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Averaging of nonstationary parabolic operators with large lower order terms

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Abstract: In this note we study the homogenization problem for a singularly perturbed non-stationary parabolic operator with lower order terms. We assume a self-similar scaling of spatial and temporal variables and prove the existence of rapidly moving coordinates in which a solution of the corresponding Cauchy problem is asymptotically given as the product of the ground state of periodic cell problem and a solution of parabolic equation with constant coefficients.

1. Introduction.

This paper deals with a homogenization problem for a non stationary parabolic equation of the form

$$\frac{\partial}{\partial t}u^\varepsilon = \frac{\partial}{\partial x_i}\Big(a_{ij}\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)\frac{\partial}{\partial x_j}u^\varepsilon\Big) + \frac{1}{\varepsilon}b_i\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)\frac{\partial}{\partial x_i}u^\varepsilon + \frac{1}{\varepsilon^2}c\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)u^\varepsilon \tag{1}$$

with a small positive parameter ε, all the coefficients being periodic both in spatial and temporal variables. We will mostly study a Cauchy problem for this equation.

The main difficulties here are coming from both fast oscillation of the coefficients not only in spatial variables but also in time and the presence of large parameters in front of the lower order terms. In particular, due to the presence of a large convection term in (1) we have to introduce rapidly moving coordinates, while the presence of a large potential in the equation results in the fast oscillation of u^ε so that a proper factorization is required. Also, since the coefficients of (1) are not stationary, the corresponding cell eigenproblem involves a parabolic operator with periodic boundary conditions in all the variables including time.

The appearance of the large factors in (1) is natural in the framework of long term behaviour of solutions to a second order parabolic equation with lower order terms. Indeed, if we consider an equation

$$\frac{\partial}{\partial s}u = \frac{\partial}{\partial z_i}\Big(a_{ij}(s,z)\frac{\partial}{\partial z_j}u\Big) + b_i(s,z)\frac{\partial}{\partial z_i}u + c(s,z)u \tag{2}$$

with periodic coefficients, and make the diffusive (self-similar) change of variables $x = z/\varepsilon$, $t = s/\varepsilon^2$, then after multiplying by ε^2 we arrive at an equation of the form (1).

Previously, the elliptic and stationary parabolic operators with lower order terms were studied in [6], [11]. It was shown that in this case the factorization with the principal eigenfunction of the elliptic cell problem reduces the original operator to that without zero order terms and with divergence free first order terms.

An example of non-stationary equations was considered in [5], it was supposed that time oscillation is slower than the spatial one, and that the effective drift is equal to zero.

Some other model problems for parabolic operators with large lower order terms were studied in [1], [9], [4], [2].

The main result of the present work is Theorem 4 which states that a solution of Cauchy problem for the equation (1) admits a representation

$$u^\varepsilon = p\Big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\Big)\exp(-\lambda_0 t/\varepsilon^2)(v^0(t, x - \frac{\bar{b}}{\varepsilon}t) + o(1)),$$

where $(\lambda_0,\ p(s,z)$ is the principal eigenpair of the periodic cell problem, $\bar{b}$ is a constant vector, $o(1)$ vanishes as $\varepsilon \to 0$, and $v^0(t,x)$ is a solution of a homogenized parabolic equation with constant coefficients and adapted initial conditions, see Section .

In order to justify this limit behaviour of u^ε we make use of asymptotic expansion technique and build three leading terms of formal asymptotic series including the initial layer, the solvability of auxiliary cell problems being studied in Section . Then the a priori estimates obtained in Section , allow us to conclude.

2. Setting the problem

Given $[0,1]^{n+1}$-periodic functions $a_{ij}(s,y)$, $b_i(s,y)$ and $c(s,y)$ such that

$$\begin{aligned} &a_{ij}\xi_i\xi_j \geq c_1|\xi|^2, \quad \xi \in \mathbb{R}^n,\ c_1 > 0;\\ &|a_{ij}| \leq c_2, \quad |b_i| \leq c_2, \quad |c| \leq c_2, \end{aligned} \tag{3}$$

we consider on the set $(0,T)\times\mathbb{R}^n$ the Cauchy problem

$$\begin{gathered} \frac{\partial}{\partial t}u^\varepsilon = \frac{\partial}{\partial x_i}\Big(a_{ij}\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)\frac{\partial}{\partial x_j}u^\varepsilon\Big) + \frac{1}{\varepsilon}b_i\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)\frac{\partial}{\partial x_i}u^\varepsilon + \frac{1}{\varepsilon^2}c\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)u^\varepsilon,\\ u^\varepsilon|_{t=0} = u_0(x) \in L^2(\mathbb{R}^n) \end{gathered} \tag{4}$$

with a small positive parameter ε, $\varepsilon \to 0$. Under the assumptions (3) this problem has for each $\varepsilon > 0$ a unique solution $u^\varepsilon \in L^2(0,T;H^1(\mathbb{R}^n)) \cap C(0,T;L^2(\mathbb{R}^n))$, no regularity of the coefficients except for the measurability being assumed.

We are aimed at studying the asymptotic behaviour of u^ε as $\varepsilon \to 0$.

3. Auxiliary problems

Our analysis relies on the following auxiliary problems. In the cylinder $Q_0 = (0, +\infty) \times \mathbb{T}^n$ consider the equation

$$\frac{\partial}{\partial s}p - \frac{\partial}{\partial z_i}a_{ij}(s,z)\frac{\partial}{\partial z_j}p - b_i(s,z)\frac{\partial}{\partial z_i}p - c(s,z)p = 0; \tag{5}$$

here and afterwards we identify periodic functions with the corresponding functions on the standard torus $\mathbb{T}^n$. For any $q(z) \in L^2(\mathbb{T}^n)$ denote by $(S_s q)(z)$ a solution $p(s,z)$ of equation (5) with the initial condition

$$p|_{s=0} = q(z).$$

Due to the Nash estimates (see [8]) there are constants $\gamma > 0$ and $C > 0$ which only depend on c_1 and c_2 in (3), such that

$$\|S_1 q\|_{C^\gamma}(\mathbb{T}^n) \le C\|q\|_{L^2(\mathbb{T}^n)}$$

Therefore, S_1 is a compact operator in $L^2(\mathbb{T}^n)$ and in $C^{\gamma/2}(\mathbb{T}^n)$, and it has the same spectrum in the both mentioned spaces.

By the maximum principle, for any $q(z) > 0$ the solution $p(s,z)$ is positive. Thus according to Theorem 9.2 and 11.5 in [7] there is a simple real eigenvalue $\lambda_0 > 0$ of the operator S_1 such that the rest of the spectrum of S_1 belongs to the disk $\lambda \in \mathbb{C} : \|\lambda\| < \lambda_0$. Moreover, the corresponding eigenfunction $q_0(z)$ is also real and, under proper normalization, strictly positive.

If we denote $\Lambda_0 = \ln \lambda_0$, then the function

$$p_0(s,z) = \exp(-\Lambda_0 s)(S_s q_0)(z)$$

is a $[0,1]^{n+1}$-periodic (that is periodic both in spatial and temporal variables) solution of the equation

$$\frac{\partial}{\partial s}p - \frac{\partial}{\partial z_i}\Big(a_{ij}(s,z)\frac{\partial}{\partial z_j}p\Big) - b_i(s,z)\frac{\partial}{\partial z_i}p - c(s,z)p = \Lambda_0 p \tag{6}$$

We summarize this in the following statement

Proposition 1 *There is a $\Lambda_0 \in R$ such that the equation (6) has a positive periodic in s and z solution. This Λ_0 is unique, and (6) has a unique, up to a multiplicative constant, solution.*

In what follows the choice of p_0 is fixed by the normalization condition

$$\int\limits_{\mathbb{T}^{n+1}} p_0(s,z)\,ds\,dz = 1 \tag{7}$$

By the same arguments the adjoint operator

$$-\frac{\partial}{\partial s}p - \frac{\partial}{\partial z_i}\Big(a_{ij}(s,z)\frac{\partial}{\partial z_j}p\Big) + \frac{\partial}{\partial z_i}(b_i(s,z)p) - c(s,z)p \tag{8}$$

has a simple eigenvalue at Λ_0, and the corresponding eigenfunction is positive. We denote it by $p_0^*(s,z)$.

The statement below is a consequence the Fredholm theorem

Proposition 2 *The equation*

$$\frac{\partial}{\partial s}v - \frac{\partial}{\partial z_i}\Big(a_{ij}(s,z)\frac{\partial}{\partial z_j}v\Big) - b_i(s,z)\frac{\partial}{\partial z_i}v - (c(s,z)+\Lambda_0)v = f(s,z), \tag{9}$$

$f \in L^2(\mathbb{T}^{n+1})$, is solvable on $\mathbb{T}^{n+1}$ if and only if

$$\int_{\mathbb{T}^{n+1}} f(s,z)p_0^*(s,z)dsdz = 0. \tag{10}$$

A solution is unique up to an additive constant.

4. A priori estimates

We factorize a solution of problem (4) as follows

$$u^\varepsilon(t,x) = \exp\big(\Lambda_0 t/\varepsilon^2\big)p_0\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)v^\varepsilon(t,x). \tag{11}$$

This allows us to get rid of the exponential growth or decay of the solution and of its fast oscillation. We are going to show that v^ε admits uniform in ε energy estimate. To this end we substitute (11) in the equation (4) and use the equation (6). This gives after straightforward rearrangements

$$\begin{gathered} p_0\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)\frac{\partial}{\partial t}v^\varepsilon = \operatorname{div}\Big(p_0\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)a\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)\nabla v^\varepsilon\Big) \\ +\frac{1}{\varepsilon}a\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)\nabla_z p_0\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)\cdot\nabla v^\varepsilon + \frac{1}{\varepsilon}p_0\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)b\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)\cdot\nabla v^\varepsilon \\ v^\varepsilon|_{t=0} = p_0^{-1}(0,\frac{x}{\varepsilon})u_0(x). \end{gathered} \tag{12}$$

In what follows for the sake of brevity we denote for a generic function $g = g(s,z)$

$$g^\varepsilon(t,x) = g\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big), \quad \nabla_z g\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big) = \nabla_z g^\varepsilon(t,x) = \nabla_z g\big(\frac{t}{\varepsilon^2},z\big)|_{z=\frac{x}{\varepsilon}},$$

$$\nabla_s g\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big) = \nabla_s g^\varepsilon(t,x) = \nabla_s g\big(s,\frac{z}{\varepsilon}\big)|_{s=\frac{t}{\varepsilon^2}}.$$

We will also use the notation

$$\tilde a(s,z) = p_0(s,z)a(s,z), \quad \tilde b(s,z) = a(s,z)\nabla_z p_0(s,z) + p_0(s,z)b(s,z).$$

Multiplying the last equation by $p_0^*\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)v^\varepsilon$ and integrating the resulting relation over $[0,T]\times\mathbb{R}^n$, we get after simple transformations

$$-\frac{1}{2\varepsilon^2}\int\limits_0^T\int\limits_{\mathbb{R}^n}\big((p_0^{*,\varepsilon}(x,t)\partial_s p_0^\varepsilon(x,t) + p_0^\varepsilon(x,t)\partial_s p_0^{*,\varepsilon}(x,t)\big)(v^\varepsilon)^2dxdt+$$

$$\frac{1}{2}\int\limits_{\mathbb{R}^n} p_0^\varepsilon(x,T)p_0^{*,\varepsilon}(x,T)(v^\varepsilon(x,T))^2dx - \frac{1}{2}\int\limits_{\mathbb{R}^n} p_0^\varepsilon(x,0)p_0^{*,\varepsilon}(x,0)(v^\varepsilon(x,0))^2dx =$$

$$-\int\limits_0^T\int\limits_{\mathbb{R}^n} p_0^\varepsilon(x,t)p_0^{*,\varepsilon}(x,t)a^\varepsilon(x,t)\nabla v^\varepsilon\cdot\nabla v^\varepsilon dxdt +$$

$$\frac{1}{2\varepsilon}\int\limits_0^T\int\limits_{\mathbb{R}^n}\big\{p_0^{*,\varepsilon}(x,t)a^\varepsilon(x,t)\nabla_z p_0^\varepsilon(x,t) - p_0^\varepsilon(x,t)a^\varepsilon(x,t)\nabla_z p_0^{*,\varepsilon}(x,t) +$$

$$b^\varepsilon(x,t)p_0^\varepsilon(x,t)p_0^{*,\varepsilon}(x,t)\big\}\nabla((v^\varepsilon)^2)dxdt.$$

Integrating by parts on the right hand side of this relation gives

$$\frac{1}{2}\int\limits_{\mathbb{R}^n} p_0^\varepsilon(x,T)p_0^{*,\varepsilon}(x,T)(v^\varepsilon(x,T))^2dx +$$

$$\int\limits_0^T\int\limits_{\mathbb{R}^n} p_0^\varepsilon(x,t)p_0^{*,\varepsilon}(x,t)a^\varepsilon(x,t)\nabla v^\varepsilon\cdot\nabla v^\varepsilon dxdt =$$

$$= \frac{1}{2}\int\limits_{\mathbb{R}^n} p_0^\varepsilon(x,0)p_0^{*,\varepsilon}(x,0)(u_0(x))^2dx + \frac{1}{2\varepsilon^2}\int\limits_0^T\int\limits_{\mathbb{R}^n}\big\{p_0^{*,\varepsilon}(x,t)\partial_s p_0^\varepsilon(x,t) +$$

$$p_0^\varepsilon(x,t)\partial_s p_0^{*,\varepsilon}(x,t) + a^\varepsilon(x,t)\nabla_z p_0^\varepsilon(x,t)\nabla_z p_0^{*,\varepsilon}(x,t)$$

$$-p_0^{*,\varepsilon}(x,t)\nabla_z(a^\varepsilon(x,t)\nabla_z p_0^\varepsilon(x,t)) - a^\varepsilon(x,t)\nabla_z p_0^\varepsilon(x,t)\nabla_z p_0^{*,\varepsilon}(x,t) +$$

$$p_0^\varepsilon(x,t)\nabla_z(a^\varepsilon(x,t)\nabla_z p_0^{*,\varepsilon}(x,t)) - p_0^\varepsilon(x,t)\nabla_z\cdot(b^\varepsilon(x,t)p_0^{*,\varepsilon}(x,t))$$

$$-p_0^{*,\varepsilon}b^\varepsilon(x,t)\nabla_z p_0^\varepsilon(x,t)\big\}(v^\varepsilon)^2dxdt = \frac{1}{2}\int\limits_{\mathbb{R}^n} p_0^\varepsilon(x,0)p_0^{*,\varepsilon}(x,0)(u_0(x))^2dx +$$

$$\frac{1}{2\varepsilon^2}\int\limits_0^T\int\limits_{\mathbb{R}^n}\big\{\partial_s p_0^\varepsilon(x,t) - \nabla_z(a^\varepsilon(x,t)\nabla_z p_0^\varepsilon(x,t)) - b^\varepsilon(x,t)\nabla_z p_0^\varepsilon(x,t)$$

$$-c^\varepsilon(x,t)p_0^\varepsilon(x,t)\big\}p_0^{*,\varepsilon}(x,t))(v^\varepsilon)^2dxdt + \frac{1}{2\varepsilon^2}\int\limits_0^T\int\limits_{\mathbb{R}^n}\big\{\partial_s p_0^{*,\varepsilon}(x,t) +$$

$$\nabla_z(a^\varepsilon(x,t)\nabla_z p_0^{*,\varepsilon}(x,t)) - \nabla_z\cdot(b^\varepsilon(x,t)p_0^{*,\varepsilon}(x,t)) + c^\varepsilon(x,t)p_0^{*,\varepsilon}(x,t)\big\}p_0^\varepsilon(x,t)(v^\varepsilon)^2dxdt$$

$$= \frac{1}{2}\int\limits_{\mathbb{R}^n} p_0^\varepsilon(x,0)p_0^{*,\varepsilon}(x,0)(u_0(x))^2dx.$$

This implies the bound

$$\|v^\varepsilon\|^2_{L^\infty(0,T;L^2(\mathbb{R}^n))} + \|v^\varepsilon\|^2_{L^2(0,T;H^1(\mathbb{R}^n))} \le C\|u_0\|^2_{L^2(\mathbb{R}^n)}. \tag{13}$$

In the presence of the right hand side the equation (12) takes the form

$$p_0^\varepsilon(t,x)\frac{\partial}{\partial t}w^\varepsilon = \mathrm{div}(p_0^\varepsilon(t,x)a^\varepsilon(t,x)\nabla w^\varepsilon)+$$
$$\frac{1}{\varepsilon}\Big(a^\varepsilon(t,x)\nabla_z p_0^\varepsilon(t,x)+p_0^\varepsilon(t,x)b^\varepsilon(t,x)\Big)\cdot\nabla w^\varepsilon + f(t,x)+\mathrm{div}f_1(t,x) \tag{14}$$
$$w^\varepsilon|_{t=0}=w_0(x).$$

In this case the following estimate can be obtained in the same way as (13)

$$\|w^\varepsilon\|^2_{L^\infty(0,T;L^2(\mathbb{R}^n))}+\|w^\varepsilon\|^2_{L^2(0,T;H^1(\mathbb{R}^n)}\le$$
$$\le C\|w_0\|^2_{L^2(\mathbb{R}^n)}+C\|f\|^2_{L^2(0,T;L^2(\mathbb{R}^n))}+\varepsilon^{-1}C\|f_1\|^2_{L^2(0,T;L^2(\mathbb{R}^n))}. \tag{15}$$

5. Asymptotic expansion

We now proceed by constructing three leading terms of the asymptotic expansion of v^ε. As usually, the asymptotic expansion technique requires some regularity of the data. Here we assume that the initial function in (4) is smooth enough, say C^2 function, and has a finite support. Non-smooth data will be discussed later on.

We represent v^ε as follows

$$v^\varepsilon \sim v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)+\varepsilon\chi\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)\nabla_x v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)+ \tag{16}$$
$$+\varepsilon^2\psi^{ij}\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)\frac{\partial^2}{\partial x_i\partial x_j}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)+\dots\ ,$$

where $\bar b$ is a constant vector, $\chi^i(s,z)$ and $\psi^{ij}(s,z)$ are periodic in s and z functions, $v^0(t,x)$ does not depend on ε. All of them are to be determined.

Identification of $\bar b$.

Substituting the expansion (16) in the equation (12) and collecting power-like terms in the resulting relation gives the following equations

$$\Big(p_0(s,z)\frac{\partial}{\partial s}\chi^k-\frac{\partial}{\partial y_i}\big(\tilde a_{ij}(s,y)\frac{\partial}{\partial y_j}\chi^k\big)-\tilde b_i(s,y)\frac{\partial}{\partial y_j}\chi^k+$$
$$\frac{\partial}{\partial z_i}\tilde a_{ik}(s,z)+\tilde b_k(s,z)+p_0(s,y)\bar b_k\Big)\frac{\partial}{\partial x_k}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)=0. \tag{17}$$

and

$$p_0(s,y)\frac{\partial}{\partial t}v^0\big(t,x-\frac{\bar b}{\varepsilon}s\big)\Big|_{s=t}=\Big(-p_0(s,z)\frac{\partial}{\partial s}\psi^{ij}+$$
$$\frac{\partial}{\partial y_k}\big(\tilde a_{kl}(s,y)\frac{\partial}{\partial y_l}\psi^{ij}\big)-\tilde b_i(s,y)\frac{\partial}{\partial y_i}\psi^{ij}\Big)\frac{\partial^2}{\partial x_i\partial x_j}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)+$$
$$\Big(\tilde a_{kl}(s,z)+\tilde a_{kj}(s,z)\frac{\partial}{\partial y_j}\chi^l(s,y)+\frac{\partial}{\partial y_i}(\tilde a_{il}(s,z)\chi^k(s,y))+ \tag{18}$$
$$\tilde b_k(s,z)\chi^l(s,y)+p_0(s,y)\bar b_k\chi^l(s,y)\Big)\frac{\partial^2}{\partial x_k\partial x_l}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big).$$

Considering (6) and (11), one can show that the vector-function $\theta(s,z) = p_0(s,z)\chi(s,z)$ satisfies the equation

$$\Big(\frac{\partial}{\partial s}\theta^k - \frac{\partial}{\partial y_i}\big(a_{ij}(s,y)\frac{\partial}{\partial y_j}\theta^k\big) - b_i(s,y)\frac{\partial}{\partial y_j}\theta^k - (c(s,z)-\Lambda_0)\theta^k + \\ + \frac{\partial}{\partial z_i}a_{ik}(s,z) + b_k(s,z) + \bar{b}_k\Big)\frac{\partial}{\partial x_k}v^0\big(t, x - \frac{\bar{b}}{\varepsilon}t\big) = 0. \tag{19}$$

By (9) this equation has a periodic solution iff

$$\bar{b} = \int\limits_{\mathbb{T}^{n+1}} (\mathrm{div}\, a(s,z) + b(s,z))p_0^*(s,z)\, dz\, ds. \tag{20}$$

Thus $\bar{b}$ is determined uniquely.

Derivation of the limit equation.

Our next step is to write down the compatibility condition for the equation (18). To this end we first observe that the function p_0^* satisfies the equation

$$\frac{\partial}{\partial s}\big(p_0(s,z)p_0^*(s,z)\big) + \mathrm{div}\big(\tilde{a}(s,z)\nabla p_0^*(s,z)\big) - \mathrm{div}\big(\tilde{b}(s,z)p_0^*(s,z)\big) = 0,$$

moreover, a solution is unique up to an additive constant. Indeed, if we denote

$$\mathcal{A} = \frac{\partial}{\partial s} - \frac{\partial}{\partial z_i}a_{ij}(s,z)\frac{\partial}{\partial z_j} - b_i(s,z)\frac{\partial}{\partial z_i} - (c(s,z)+\Lambda_0),$$

$$\tilde{\mathcal{A}} = p_0\frac{\partial}{\partial s} - \frac{\partial}{\partial z_i}\tilde{a}_{ij}(s,z)\frac{\partial}{\partial z_j} - \tilde{b}_i(s,z)\frac{\partial}{\partial z_i},$$

then, by the definition of $\tilde{\mathcal{A}}$, we have $\tilde{\mathcal{A}}g = \mathcal{A}(p_0 g)$, therefore the kernels of the adjoint operators coincide.

By the Fredholm theorem the equation (18) has a periodic solution $\psi = \psi^{ij}(s,z)$ if the following relation holds

$$\Big(\int\limits_{\mathbb{T}^{n+1}} p_0(s,y)p_0^*(s,y)\, dsdy\Big)\frac{\partial}{\partial t}v^0\big(t, x - \frac{\bar{b}}{\varepsilon}r\big)\Big|_{r=t} = \\ \Big(\int\limits_{\mathbb{T}^{n+1}} p_0^*(s,z)\Big\{\tilde{a}_{kl}(s,z) + \tilde{a}_{kj}(s,z)\frac{\partial}{\partial y_j}\chi^l(s,y) + \frac{\partial}{\partial y_i}(\tilde{a}_{il}(s,z)\chi^k(s,y)) + \\ \tilde{b}_k(s,z)\chi^l(s,y) + p_0(s,y)\bar{b}_k\chi^l(s,y)\Big\}\, dsdz\Big)\frac{\partial^2}{\partial x_k \partial x_l}v^0\big(t, x - \frac{\bar{b}}{\varepsilon}t\big).$$

Thus we end up with the following equation for the function $v^0(t,x)$):

$$\bar{\sigma}\frac{\partial}{\partial t}v^0(t,x) = \bar{a}^{ij}\frac{\partial^2}{\partial x_i \partial x_j}v^0(t,x) \tag{21}$$

with

$$\bar{\sigma} = \int\limits_{\mathbb{T}^{n+1}} p_0(s,y)p_0^*(s,y)\,dsdy,$$

$$\begin{aligned}\bar{a}_{ij} = \int\limits_{\mathbb{T}^{n+1}} p_0^*(s,z)\Big\{\tilde{a}_{ij}(s,z) + \tilde{a}_{ik}(s,z)\frac{\partial}{\partial y_k}\chi^j(s,y) + \\ \frac{\partial}{\partial y_k}(\tilde{a}_{ki}(s,z)\chi^j(s,y)) + \tilde{b}_i(s,z)\chi^j(s,y) + p_0(s,y)\bar{b}_i\chi^j(s,y)\Big\}\,dsdz.\end{aligned} \tag{22}$$

Since both p_0 and p_0^* are strictly positive functions, we have $\bar{\sigma} > 0$.

Ellipticity of the effective equation.

Let us show that the effective matrix $\bar{a}_{ij}$ is positive definite. To this end we are going to show that $\hat{a}\xi\cdot\xi = \bar{a}\xi\cdot\xi$ for any $\xi\in\mathbb{R}^n$ with

$$\hat{a}_{ij} = \int\limits_{\mathbb{T}^{n+1}} p_0^*(s,z)(I+\nabla\chi(s,z))^t\tilde{a}(s,z)(I+\nabla\chi(s,z))\,dzds, \tag{23}$$

where the index t stands for an adjoint matrix. Indeed,

$$\hat{a}_{ij}\xi^i\xi^j = \int\limits_{\mathbb{T}^{n+1}} \xi^i\xi^j p_0^*(\delta_{ik} + \frac{\partial}{\partial z_k}\chi^i)\tilde{a}_{kl}(\delta_{lj} + \frac{\partial}{\partial z_l}\chi^j)\,dzds =$$

$$\int\limits_{\mathbb{T}^{n+1}}\xi^i\xi^j p_0^*\Big\{\tilde{a}_{ij} + \tilde{a}_{ik}\frac{\partial}{\partial y_k}\chi^j\Big\}\,dzds + \int\limits_{\mathbb{T}^{n+1}} \xi^i\xi^j p_0^*\Big\{\tilde{a}_{lj} + \tilde{a}_{lk}\frac{\partial}{\partial y_k}\chi^j\Big\}\frac{\partial}{\partial y_l}\chi^i\,dzds.$$

The second integral on the r.h.s. can be rearranged as follows

$$\int\limits_{\mathbb{T}^{n+1}} \xi^i\xi^j p_0^*\Big\{\tilde{a}_{lj} + \tilde{a}_{lk}\frac{\partial}{\partial y_k}\chi^j\Big\}\frac{\partial}{\partial y_l}\chi^i\,dzds = -\int\limits_{\mathbb{T}^{n+1}} \xi^i\xi^j\chi^i\frac{\partial}{\partial y_l}p_0^*\Big\{\tilde{a}_{lj} + \tilde{a}_{lk}\frac{\partial}{\partial y_k}\chi^j\Big\}\,dzds$$

$$-\int\limits_{\mathbb{T}^{n+1}} \xi^i\xi^j p_0^*\chi^i\frac{\partial}{\partial y_l}\Big\{\tilde{a}_{lj} + \tilde{a}_{lk}\frac{\partial}{\partial y_k}\chi^j\Big\}\,dzds = \int\limits_{\mathbb{T}^{n+1}} \xi^i\xi^j p_0^*\frac{\partial}{\partial y_l}\big(\tilde{a}_{lj}\chi^i\big)\,dzds +$$

$$-\frac{1}{2}\int\limits_{\mathbb{T}^{n+1}} \xi^i\xi^j\tilde{a}_{lk}\frac{\partial}{\partial y_l}p_0^*\frac{\partial}{\partial y_k}(\chi^i\chi^j)\,dzds - \int\limits_{\mathbb{T}^{n+1}} \xi^i\xi^j p_0^*\chi^i\Big(p_0\frac{\partial}{\partial s}\chi^j -$$

$$-\tilde{b}_j - \tilde{b}_k\frac{\partial}{\partial y_k}\chi^j - p_0\bar{b}_i\Big)\,dzds = \int\limits_{\mathbb{T}^{n+1}} \xi^i\xi^j p_0^*\Big\{\frac{\partial}{\partial y_l}\big(\tilde{a}_{lj}\chi^i\big) + \chi^i\tilde{b}_j + \chi^i p_0\bar{b}_j\Big\}\,dzds +$$

$$+\frac{1}{2}\int\limits_{\mathbb{T}^{n+1}} \xi^i\xi^j\chi^i\chi^j\Big\{\frac{\partial}{\partial s}(p_0p_0^*) + \frac{\partial}{\partial y_k}\Big(\tilde{a}_{lk}\frac{\partial}{\partial y_l}p_0^*\Big) - \frac{\partial}{\partial y_k}(\tilde{b}_kp_0^*)\Big\}\,dzds =$$

$$\int\limits_{\mathbb{T}^{n+1}} \xi^i\xi^j p_0^*\Big\{\frac{\partial}{\partial y_l}\big(\tilde{a}_{lj}\chi^i\big) + \chi^i\tilde{b}_j + \chi^i p_0\bar{b}_i\Big\}\,dzds$$

Combining the last two relations we conclude that $\hat{a}\xi \cdot \xi = \bar{a}\xi \cdot \xi$. The positive definiteness of $\hat{a}$ is evident.

The effective initial condition.

Also, we have to provide an initial condition for the function v^0. Since $v^\varepsilon(0,x) = p_0^{-1}\big(0, \frac{x}{\varepsilon}\big)$ is a rapidly oscillating function, we cannot just borrow the initial conditions from (4). Instead we consider the following auxiliary Cauchy problem

$$\frac{\partial}{\partial s}\zeta_0 = \mathrm{div}\big(\tilde{a}(s,z)\nabla\zeta_0\big) + \tilde{b}(s,z)\nabla\zeta_0, \quad (s,z) \in (0,\infty) \times \mathbb{T}^n \tag{24}$$
$$\zeta_0(0,z) = p_0^{-1}(0,z).$$

It is known (see [8]) that this problem has a unique solution. Moreover, using the maximum principle one can show that this solution converges at exponential rate to a constant which in fact is equal to

$$\bar{p} = \int\limits_{\mathbb{T}^n} p_0^{-1}(0,z) p_0^*(0,z) dz.$$

In other words, there is a $\gamma > 0$ such that

$$\|\zeta_0(s,\cdot) - \bar{p}\|_{L^\infty(T^n)} \le Ce^{-\gamma s}, \qquad \int_t^{t+1} \|\nabla\zeta_0(s,\cdot)\|_{L^2(T^n)} ds \le Ce^{-\gamma t} \tag{25}$$

for all positive t and s.

The limit equation (21) should now be equipped with the initial condition

$$v^0(0,x) = \bar{p}u_0(x). \tag{26}$$

Main results

Let us recall the definition of $\bar{b}$, $\bar{\sigma}$ and $\bar{a}$:

$$\bar{b} = \int\limits_{\mathbb{T}^{n+1}} (\mathrm{div}\, a(s,z) + b(s,z)) p_0^*(s,z)\, dz\, ds,$$

$$\bar{\sigma} = \int\limits_{\mathbb{T}^{n+1}} p_0(s,y) p_0^*(s,y)\, dsdy,$$

$$\bar{a}_{ij} = \int\limits_{\mathbb{T}^{n+1}} p_0^*(s,z)(I + \nabla\chi(s,z))^t \tilde{a}(s,z)(I + \nabla\chi(s,z))\, dzds.$$

The asymptotic behaviour of v^ε is described by the following statement.

Theorem 3 *Under our standing assumptions the solution v^ε of problem (12) satisfies the following limit relation*

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\mathbb{R}^n} \Big(v^\varepsilon(t,x) - v^0\big(t, x - \frac{\bar{b}}{\varepsilon}t\big)\Big)^2 dxdt \; = \; 0.$$

where v^0 is a solution of Cauchy problem

$$\bar\sigma\frac{\partial}{\partial t}v^0(t,x)=\bar a^{ij}\frac{\partial^2}{\partial x_i\partial x_j}v^0(t,x)\qquad\text{in } (0,T)\times\mathbb{R}^n,$$
$$v^0(0,x)=\bar p u_0(x). \tag{27}$$

Proof:
We combine the interior expansion (16) with the initial layer of the form

$$\Big(\tilde\zeta_0\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)-1\Big)v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)+\varepsilon\zeta_1\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)\nabla v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big),$$

where $\tilde\zeta_0(s,z)=\bar p^{-1}\zeta_0(s,z)$, $\zeta_0(s,z)$ is a solution of (24) and ζ_1 is to be determined. Denote

$$V^\varepsilon(t,x)=v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)+\varepsilon\chi\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)\nabla_x v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)+$$
$$+\varepsilon^2\psi^{ij}\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)\frac{\partial^2}{\partial x_i\partial x_j}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)+\Big(\tilde\zeta_0\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)-1\Big)v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)+$$
$$+\varepsilon\zeta_1\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)\nabla v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)$$

Substituting this ansatz in the equation (12) and considering (17), (18) and (24), one gets

$$\frac{\partial}{\partial t}V^\varepsilon-\frac{\partial}{\partial x_i}\Big(\tilde a_{ij}\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)\frac{\partial}{\partial x_j}V^\varepsilon\Big)-\frac{1}{\varepsilon}\tilde b_i\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)\frac{\partial}{\partial x_i}V^\varepsilon=$$
$$=\varepsilon\chi^{i,\varepsilon}\frac{\partial^2}{\partial x_i\partial t}v^0\big(t,x-\frac{\bar b}{\varepsilon}r\big)\big|_{r=t}-\varepsilon\tilde a^\varepsilon_{ij}\chi^{k,\varepsilon}\frac{\partial^3}{\partial x_i\partial x_j\partial x_k}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)-$$
$$-\varepsilon\tilde a^\varepsilon_{ij}\frac{\partial}{\partial z_j}\psi^{kl,\varepsilon}\frac{\partial^3}{\partial x_i\partial x_k\partial x_l}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)-\varepsilon\frac{\partial}{\partial z_i}\big(\tilde a^\varepsilon_{ij}\psi^{kl,\varepsilon}\big)\frac{\partial^3}{\partial x_j\partial x_k\partial x_l}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)$$
$$-\varepsilon\tilde b^\varepsilon_i\psi^{kl,\varepsilon}\frac{\partial^3}{\partial x_i\partial x_k\partial x_l}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)-\varepsilon\bar b_i\psi^{kl,\varepsilon}\frac{\partial^3}{\partial x_i\partial x_k\partial x_l}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)$$
$$-\varepsilon^2\psi^{kl,\varepsilon}\frac{\partial^3}{\partial x_k\partial x_l\partial t}v^0\big(t,x-\frac{\bar b}{\varepsilon}r\big)\big|_{r=t}-\varepsilon^2\psi^{kl,\varepsilon}\tilde a^\varepsilon_{ij}\frac{\partial^4}{\partial x_k\partial x_l\partial x_i\partial x_j}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)$$
$$-\varepsilon^{-1}\tilde a^\varepsilon_{ij}\frac{\partial}{\partial z_j}\zeta^\varepsilon_0\frac{\partial}{\partial x_i}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)-\varepsilon^{-1}\frac{\partial}{\partial z_i}\Big(\tilde a^\varepsilon_{ij}(\zeta^\varepsilon_0-\bar p)\Big)\frac{\partial}{\partial x_j}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)$$
$$-\varepsilon^{-1}\tilde b^\varepsilon_i(\zeta^\varepsilon_0-\bar p)\frac{\partial}{\partial x_i}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)-\varepsilon^{-1}\bar b_i(\zeta^\varepsilon_0-\bar p)\frac{\partial}{\partial x_i}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)+$$
$$+(\zeta^\varepsilon_0-\bar p)\frac{\partial}{\partial t}v^0\big(t,x-\frac{\bar b}{\varepsilon}r\big)\big|_{r=t}-\tilde a^\varepsilon_{ij}(\zeta^\varepsilon_0-\bar p)\frac{\partial^2}{\partial x_i\partial x_j}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)$$
$$+\varepsilon^{-1}\frac{\partial}{\partial s}\zeta^\varepsilon_{1,k}\frac{\partial}{\partial x_k}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)-\varepsilon^{-1}\frac{\partial}{\partial z_i}\Big(\tilde a^\varepsilon_{ij}\frac{\partial}{\partial z_j}\zeta^\varepsilon_{1,k}\Big)\frac{\partial}{\partial x_k}v^0\big(t,x-\frac{\bar b}{\varepsilon}t\big)+$$

$$-\tilde{a}_{ij}^{\varepsilon}\frac{\partial}{\partial z_j}\zeta_{1,k}^{\varepsilon}\frac{\partial^2}{\partial x_i\partial x_k}v^0\big(t,x-\frac{\bar{b}}{\varepsilon}t\big)-\frac{\partial}{\partial z_i}\Big(\tilde{a}_{ij}^{\varepsilon}\zeta_{1,k}^{\varepsilon}\Big)\frac{\partial^2}{\partial x_j\partial x_k}v^0\big(t,x-\frac{\bar{b}}{\varepsilon}t\big)$$
$$-\tilde{b}_i^{\varepsilon}\zeta_{1,k}^{\varepsilon}\frac{\partial^2}{\partial x_i\partial x_k}v^0\big(t,x-\frac{\bar{b}}{\varepsilon}t\big)+\varepsilon\zeta_{1,k}^{\varepsilon}\frac{\partial^2}{\partial t\partial x_k}v^0\big(t,x-\frac{\bar{b}}{\varepsilon}r\big)\big|_{r=t}-$$
$$-\varepsilon\tilde{a}_{ij}^{\varepsilon}\zeta_{1,k}^{\varepsilon}\frac{\partial^3}{\partial x_i\partial x_j\partial x_k}v^0\big(t,x-\frac{\bar{b}}{\varepsilon}t\big).$$

The structure of the right hand side here suggests that the function ζ_1 should satisfy the equation

$$\frac{\partial}{\partial s}\zeta_{1,k}-\frac{\partial}{\partial z_i}\Big(\tilde{a}_{ij}(s,z)\frac{\partial}{\partial z_j}\zeta_{1,k}\Big)=\tilde{a}_{kj}(s,z)\frac{\partial}{\partial z_j}\tilde{\zeta}_0(s,z)+ \tag{28}$$
$$+\frac{\partial}{\partial z_i}\Big(\tilde{a}_{ik}(s,z)(\tilde{\zeta}_0(s,z)-1)\Big)+\tilde{b}_k(s,z)(\tilde{\zeta}_0(s,z)-1)+\bar{b}_k(\tilde{\zeta}_0(s,z)-1),$$

where $(s,z)\in(0,+\infty)\times\mathbb{T}^n$. Due to (25) the terms on the right hand side decay exponentially. Namely,

$$\int_\tau^{\tau+1}\|\tilde{a}_{ik}(s,\cdot)(\tilde{\zeta}_0(s,\cdot)-1)\|_{L^2(\mathbb{T}^n)}ds\le C\varepsilon^{-\gamma\tau},$$
$$\Big\|\frac{\partial}{\partial z_i}\Big(\tilde{a}_{ik}(s,\cdot)(\tilde{\zeta}_0(s,\cdot)-1)\Big)+(\tilde{b}_k(s,\cdot)+\bar{b})(\tilde{\zeta}_0(s,\cdot)-1)\Big\|_{W^{-1,\infty}(\mathbb{T}^n)}\le C\varepsilon^{-\gamma s}.$$

According to [] this implies that the equation (28) with the initial condition $\zeta_1(0,z)=0$, has a solution that stabilizes at the exponential rate to a constant. Subtracting this constant, we obtain a solution of (28) that tends to zero at the exponential rate, as $t\to+\infty$. For this solution we keep the notation ζ_1. Finally, under this choice of ζ_1, we have

$$\frac{\partial}{\partial t}V^{\varepsilon}-\frac{\partial}{\partial x_i}\Big(\tilde{a}_{ij}\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)\frac{\partial}{\partial x_j}V^{\varepsilon}\Big)-\frac{1}{\varepsilon}\tilde{b}_i\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)\frac{\partial}{\partial x_i}V^{\varepsilon}=$$
$$=\varepsilon R_1^{\varepsilon}(x,t)+\varepsilon^2\nabla R_2^{\varepsilon}(x,t)$$
$$+(\zeta_0^{\varepsilon}-\bar{p})\frac{\partial}{\partial t}v^0\big(t,x-\frac{\bar{b}}{\varepsilon}r\big)\big|_{r=t}-\tilde{a}_{ij}^{\varepsilon}(\zeta_0^{\varepsilon}-\bar{p})\frac{\partial^2}{\partial x_i\partial x_j}v^0\big(t,x-\frac{\bar{b}}{\varepsilon}t\big)$$
$$-\tilde{a}_{ij}^{\varepsilon}\frac{\partial}{\partial z_j}\zeta_{1,k}^{\varepsilon}\frac{\partial^2}{\partial x_i\partial x_k}v^0\big(t,x-\frac{\bar{b}}{\varepsilon}t\big)-\frac{\partial}{\partial z_i}\Big(\tilde{a}_{ij}^{\varepsilon}\zeta_{1,k}^{\varepsilon}\Big)\frac{\partial^2}{\partial x_j\partial x_k}v^0\big(t,x-\frac{\bar{b}}{\varepsilon}t\big)$$
$$-\tilde{b}_i^{\varepsilon}\zeta_{1,k}^{\varepsilon}\frac{\partial^2}{\partial x_i\partial x_k}v^0\big(t,x-\frac{\bar{b}}{\varepsilon}t\big)+\varepsilon\zeta_{1,k}^{\varepsilon}\frac{\partial^2}{\partial t\partial x_k}v^0\big(t,x-\frac{\bar{b}}{\varepsilon}r\big)\big|_{r=t}-$$
$$-\varepsilon\tilde{a}_{ij}^{\varepsilon}\zeta_{1,k}^{\varepsilon}\frac{\partial^3}{\partial x_i\partial x_j\partial x_k}v^0\big(t,x-\frac{\bar{b}}{\varepsilon}t\big)=$$
$$=\varepsilon R_1^{\varepsilon}(x,t)+\varepsilon^2\nabla R_2^{\varepsilon}(x,t)+R_3^{\varepsilon}(x,t)+\varepsilon\nabla R_4^{\varepsilon}(x,t),$$

here

$$\|R_1^{\varepsilon}\|_{L^2((0,T)\times\mathbb{R}^n)}\le C,\quad \|R_2^{\varepsilon}\|_{L^\infty((0,T)\times\mathbb{R}^n)}\le C,$$

and R_3^{ε}, R_4^{ε} are initial layer functions satisfying the estimates

$$\|R_3^{\varepsilon}\|_{L^2((0,T)\times\mathbb{R}^n)}\le C\varepsilon,\quad \|R_4^{\varepsilon}\|_{L^2((0,T)\times\mathbb{R}^n)}\le C\varepsilon.$$

Considering the initial condition

$$V^\varepsilon(0,x) = v^0(0,x) + \varepsilon\chi\big(0,\frac{x}{\varepsilon}\big)\nabla_x v^0(0,x) +$$

$$+\varepsilon^2\psi^{ij}\big(0,\frac{x}{\varepsilon}\big)\frac{\partial^2}{\partial x_i \partial x_j}v^0(0,x) + \Big(\tilde{\zeta}_0\big(0,\frac{x}{\varepsilon}\big) - 1\big)v^0(0,x) +$$

$$+\varepsilon\zeta_1\big(0,\frac{x}{\varepsilon}\big)\nabla v^0(0,x) = p_0(0,\frac{x}{\varepsilon})u_0(x) + O(\varepsilon),$$

the estimate (15) gives

$$\|v^\varepsilon - V^\varepsilon\|^2_{L^\infty(0,T;L^2(\mathbb{R}^n))} + \|v^\varepsilon - V^\varepsilon\|^2_{L^2(0,T;H^1(\mathbb{R}^n)} \le C\varepsilon.$$

This yields the desired statement.

■

Next we want to show that the statement of last theorem holds true for any initial condition $u_0(x) \in L^2(\mathbb{R}^n)$. To this end it suffices to approximate $u_0(x)$ in $L^2(\mathbb{R}^n)$ by the sequence of C_0^∞ functions and to use the estimate (13).

As a consequence of the above statements we obtain the following result for the Cauchy problem (4).

Theorem 4 *A solution $u^\varepsilon(t,x)$ of problem (4) admits the following representation*

$$u^\varepsilon(t,x) = e^{\Lambda_0 t/\varepsilon^2} p_0\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)\Big(v^0\big(t, x - \frac{\bar{b}}{\varepsilon}t\big) + r^\varepsilon(t,x)\Big),$$

where

$$\lim_{\varepsilon\to 0}\|r^\varepsilon\|_{L^2((0,T)\times\mathbb{R}^n)} = 0,$$

and $v^0(t,x)$ is a solution to problem (27).

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