^{1-\gamma}(\xi_\varepsilon^\gamma)^2\right)^{1/\gamma}\right].
\end{aligned}$$

The result clearly follows.

We can now proceed with the

Proof of Theorem 1 We treat the sequence $\{u_2^\varepsilon(t,x)\}$ only, and we delete the index 2, as well as the parameters t and x. It suffices to show that for any $\varphi\in C_b(\mathbb{R})$, φ increasing, as $\varepsilon\to 0$,

$$E\varphi\left(\mathbb{E}[g(X)\exp(Y^\varepsilon)]\right)\to E\varphi\left(\mathbb{E}[g(X)\exp(Y)]\right), \tag{11}$$

where $X=X_t^x=x+B_t$,

$$\begin{aligned}
Y^\varepsilon &= 2\bar{c}\left[\int_x^{x+B_t}W_\varepsilon(y)dy-\int_0^t W_\varepsilon(x+B_s)dB_s\right], \\
Y &= 2\bar{c}\left[\int_x^{x+B_t}W(y)dy-\int_0^t W(x+B_s)dB_s\right].
\end{aligned}$$

We note that for all $M>0$, $\varepsilon>0$, $0<\gamma<1/2$,

$$\mathbb{E}[g(X)\exp(Y^\varepsilon)]=\mathbb{E}[g(X)\{\exp(Y^\varepsilon)\wedge M\}]+\rho_{\varepsilon,M},$$

and

$$\begin{aligned}
0\le\rho_{\varepsilon,M} &= \mathbb{E}[g(X)\{\exp(Y^\varepsilon)-M\}\mathbf{1}_{\exp(Y^\varepsilon)>M}] \\
&\le M^{-1}\mathbb{E}[g(X)\{\exp(Y^\varepsilon)\}^2] \\
&\le \frac{\|g\|_\infty}{M}\Psi_\gamma(\xi_\varepsilon^\gamma),
\end{aligned}$$

where we have used Proposition 8. Consequently

$$\begin{aligned}
E\varphi\left(\mathbb{E}[g(X)\{\exp(Y^\varepsilon)\wedge M\}]\right) &\le E\varphi\left(\mathbb{E}[g(X)\exp(Y^\varepsilon)]\right) \\
&\le E\varphi\left(\mathbb{E}[g(X)\{\exp(Y^\varepsilon)\wedge M\}]+\frac{\|g\|_\infty}{M}\Psi_\gamma(\xi_\varepsilon^\gamma)\right).
\end{aligned}$$

Since both $\{W^\varepsilon,\ \varepsilon\le 1\}$ and $\{\xi_\varepsilon^\gamma,\ \varepsilon\le 1\}$ are tight, we have the convergence in law $(W^{\varepsilon_n},\xi_{\varepsilon_n}^\gamma)\Rightarrow(W,Z)$ along some subsequence $\varepsilon_n\to 0$. Moreover for each $M>0$,

$$\mathbb{E}[g(X)\{\exp(Y^\varepsilon)\wedge M\}]=\Phi_M(W^\varepsilon),$$

where

$$\Phi_M(f):=\mathbb{E}\left\{g(X)\left[\exp\left(2\bar{c}\int_x^{x+B_t}f(y)dy-2\bar{c}\int_0^t f(x+B_s)dB_s\right)\wedge M\right]\right\}$$

is a continuous mapping from $C(\mathbb{R})$ into $\mathbb{R}$ (see Lemma 9 below), hence we can take the limit in the above along the subsequence $\{\varepsilon_n\}$, yielding

$$\begin{aligned}
E\varphi\left(\mathbb{E}[g(X)\{\exp(Y)\wedge M\}]\right) &\le \liminf_{n\to\infty} E\varphi\left(\mathbb{E}[g(X)\exp(Y^{\varepsilon_n})]\right)\\
&\le \limsup_{n\to\infty} E\varphi\left(\mathbb{E}[g(X)\exp(Y^{\varepsilon_n})]\right)\\
&\le E\varphi\left(\mathbb{E}[g(X)\{\exp(Y)\wedge M\}] + \frac{\|g\|_\infty}{M}\Psi_\gamma(Z)\right).
\end{aligned}$$

Taking now the limit as $M\to\infty$, we conclude that (11) holds along the subsequence $\{\varepsilon_n\}$. But from any subsequence of the left hand side of (11), we can extract a further subsequence which converges to the right hand side of (11). So the whole sequence converges, and the theorem is established.

Lemma 9. *Whenever $f_n \to f$ in $C(\mathbb{R})$, (i. e. uniformly on compacts),*

$$\int_x^{x+B_t} f_n(y)dy \to \int_x^{x+B_t} f(y)dy \quad \mathbb{P}\ a.\ s.,$$
$$\int_0^t f_n(x+B_s)dB_s \to \int_0^t f(x+B_s)dB_s \quad in\ \mathbb{P}\ probability.$$

Proof The first statement is obvious. The second follows from the following inequality

$$\begin{aligned}
&\mathbb{E}\left[1\wedge\left|\int_0^t [f(x+B_s)-f_n(x+B_s)]dB_s\right|\right]\\
&\qquad\le 3\mathbb{E}\left[1\wedge\left(\int_0^t |f(x+B_s)-f_n(x+B_s)|^2 ds\right)^{1/2}\right],
\end{aligned}$$

and the fact that, as $n\to\infty$,

$$\int_0^t |f(x+B_s)-f_n(x+B_s)|^2 ds \to 0 \quad \mathbb{P}\ \text{a. s.}$$

5. The PDEs for the limits u_1 and u_2

Formally, the limiting SPDE for u_2 reads

$$\begin{cases}
\dfrac{\partial u_2}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u_2}{\partial x^2}(t,x) + \bar{c}u_2(t,x)\circ W(dx), \quad t\ge 0,\ x\in\mathbb{R};\\
u_2(0,x) = g(x), \quad x\in\mathbb{R},
\end{cases} \tag{12}$$

where the above stochastic integral is understood in the sense of the anticipative Stratonovich integral (see [6] or [5]). But it seems very hard to study such an equation. Therefore, we will now change our point of view, by rewriting the original equation (2) in a different way. Note that the last term in (2) reads

$$\bar{c}W'_\varepsilon(x)u_2^\varepsilon(t,x) = \bar{c}\frac{\partial(W_\varepsilon u_2^\varepsilon)}{\partial x}(t,x) - \bar{c}W_\varepsilon(x)\frac{\partial u_2^\varepsilon}{\partial x}(t,x)$$

Consequently equation (2) can be rewritten as

$$\begin{aligned}&\frac{\partial u_2^\varepsilon}{\partial t}(t,x)=\frac{1}{2}\frac{\partial^2 u_2^\varepsilon}{\partial x^2}(t,x)+\overline{c}\frac{\partial(W_\varepsilon u_2^\varepsilon)}{\partial x}(t,x)-\overline{c}W_\varepsilon(x)\frac{\partial u_2^\varepsilon}{\partial x}(t,x),\quad t\ge 0,\ x\in\mathbb{R};\\&u_2^\varepsilon(0,x)=g(x),\quad x\in\mathbb{R}.\end{aligned}\tag{13}$$

Hence the limiting equation can be rewritten as follows

$$\begin{aligned}&\frac{\partial u_2}{\partial t}(t,x)=\frac{1}{2}\frac{\partial^2 u_2}{\partial x^2}(t,x)+\overline{c}\frac{\partial(W u_2)}{\partial x}(t,x)-\overline{c}W(x)\frac{\partial u_2}{\partial x}(t,x),\quad t\ge 0,\ x\in\mathbb{R};\\&u_2(0,x)=g(x),\quad x\in\mathbb{R}.\end{aligned}\tag{14}$$

and similarly for u_1

$$\begin{aligned}&\frac{\partial u_1}{\partial t}(t,x)=\frac{1}{2}\frac{\partial^2 u_1}{\partial x^2}(t,x)+\overline{c}\frac{\partial(W u_1)}{\partial x}(t,x)-\overline{c}W(x)\frac{\partial u_1}{\partial x}(t,x),\quad t\ge 0,\ a<x<b;\\&u_1(0,x)=g(x),\ a<x<b,\quad u_1(t,a)=g(a),\ u_1(t,b)=g(b),\ t\ge 0.\end{aligned}\tag{15}$$

Our aim in the remaining of this section is to establish the two following results

Theorem 10. *The parabolic PDE* (15) *has a unique solution* $u_1\in L^2_{loc}(\mathbb{R}_+;H^1(a,b))$ *a.s., which is given by the Feynman–Kac formula*

$$u_1(t,x):=\mathbb{E}\left[g(X^x_{t\wedge\tau_x})\exp\left(\overline{c}\int_{\mathbb{R}}L^{y-x}_{t\wedge\tau_x}W(dy)\right)\right]\tag{16}$$

Theorem 11. *The parabolic PDE* (14) *has a solution* $u_2\in L^2_{loc}(\mathbb{R}_+;H^1_{loc}(\mathbb{R}))$ *a.s., which is given by the Feynman–Kac formula*

$$u_2(t,x):=\mathbb{E}\left[g(X^x_t)\exp\left(\overline{c}\int_{\mathbb{R}}L^{y-x}_{t}W(dy)\right)\right]\tag{17}$$

Proof of theorem 10: Recall the definition (8) of the smooth approximation $\{W_n\}$ of the Wiener process $\{W\}$, and consider the sequence of "approximating" PDEs

$$\begin{aligned}&\frac{\partial u_1^n}{\partial t}(t,x)=\frac{1}{2}\frac{\partial^2 u_1^n}{\partial x^2}(t,x)+\overline{c}W_n'(x)u_1^n(t,x),\quad t\ge 0,\ a<x<b;\\&u_1^n(0,x)=g(x),\ a<x<b,\quad u_1^n(t,a)=g(a),\ u_1^n(t,b)=g(b),\ t\ge 0.\end{aligned}\tag{18}$$

The Feynman-Kac formula (see e.g.[3]) yields

$$\begin{aligned}u_1^n(t,x)&=\mathbb{E}\left[g(x+B_{t\wedge\tau_x})\exp\left(\overline{c}\int_0^{t\wedge\tau_x}W_n'(x+B_s)ds\right)\right]\\&=\mathbb{E}\left[g(x+B_{t\wedge\tau_x})\exp\left(\overline{c}\int_{\mathbb{R}}W_n'(z)L^{z-x}_{t\wedge\tau_x}dz\right)\right],\end{aligned}\tag{19}$$

where $\{B\}$ stands for a standard Brownian motion defined on $(\Omega,\mathcal{F},\mathbb{P})$ and L^y_s denotes it's local time at time s and point y.

As in the proof of Lemma 4, we get

$$\int_{\mathbb{R}} W_n'(z) L_{t\wedge\tau_x}^{z-x} dz \to \int_{\mathbb{R}} L_{t\wedge\tau_x}^{z-x} W(dz),$$

in $L^2(\Sigma)$, $\mathbb{P}$ a.s., as $n \to \infty$. This allows us to take the limit in (19), provided we establish a uniform integrability estimate analogous to that contained in Lemma 5 and Proposition 8. The argument is essentially the same as that in section **4.**, but simpler than there, and we do not repeat it.

Moreover, the u_1^n equation can be rewritten as

$$\begin{aligned} \frac{\partial u_1^n}{\partial t}(t,x) &= \frac{1}{2}\frac{\partial^2 u_1^n}{\partial x^2}(t,x) + \bar{c}\frac{\partial (W_n u_1^n)}{\partial x}(t,x) - \bar{c} W_n(x)\frac{\partial u_1^n}{\partial x}(t,x), \\ u_1^n(0,x) &= g(x). \end{aligned} \tag{20}$$

Now since for each $\sigma \in \Sigma$, there exists $K(\sigma) < \infty$ such that

$$\sup_{x\in[a,b]} |W(x,\sigma)| \le K(\sigma), \qquad \sup_{x\in[a,b], n\in\mathbb{N}} |W_n(x,\sigma)| \le K(\sigma),$$

it is an easy matter to show that

$$u_1^n \to u_1 \quad \text{in the space } L^2_{\text{loc}}(\mathbb{R}_+; H^1(a,b)) \text{ a.s.},$$

where u_1 is the unique solution in the space $L^2_{\text{loc}}(\mathbb{R}_+; H^1(a,b))$ of the PDE (15).

Proof of theorem 11: The general strategy is the same as that of the proof of Theorem 10. We need to show that u_2, given by the formula

$$u_2(t,x) = \mathbb{E}\left[g(X_t^x) \exp\left(\bar{c} \int_{\mathbb{R}} L_t^{y-x} W(dy) \right) \right],$$

belongs to $L^2_{\text{loc}}(\mathbb{R}_+; H^1_{\text{loc}}(\mathbb{R}))$, and solves the parabolic PDE (14).

For that sake, we define a new approximation of the Wiener process W, as

$$\bar{W}_n(x) = [W_n(x) \wedge n] \vee (-n),$$

where $W_n(x)$ has been defined in (8). Let u_2^n denote the solution of the approximating PDE

$$\begin{aligned} \frac{\partial u_2^n}{\partial t}(t,x) &= \frac{1}{2}\frac{\partial^2 u_2^n}{\partial x^2}(t,x) + \bar{c}\bar{W}_n'(x) u_2^n(t,x), \quad t \ge 0,\ x \in \mathbb{R}; \\ u_2^n(0,x) &= g(x),\ x \in \mathbb{R}. \end{aligned} \tag{21}$$

Again from the Feynman–Kac formula,

$$\begin{aligned} u_2^n(t,x) &= \mathbb{E}\left[g(x+B_t) \exp\left(\bar{c} \int_0^t \bar{W}_n'(x+B_s) ds \right) \right] \\ &= \mathbb{E}\left[g(x+B_t) \exp\left(\bar{c} \int_{\mathbb{R}} \bar{W}_n'(z) L_t^{z-x} dz \right) \right], \end{aligned} \tag{22}$$

where $\{B\}$ stands for a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and L^y_s denotes it's local time at time s and point y.

As in the proof of Theorem 10, as $n \to \infty$,

$$\mathbb{E}\left[g(x+B_t)\exp\left(\bar{c}\int_{\mathbb{R}} \bar{W}'_n(z)L_t^{z-x}dz\right)\right] \to \mathbb{E}\left[g(x+B_t)\exp\left(\bar{c}\int_{\mathbb{R}} L_t^{z-x}W(dz)\right)\right].$$

For each $M>0$, we now write an equation satisfied by u_2^n on $\mathbb{R}_+ \times [-M, M]$.

$$\begin{aligned}
&\frac{\partial u_2^n}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 u_2^n}{\partial x^2}(t,x) + \bar{c}\frac{\partial (W_n u_2^n)}{\partial x}(t,x) - \bar{c}W_n(x)\frac{\partial u_2^n}{\partial x}(t,x),\\
&\qquad t>0, \quad -M<x<M\\
&u_2^n(0,x) = g(x), \quad u_2^n(t,-M) = \xi^n(t,-M),\ u_2^n(t,M) = \xi^n(t,M),
\end{aligned} \tag{23}$$

where ξ^n denotes the right hand side of (22). It is now easy to show that

$$v_2^n(t,x) := u_2^n(t,x) - x\frac{\xi^n(t,M)-\xi^n(t,-M)}{2M} - \frac{\xi^n(t,M)+\xi^n(t,-M)}{2}$$

solves the equation (23) but with homogeneous Dirichlet boundary conditions. Now v_2^n converges strongly in $L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1_0(-M,M))$, P a. s., towards the solution of the parabolic PDE

$$\begin{aligned}
&\frac{\partial v_2}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 v_2}{\partial x^2}(t,x) + \bar{c}\frac{\partial (W v_2)}{\partial x}(t,x) - \bar{c}W(x)\frac{\partial v_2}{\partial x}(t,x),\\
&\qquad t\ge 0,\ -M<x<M,\ ;\\
&v_2(0,x) = g(x),\ -M<x<M, \quad v_2(t,-M) = v_2(t,M) = 0.
\end{aligned} \tag{24}$$

We conclude the $u_2 := \lim_{n\to\infty} u_2^n$ belongs to the space $L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1_{\mathrm{loc}}(\mathbb{R}))$ a. s., and it satisfies (14) in the variational sense, i.e. for any $t>0$, any $\varphi \in C^2(\mathbb{R})$ with compact support, and a. s. ($<.,.>$ denotes the scalar product in $L^2(\mathbb{R})$),

$$\begin{aligned}
&< u_2(t), \varphi >=< g, \varphi >\\
&+\int_0^t \left[\frac{1}{2} < u_2(s), \varphi'' > -\bar{c} < W u_2(s), \varphi' > -\bar{c} < W\frac{\partial u_2}{\partial x}(s), \varphi >\right] ds.
\end{aligned}$$

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