Матем. Сборник
Том 115(157) (1981), Вып. 3

Math. USSR Sbornik
Vol. 43(1982), No. 3

A SCATTERING PROBLEM IN LAMINAR MEDIA

UDC 517.9

A. L. PJATNICKIĬ

ABSTRACT. The scattering problem in a laminar medium

$$\Delta u(x) + k^2 q(x_1,\dots,x_n, x_1/\varepsilon)u(x) = 0$$

with a radiation condition at infinity is considered. The potential $q(x, y)$ is periodic in the variable y. Here k is a large parameter, and ε is a small parameter with $k \sim \varepsilon^{-\alpha}$, $\alpha > 1$.

In this paper a formal asymptotic expansion of the solution of this problem is found. To construct it an operator analogous to the canonical Maslov operator is used which acts on a certain Lagrangian manifold not depending on ε. An analogous problem for the Schrödinger equation in a laminar medium is solved.

Bibliography: 10 titles.

In the space $\mathbf{R}^n$ we consider the scattering problem

$$\Delta u_{k,\varepsilon}(x) + k^2 q\left(x_1,\dots,x_n, \frac{x_1}{\varepsilon}\right)u_{k,\varepsilon}(x) = 0 \tag{1}$$

with Sommerfeld radiation conditions at infinity. Here k is a large parameter, and ε is a small parameter. The potential $q(x_1,\dots,x_n, y)$ is periodic in the variable y, is everywhere positive, and goes to one uniformly with respect to $y \in \mathbf{R}^1$:

$$q(x, y)\,|_{|x|>\alpha} = 1.$$

In this paper an asymptotic expansion of the solution $u_{k,\varepsilon}(x)$ is constructed as $k \to \infty$ and $\varepsilon \to 0$ with $k \sim \varepsilon^{-\alpha}$, where $\alpha > 1$. The latter condition implies that there are many wave lengths in a single layer of the medium.

We note that for potentials not depending on ε an asymptotic expansion was constructed in [1] and justified in [2]–[4]. In the physical literature the transmission of a wave through a laminar medium is studied in [5].

In this paper the asymptotic expansion is constructed by means of an operator that is an analogue of the canonical operator (see [1] and [6]) and acts on a certain Lagrangian manifold Λ^n, while this manifold does not depend on ε. We call it the *averaged manifold.* At points of $\mathbf{R}^n$ not lying on caustics of the manifold Λ^n the asymptotic expansion of $u_{k,\varepsilon}$ is a finite sum of asymptotic series of the form

$$e^{ikS_\varepsilon(x,y)} \sum_{j=0}^{\infty} (\varepsilon k)^{-j} \varphi_\varepsilon^j(x, y)\,|_{y=x_1/\varepsilon}, \tag{2}$$

1980 *Mathematics Subject Classification.* Primary 35P25, 35C10; Secondary 35J05.

where

$$S_\varepsilon(x, y) = S_0(x) + \varepsilon S_1(x, y) + \cdots, \qquad \varphi_\varepsilon^j(x, y) = \varphi_0^j(x, y) + \varepsilon\varphi_1^j(x, y) + \cdots.$$

It is easy to see that in the argument of the exponential function in (2) all terms with indices greater than α tend to zero as $k \to \infty$ and $\varepsilon \to 0$, $k \sim \varepsilon^{-\alpha}$, and that

$$e^{ik\varepsilon^l S_l(x,y)} = 1 + O(\varepsilon^{l-\alpha}).$$

The asymptotic expansion (2) is therefore well defined.

The averaged Lagrangian manifold Λ^n is constructed in the following manner. We consider the Hamiltonian system of ordinary equations with Hamiltonian

$$\overline{H}(x, p) = p_1 - M\left\{\sqrt{q(x, y) - |p'|^2}\right\}. \tag{3}$$

Here $M\{f\}$ denotes the average of a function f periodic in the variable y over a period, and p' denotes the vector $(p_2, \ldots, p_n)$. The family of trajectories of this system of equations with the initial conditions

$$x_1|_{t=0} = -a, \quad p_1|_{t=0} = 1, \quad p'|_{t=0} = 0 \tag{4}$$

forms a Lagrangian manifold Λ^n which we call the averaged manifold. Here the following condition on the potential $q(x, y)$ occurs: any solution of the Hamiltonian system with Hamiltonian (3) and initial conditions (4) for all t and y satisfies the inequality

$$q(x(t), y) - |p'(t)|^2 > 0. \tag{5}$$

This condition is henceforth assumed satisfied. We observe that this is clearly the case if the region in which $q(x, y)$ differs from one is sufficiently small or if $q(x, y)$ depends weakly on the variable y.

In the last section an analogous expansion is constructed and justified for the Schrödinger equation

$$ih\frac{\partial}{\partial t}u = \frac{h^2}{2m}\Delta u + v\left(x_1, \ldots, x_n, \frac{x_1}{\varepsilon}\right)u.$$

Here conditions are imposed on the potential $v(x_1, \ldots, x_n, y)$ which are analogous to the conditions on $q(x, y)$.

In exactly the same way it is possible to construct a formal asymptotic expansion for any λ^{-1}-pseudodifferential operator with a symbol $L(x, y, p)$ that is periodic in the variable $y = x_1/\varepsilon$.

§1. A special case

In this section we consider a situation in which the manifold projects injectively onto the plane $x_1, \ldots, x_n$.

For simplicity we consider a two-dimensional problem, since the case of a greater number of variables differs in no practical way from the two-dimensional case.

We investigate the following scattering problem:

$$\Delta u_{k,\varepsilon}(x) + k^2 q(x, x_1/\varepsilon)u_{k,\varepsilon}(x) = 0,$$
$$u_{k,\varepsilon}(x) = e^{ikx_1} + w_{k,\varepsilon}(x), \qquad w_{k,\varepsilon}(x) = O(r^{(1-n)/2}e^{ikr}). \tag{1'}$$

Here $x \in \mathbf{R}^2$, $r = |x|$, k is a large parameter, and ε is a small parameter with $k \sim \varepsilon^{-\alpha}$, $\alpha > 1$. The function $q(x, y)$ is infinitely smooth, positive, periodic in the variable y, and goes to one uniformly with respect to y:

$$q(x, y)\,|_{|x|>\alpha} = 1.$$

It is also assumed that condition (5) formulated in the Introduction is satisfied.

THEOREM 1. *Suppose that the Lagrangian manifold Λ^n projects injectively onto the plane x_1, x_2. Then the solution of problem (1′) admits a formal asymptotic expansion, and this expansion has the form*

$$| u_{k,\varepsilon}(x) \sim e^{ikS_\varepsilon(x,y)}\left(\varphi_\varepsilon^0(x, y) + (\varepsilon k)^{-1}\varphi_\varepsilon^1(x, y) + \cdots\right)|_{y=x_1/\varepsilon}, \tag{2′}$$

where, just as in (2), the function $S_\varepsilon(x, y)$ and all the functions $\varphi_\varepsilon^j(x, y)$ are asymptotic series in powers of ε.

PROOF. In place of the variable x_1/ε we introduce the independent variable y. Then the operators $\partial/\partial x_1$ and $\partial^2/\partial x_1^2$ in the new variables take the form

$$\frac{\partial}{\partial x_1}\chi\left(x, \frac{x_1}{\varepsilon}\right) = \left(\frac{\partial}{\partial x_1}\chi(x, y) + \varepsilon^{-1}\frac{\partial}{\partial y}\chi(x, y)\right)\Bigg|_{y=x_1/\varepsilon},$$

$$\frac{\partial^2}{\partial x_1^2}\chi\left(x, \frac{x_1}{\varepsilon}\right) = \left(\frac{\partial^2}{\partial x_1^2}\chi(x, y) + 2\varepsilon^{-1}\frac{\partial^2}{\partial x_1 \partial y}\chi(x, y) + \varepsilon^{-2}\frac{\partial^2}{\partial y^2}\chi(x, y)\right)\Bigg|_{y=x_1/\varepsilon}.$$

We now substitute the expansion (2′) into (1′) and equate coefficients of like powers of k and ε. The group of equations for k^2 and all possible powers of ε must be used to find the terms of the expansion of the phase, while the remaining equations, which are simpler, make it possible to find the amplitude in (2′). We begin with the equation for k^2 and ε^0:

$$\left(\frac{\partial S_0(x)}{\partial x_1} + \frac{\partial S_1(x, y)}{\partial y}\right)^2 + \left(\frac{\partial S_0(x)}{\partial x_2}\right)^2 = q(x, y). \tag{6}$$

We seek all the functions $S_j(x, y)$ as periodic functions in the variable y.

We rewrite (6) in the form

$$\frac{\partial S_0(x)}{\partial x_1} + \frac{\partial S_1(x, y)}{\partial y} = \sqrt{q(x, y) - \left(\frac{\partial S_0(x)}{\partial x_2}\right)^2}. \tag{7}$$

We assume that the expression under the square root sign does not vanish, and therefore this transformation is legitimate. It will be clear below that this assumption is justified. The function $S_1(x, y)$ must be a periodic function of y. It is clear that the solvability condition for (7) in functions periodic in y is the equality

$$\frac{\partial S_0(x)}{\partial x_1} = M\left\{\sqrt{q(x, y) - \left(\frac{\partial S_0(x)}{\partial x_2}\right)^2}\right\}. \tag{8}$$

Thus, for $S_0(x)$ we obtain a nonlinear equation of first order. We define the boundary condition for $S_0(x)$ as follows:

$$S_0(x)\,|_{x_1=-a} = -a.$$

It is henceforth convenient to denote the square root $\sqrt{q(x, y) - p_2^2}$ by $Q(x, y, p_2)$. Solution of (8) now reduces to integrating along a trajectory of the Hamiltonian system

$$\dot{x}_1 = \frac{\partial \overline{H}(x, p)}{\partial p_1} = 1, \qquad \dot{x}_2 = \frac{\partial \overline{H}(x, p)}{\partial p_2} = M\left\{ \frac{p_2}{Q(x, y, p_2)} \right\},$$

$$\dot{p}_i = -\frac{\partial \overline{H}(x, p)}{\partial x_i} = \frac{1}{2} M\left\{ \frac{\partial q(x, y)}{\partial x_i} \cdot (Q(x, y, p_2))^{-1} \right\},$$

$$x_1 |_{t=0} = -a, \quad p_1 |_{t=1} = 1, \quad p_2 |_{t=0} = 0, \tag{9}$$

with Hamiltonian $\overline{H}(x, p)$, while this trajectory can be replaced by any other curve joining the given points and lying on the Lagrangian manifold Λ^n (see [1] and [6]). Because of the assumptions made in this section concerning Λ^n the function $S_0(x)$ is defined on all $\mathbf{R}^n$ and is equal to

$$S_0(x) = -a + \int_{x_0}^{x} p\, dx.$$

After $S_0(x)$ has been found, using (8), we can write (7) in the form

$$\frac{\partial}{\partial y} S_1(x, y) = Q\left(x, y, \frac{\partial S_0(x)}{\partial x_2} \right) - M\left\{ Q\left(x, y, \frac{\partial S_0(x)}{\partial x_2} \right) \right\},$$

whence we immediately find $S_1(x, y)$ up to a function not depending on y. It will be convenient for us to write it in the form

$$S_1(x, y) = \mathring{S}_1(x, y) + \mu_1(x) = \int_{y_0}^{y} \left(Q\left(x, z, \frac{\partial S_0}{\partial x_2} \right) - M\left\{ Q\left(x, y, \frac{\partial S_0}{\partial x_2} \right) \right\} \right) dz$$

$$- M\left\{ \int_{y_0}^{y} \left(Q\left(x, z, \frac{\partial S_0}{\partial x_2} \right) - M\left\{ Q\left(x, y, \frac{\partial S_0}{\partial x_2} \right) \right\} \right) dz \right\} + \mu_1(x).$$

It is clear that $M\{\mathring{S}_1(x, y)\} = 0$.

We now consider the equation for k^2 and ε^1:

$$\left(\frac{\partial S_0(x)}{\partial x_1} + \frac{\partial S_1(x, y)}{\partial y} \right)\left(\frac{\partial S_1(x, y)}{\partial x_1} + \frac{\partial S_2(x, y)}{\partial y} \right) + \frac{\partial S_0(x)}{\partial x_2} \cdot \frac{\partial S_1(x, y)}{\partial x_2} = 0. \tag{10}$$

Since

$$\frac{\partial S_0(x)}{\partial x_1} + \frac{\partial S_1(x, y)}{\partial y} = Q\left(x, y, \frac{\partial S_0(x)}{\partial x_2} \right) > 0,$$

both sides of (10) can be divided by $Q = Q(x, y, \partial S_0/\partial x_2)$:

$$\frac{\partial S_1(x, y)}{\partial x_1} + \frac{\partial S_2(x, y)}{\partial y} + \frac{\partial S_0(x)}{\partial x_2} \cdot \frac{\partial S_1(x, y)}{\partial x_2} \cdot \left(Q\left(x, y, \frac{\partial S_0(x)}{\partial x_2} \right) \right)^{-1} = 0.$$

We take the mean of both sides of the last equality over a period:

$$\frac{\partial \mu_1(x)}{\partial x_1} + M\left\{ \frac{\partial S_0(x)}{\partial x_2} Q^{-1} \right\} \frac{\partial \mu_1(x)}{\partial x_2} = -M\left\{ \frac{\partial S_0(x)}{\partial x_2} \frac{\partial \mathring{S}_1(x, y)}{\partial x_2} Q^{-1} \right\}.$$

Using the original Hamiltonian system (9), the last equation can be transformed to the form

$$\frac{d\mu_1}{dt} = -M\left\{p_2(t)\cdot\frac{\partial \mathring{S}_1(x(t), y)}{\partial x_2}\cdot\left(Q(x(t), y, p_2(t))\right)^{-1}\right\}.$$

It is natural to take zero as the initial condition for μ_1.

Continuing this process, from the next equations for k^2 we find all functions

$$S_j(x, y) = \mathring{S}_j(x, y) + \mu_j(x), \quad M\{\mathring{S}_j(x, y)\} = 0, \quad \mu_j|_{t=0} = 0.$$

We seek the functions φ_i^0 from the group of equations for the first power of k. The first of these is k^1 and ε^{-1}:

$$2\frac{\partial\varphi_0^0(x, y)}{\partial y}\left(\frac{\partial S_0(x)}{\partial x_1} + \frac{\partial S_1(x, y)}{\partial y}\right) + \varphi_0^0(x, y)\frac{\partial^2 S_1(x, y)}{\partial y^2} = 0.$$

Dividing this equation by $(\partial S_0/\partial x_1 + \partial S_1/\partial y)^{1/2}$, we rewrite it as follows:

$$2\frac{\partial}{\partial y}\left[\varphi_0^0(x, y)\left(\frac{\partial S_0(x)}{\partial x_1} + \frac{\partial S_1(x, y)}{\partial y}\right)^{1/2}\right] = 0,$$

whence we easily find that

$$\varphi_0^0(x, y) = c(x)\left(\frac{\partial S_0(x)}{\partial x_1} + \frac{\partial S_1(x, y)}{\partial y}\right)^{-1/2}. \tag{11}$$

Here $c(x)$ is an arbitrary function of x. In order to find it, we use the equation for ε^0:

$$2\frac{\partial\varphi_0^0}{\partial x_1}\left(\frac{\partial S_0}{\partial x_1} + \frac{\partial S_1}{\partial y}\right) + 2\frac{\partial\varphi_1^0}{\partial y}\left(\frac{\partial S_0}{\partial x_1} + \frac{\partial S_1}{\partial y}\right) + 2\frac{\partial\varphi_0^0}{\partial y}\left(\frac{\partial S_1}{\partial x_1} + \frac{\partial S_2}{\partial y}\right)$$
$$+\varphi_0^0\left(\frac{\partial^2 S_0}{\partial x_1^2} + 2\frac{\partial^2 S_1}{\partial x_1\partial y} + \frac{\partial^2 S_2}{\partial y^2}\right) + \varphi_1^0\frac{\partial^2 S_1}{\partial y^2} + 2\frac{\partial\varphi_0^0}{\partial x_2}\frac{\partial S_0}{\partial x_2} + \varphi_0^0\frac{\partial^2 S_0}{\partial x_2^2} = 0.$$

Dividing both sides by $(\partial S_0/\partial x_1 + \partial S_1/\partial y)^{1/2}$ and making simple transformations, we find, using (11), that

$$2\frac{\partial}{\partial y}\left[\varphi_1^0\left(\frac{\partial S_0}{\partial x_1} + \frac{\partial S_1}{\partial y}\right)^{1/2}\right] + 2\frac{\partial}{\partial x_1}\left[\varphi_0^0\left(\frac{\partial S_0}{\partial x_1} + \frac{\partial S_1}{\partial y}\right)^{1/2}\right]$$
$$+2c(x)\left(\frac{\partial S_0}{\partial x_1} + \frac{\partial S_1}{\partial y}\right)^{-1/2}\left(\frac{\partial}{\partial y}\left(\frac{\partial S_0}{\partial x_1} + \frac{\partial S_1}{\partial y}\right)^{-1/2}\right)\left(\frac{\partial S_1}{\partial x_1} + \frac{\partial S_2}{\partial y}\right)$$
$$+c(x)\left(\frac{\partial S_0}{\partial x_1} + \frac{\partial S_1}{\partial y}\right)\left(\frac{\partial^2 S_1}{\partial x_1\partial y} + \frac{\partial^2 S_2}{\partial y^2}\right)$$
$$+2\left(\frac{\partial}{\partial x_2}c(x)\right)\frac{\partial S_0}{\partial x_2}\left(\frac{\partial S_0}{\partial x_1} + \frac{\partial S_1}{\partial y}\right)^{-1}$$
$$+c(x)\left[2\left(\frac{\partial S_0}{\partial x_1} + \frac{\partial S_1}{\partial y}\right)^{-1/2}\left(\frac{\partial}{\partial x_2}\left(\frac{\partial S_0}{\partial x_1} + \frac{\partial S_1}{\partial y}\right)^{-1/2}\right)\frac{\partial S_0}{\partial x_2}\right.$$
$$\left.+\left(\frac{\partial S_0}{\partial x_1} + \frac{\partial S_1}{\partial y}\right)^{-1}\frac{\partial^2 S_0}{\partial x_2^2}\right] = 0. \tag{12}$$

Applying the equality

$$\frac{\partial}{\partial z}\left(f(z)^{-1}\right) = 2f(z)^{-1/2}\frac{\partial}{\partial z}\left(f(z)^{-1/2}\right),$$

we transform (12) to the form

$$2\frac{\partial}{\partial y}\left[\varphi_1^0\left(\frac{\partial S_0}{\partial x_1}+\frac{\partial S_1}{\partial y}\right)^{1/2}\right]+2\frac{\partial}{\partial x_1}c(x)+c(x)\frac{\partial}{\partial y}\left[\left(\frac{\partial S_1}{\partial x_1}+\frac{\partial S_2}{\partial y}\right)\left(\frac{\partial S_0}{\partial x_1}+\frac{\partial S_1}{\partial y}\right)^{-1}\right]$$
$$+2\left(\frac{\partial}{\partial x_2}c(x)\right)\frac{\partial S_0}{\partial x_2}\left(\frac{\partial S_0}{\partial x_1}+\frac{\partial S_1}{\partial y}\right)^{-1}+c(x)\frac{\partial}{\partial x_2}\left[\frac{\partial S_0}{\partial x_2}\left(\frac{\partial S_0}{\partial x_1}+\frac{\partial S_1}{\partial y}\right)\right]=0.$$

The solvability condition for this equation is that the mean of both sides be equal to zero:

$$2\frac{\partial}{\partial x_1}c(x)+2M\left\{\frac{\partial S_0}{\partial x_2}Q^{-1}\right\}\frac{\partial}{\partial x_2}c(x)+c(x)\frac{\partial}{\partial x_2}M\left\{\frac{\partial S_0}{\partial x_2}Q^{-1}\right\}=0.$$

The last equation reduces to an equation along trajectories

$$2\frac{d}{dt}c(x(t))+c(x(t))\frac{\partial}{\partial x_2}M\left\{\frac{\partial S_0}{\partial x_2}\left(Q\left(x(t),y,\frac{\partial S_0}{\partial x_2}\right)\right)^{-1}\right\}=0. \tag{13}$$

According to the Liouville theorem (see [1]), the determinant

$$J(t,\xi)=\det\frac{\partial(x_1,x_2)}{\partial(t,\xi)},\qquad \xi=x_2|_{t=0},$$

satisfies the equation

$$\frac{d}{dt}\ln J(t,\xi)=\frac{\partial}{\partial x_2}M\left\{\frac{\partial S_0}{\partial x_2}\left(Q\left(x(t),y,\frac{\partial S_0}{\partial x_2}\right)\right)^{-1}\right\},$$

which together with (13) leads to the following equation for $c(x)$:

$$\frac{1}{\sqrt{J}}\frac{d}{dt}\left(c(x)\sqrt{J}\right)=0.$$

Recalling that $c|_{t=0}=1$, we find that

$$c(x)=\sqrt{|J|^{-1}}.$$

All terms of the asymptotics of the amplitude $\varphi_j^i(x,y)$ are constructed in a similar way.

The principal term of the expansion we have constructed has the form

$$u_{k,\varepsilon}(x)\sim\exp\left(ik\left(S_0(x)+\cdots+\varepsilon^l S_l\left(x,\frac{x_1}{\varepsilon}\right)\right)\right)\left(|J|\,Q\left(x,\frac{x_1}{\varepsilon},\frac{\partial S_0}{\partial x_2}\right)\right)^{-1/2}.$$

REMARK. The principal term of the phase function $S_0(x)$ does not correspond to any differential operator.

§2. Construction of the asymptotics in the general case

In this section the asymptotic expansion of problem (1) will be constructed under the assumption that the Lagrangian manifold Λ^n is a manifold of general position in $\mathbf{R}^{2n}$.

Lemma 1. *Let M^n be a Lagrangian manifold in $\mathbf{R}^{2n}$, let (x^0, p^0) be a point on this manifold, and suppose that in a neighborhood of this point on M^n there are given coordinates $\xi_1, \ldots, \xi_n$. Let the inequality $\partial x_1/\partial\xi_1 > 0$ be satisfied at the point (x^0, p^0). Then in a neighborhood of (x^0, p^0) it is possible to choose canonical coordinates so that p_1 is not a coordinate.*

Proof. We first prove the lemma in the three-dimensional case. Suppose that in a neighborhood of (x^0, p^0) on M^3 there are coordinates θ, ξ, η. The fact that M^3 is Lagrangian ensures that

$$\frac{\partial x_1}{\partial\theta}\frac{\partial p_1}{\partial\xi} + \frac{\partial x_2}{\partial\theta}\frac{\partial p_2}{\partial\xi} + \frac{\partial x_3}{\partial\theta}\frac{\partial p_3}{\partial\xi} - \frac{\partial x_1}{\partial\xi}\frac{\partial p_1}{\partial\theta} - \frac{\partial x_2}{\partial\xi}\frac{\partial p_2}{\partial\theta} - \frac{\partial x_3}{\partial\xi}\frac{\partial p_3}{\partial\theta} = 0,$$

$$\frac{\partial x_1}{\partial\theta}\frac{\partial p_1}{\partial\eta} + \frac{\partial x_2}{\partial\theta}\frac{\partial p_2}{\partial\eta} + \frac{\partial x_3}{\partial\theta}\frac{\partial p_3}{\partial\eta} - \frac{\partial x_1}{\partial\eta}\frac{\partial p_1}{\partial\theta} - \frac{\partial x_2}{\partial\eta}\frac{\partial p_2}{\partial\theta} - \frac{\partial x_3}{\partial\eta}\frac{\partial p_3}{\partial\theta} = 0,$$

$$\frac{\partial x_1}{\partial\xi}\frac{\partial p_1}{\partial\eta} + \frac{\partial x_2}{\partial\xi}\frac{\partial p_2}{\partial\eta} + \frac{\partial x_3}{\partial\xi}\frac{\partial p_3}{\partial\eta} - \frac{\partial x_1}{\partial\eta}\frac{\partial p_1}{\partial\xi} - \frac{\partial x_2}{\partial\eta}\frac{\partial p_2}{\partial\xi} - \frac{\partial x_3}{\partial\eta}\frac{\partial p_3}{\partial\xi} = 0. \tag{14}$$

We multiply the first equality by $\partial x_2/\partial\eta$, the second by $\partial x_2/\partial\xi$, the third by $\partial x_2/\partial\theta$, and we add them:

$$\det\frac{\partial(x_1, p_1, x_2)}{\partial(\theta, \xi, \eta)} = \det\frac{\partial(x_2, x_3, p_3)}{\partial(\theta, \xi, \eta)}. \tag{15}$$

In exactly the same way, multiplying (14) by the derivatives of x_3, p_2, and p_3 in place of x_2, we obtain three equalities analogous to (15).

We now suppose that in a neighborhood of (x^0, p^0) it is not possible to take x_1 as the first canonical coordinate. The first coordinate is then p_1. It may be assumed with no loss of generality that the two other coordinates are x_2 and x_3. Since the variables x_1, x_2, and x_3 cannot be taken as coordinates, we have

$$\det\frac{\partial(x_1, x_2, x_3)}{\partial(\theta, \xi, \eta)}(x^0, p^0) = 0;$$

therefore

$$\nabla x_1 = \alpha\nabla x_2 + \beta\nabla x_3, \qquad \nabla = \left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}\right),$$

and at least one of the numbers α and β is nonzero by the hypotheses of the lemma. Suppose that $\alpha \neq 0$. Then

$$\det\frac{\partial(x_1, p_1, x_3)}{\partial(\theta, \xi, \eta)} \neq 0.$$

Hence by (15)

$$\det\frac{\partial(x_2, p_2, x_3)}{\partial(\theta, \xi, \eta)} \neq 0.$$

Therefore,

$$\det\frac{\partial(x_1, p_2, x_3)}{\partial(\theta, \xi, \eta)} = \alpha\det\frac{\partial(x_2, p_2, x_3)}{\partial(\theta, \xi, \eta)} \neq 0,$$

i.e., in a neighborhood of (x^0, p^0) it is possible to take x_1, p_2, x_3 as coordinates on M^n. We have obtained a contradiction.

In a space of higher dimension the lemma is proved similarly by using the following simple proposition which makes it possible to pass from equalities of the type (14) to equalities of the type (15).

Proposition. *Suppose there is given an $n \times n$ matrix A. Fix any two rows of this matrix with indices i and j. Then there is the following formula for computing the determinant of this matrix*:

$$\det(a_{kl}) = \sum_{1 \leqslant s < m \leqslant n} (-1)^{i+j+s+m} \begin{vmatrix} a_{si} & a_{mi} \\ a_{sj} & a_{mj} \end{vmatrix} A_{ijms}.$$

Here A_{ijms} are the complementary minors of order $(n-2) \times (n-2)$.

This proposition is a special case of a theorem of Laplace (see [7]).

It is henceforth assumed that on Λ^n there are given canonical coordinates, and the variable p_1 is not a coordinate in any local chart. We also suppose that the manifold Λ^n is in general position, i.e., the set of points of this manifold in a neighborhood of which it is not possible to take $x_1, \dots, x_n$ as coordinates has dimension no more than $n-1$.

In a singular local chart, i.e., in a local chart where certain p_j enter as canonical coordinates we seek a solution of (1) in the form

$$u(x) = F^{-1}_{k, p_{\bar I} \to x_{\bar I}} \tilde u(x_I, p_{\bar I}).$$

Here x_I, $p_{\bar I}$ are coordinates in the corresponding local chart, $I \cap \bar I = \varnothing$ and $I \cup \bar I = \{1, \dots, n\}$. Here $F^{-1}_{k, p_{\bar I} \to x_{\bar I}}$ denotes the operator of fast Fourier transform

$$F^{-1}_{k, p_{\bar I} \to x_{\bar I}} \tilde u(x_I, p_{\bar I}) = \left(\frac{k}{2\pi}\right)^{|\bar I|/2} \int_{R^{|\bar I|}} e^{ikx_{\bar I}p_{\bar I}} \tilde u(x_I, p_{\bar I})\, dp_{\bar I}.$$

The original equation (1) can now be rewritten as follows:

$$F_{k, x_{\bar I} \to p_{\bar I}} \left[\Delta + q\left(x, \frac{x_1}{\varepsilon}\right)\right] F^{-1}_{k, p_{\bar I} \to x_{\bar I}} \tilde u_{k,\varepsilon}(x_I, p_{\bar I}) = 0. \tag{16}$$

As before, we seek a solution $\tilde u_{k,\varepsilon}(x_I, p_{\bar I})$ of this equation in the form

$$\tilde u_{k,\varepsilon}(x_I, p_{\bar I}) \sim e^{ik\tilde S_\varepsilon(x_I, p_{\bar I}, y)} \sum_{j=0}^{\infty} (\varepsilon k)^{-j} \varphi^j_\varepsilon(x_I, p_{\bar I}, y)\big|_{y = x_1/\varepsilon}. \tag{17}$$

For simplicity we again consider the two-dimensional case. The general case can be studied in a completely similar way.

Theorem 2. *The action of the operator* (16) *on a function $\tilde u_{k,\varepsilon}(x_I, p_{\bar I})$ defined by the asymptotic series* (17) *can be represented by the formula*

$$F_{k, x_2 \to p_2}\left[\Delta + q\left(x, \frac{x_1}{\varepsilon}\right)\right] F^{-1}_{k, p_2 \to x_2} \tilde u_{k,\varepsilon}(x_1, p_2)$$
$$\sim e^{ik\tilde S_\varepsilon(x_1, p_2, y)} \left[\left(\sum_{j=0}^{\infty} (\varepsilon k)^{-j} \mathcal{Q}^j_\varepsilon\right)\left(\sum_{l=0}^{\infty} (\varepsilon k)^{-l} \tilde\varphi^l_\varepsilon(x_1, p_2, y)\right)\right]\Bigg|_{y = x_1/\varepsilon},$$

$$\mathcal{Q}^j_\varepsilon \sim \sum_{s=0}^{\infty} \varepsilon^s \mathcal{Q}^j_s, \quad \tilde\varphi^l_\varepsilon \sim \sum_{s=0}^{\infty} \varepsilon^s \tilde\varphi^l_s, \quad \tilde S_\varepsilon \sim \sum_{s=0}^{\infty} \varepsilon^s \tilde S_s,$$

where the $\mathcal{Q}_s^j$ are differential operators of order j. Here

$$\mathcal{Q}_0^0\tilde{\varphi} = \left[\left(\frac{\partial \tilde{S}_0}{\partial x_1} + \frac{\partial \tilde{S}_1}{\partial y}\right)^2 + p_2^2 - q\left(x_1, -\frac{\partial \tilde{S}_0}{\partial p_2}, y\right)\right]\tilde{\varphi},$$

$$\mathcal{Q}_1^0\tilde{\varphi} = \left[2\left(\frac{\partial \tilde{S}_0}{\partial x_1} + \frac{\partial S_1}{\partial y}\right)\left(\frac{\partial \tilde{S}_1}{\partial x_1} + \frac{\partial \tilde{S}_2}{\partial y}\right) + \frac{\partial}{\partial x_2} q(x_1, x_2, y)\,|_{x_2 = -\partial \tilde{S}_0/\partial p_2} \frac{\partial \tilde{S}_1}{\partial p_2}\right]\tilde{\varphi},$$

$$\mathcal{Q}_0^1\tilde{\varphi} = 2\left(\frac{\partial}{\partial y}\tilde{\varphi}\right)\left(\frac{\partial \tilde{S}_0}{\partial x_1} + \frac{\partial \tilde{S}_1}{\partial y}\right) + \tilde{\varphi}\frac{\partial^2 \tilde{S}_1}{\partial y^2},$$

$$\mathcal{Q}_1^1\tilde{\varphi} = \left[2\left(\frac{\partial}{\partial x_1}\tilde{\varphi}\right)\left(\frac{\partial \tilde{S}_0}{\partial x_1} + \frac{\partial \tilde{S}_1}{\partial y}\right) + 2\left(\frac{\partial}{\partial y}\tilde{\varphi}\right)\left(\frac{\partial \tilde{S}}{\partial x_1} + \frac{\partial \tilde{S}_2}{\partial y}\right)\right.$$

$$+\tilde{\varphi}\left(\frac{\partial^2 \tilde{S}_0}{\partial x_1^2} + 2\frac{\partial^2 \tilde{S}_1}{\partial x_1 \partial y} + \frac{\partial^2 \tilde{S}_2}{\partial y^2}\right)$$

$$\left. + \frac{1}{2}\left(\frac{\partial^2}{\partial x_2^2} q(x_1, x_2, y)\right)\frac{\partial^2 \tilde{S}_0}{\partial p_2^2}\tilde{\varphi} + \frac{\partial \tilde{\varphi}}{\partial p_2}\frac{\partial}{\partial x_2} q(x_1, x_2, y)\right]\Bigg|_{x_2 = -\partial \tilde{S}_0/\partial p_2}.$$

PROOF. In place of x_1/ε we introduce the independent variable y, and we consider y and ε as independent quantities. In (17) it is then possible to extend ε to the interval $(-\varepsilon_0, \varepsilon_0)$. We are now in the situation where the method of stationary phase with a parameter is applicable (see [8], Theorem 2.2). The remainder of the theorem is proved just as Theorem 5 of [1]. For $\varepsilon > 0$ we obtain the required formulas.

In order to construct solutions of the equations obtained in Theorem 2 it is necessary to obtain a collection of initial conditions in the singular chart. To this end it is necessary to study how the fast Fourier transform acts on functions of the form (17).

THEOREM 3. *Suppose that some neighborhood of the manifold Λ^2 projects injectively onto the planes x_1, x_2 and x_1, p_2. Then the following transition formulas hold*:

$$F_{k, x_2 \to p_2}\left(e^{ikS_\varepsilon(x, x_1/\varepsilon)} \sum_{j=0}^{\infty} (\varepsilon k)^{-j} \varphi_\varepsilon^j\left(x, \frac{x_1}{\varepsilon}\right)\right)$$

$$\sim e^{ik\tilde{S}_\varepsilon(x_1, p_2, x_1/\varepsilon)} \sum_{j=0}^{\infty} (\varepsilon k)^{-j} \tilde{\varphi}_\varepsilon^j\left(x_1, p_2, \frac{x_1}{\varepsilon}\right),$$

where

$$\tilde{S}_0(x_1, p_2) = -x_2 p_2 + S_0(x), \qquad p_2 = \partial S_0/\partial x_2,$$

$$\tilde{S}_1(x_1, p_2, y) = S_1(x, y),$$

$$\tilde{S}_2(x_1, p_2, y) = S_2(x, y) - \frac{1}{2}\left(\frac{\partial S_1}{\partial x_2}\right)^2\left(\frac{\partial^2 S_0}{\partial x_2^2}\right)^{-1},$$

$$\tilde{S}_m(x_1, p_2, y) = S_m(x, y) + v_m(S_0, \ldots, S_{m-1}),$$

$$\tilde{\varphi}_0^0(x_1, p_2, y) = \left|\frac{\partial^2 S_0}{\partial x_2^2}\right|^{-1/2} \varphi_0^0(x, y) e^{-i\pi/2\,\text{inerdex}\,\partial^2 S_0/\partial x_2^2}.$$

Here the v_m are linear fractional functions of the derivatives of S_j with denominators that are powers of $\partial^2 S_0/\partial x_2^2$. The functions $\tilde{\varphi}_l^s$ are expressed in terms of φ_j^h, $h \leq s, j \leq l$, by means of *differential operators with coefficients depending on* S_j. *The expression* inerdex A *denotes the number of negative eigenvalues of the matrix* A. *In the present two-dimensional case this is* 1 *if* $\partial^2 S_0/\partial x_2^2 < 0$ *and is* 0 *otherwise.*

PROOF. It is necessary to investigate the integral

$$I = \left(\frac{k}{2\pi}\right)^{1/2} \int e^{ik(x_2 p_2 - S_0(x) - \cdots - \varepsilon^N S_N(x, x_1/\varepsilon))}\left(\varphi_0^0\left(x, \frac{x_1}{\varepsilon}\right) + \cdots\right) dx_2.$$

In order to apply the method of stationary phase to this integral it is necessary to find a root of the equation

$$p_2 = \frac{\partial S_0}{\partial x_2} + \varepsilon \frac{\partial S_1}{\partial x_2} + \cdots + \varepsilon^N \frac{\partial S_N}{\partial x_2}. \tag{18}$$

Since the neighborhood considered projects injectively both onto the plane x_1, x_2 and onto x_1, p_2, the equation

$$p_2 = \frac{\partial S_0}{\partial x_2} \tag{19}$$

has a unique solution for p_2 in the corresponding region, and $\partial^2 S_0/\partial x_2^2 \neq 0$. As in Theorem 2, we use the following technique: we assume that $y = x_1/\varepsilon$ is an independent variable, and we extend ε to the interval $(-\varepsilon_0, \varepsilon_0)$. Because of the implicit function theorem, for the same p_2 as in (19) equation (18) has for $|\varepsilon| < \varepsilon_0$ a unique solution provided that ε_0 is sufficiently small. For this solution there is the representation

$$x_2^\varepsilon = x_2^0 + \varepsilon\alpha_1 + \cdots + \varepsilon^N \alpha_N + O(\varepsilon^N). \tag{20}$$

Here x_2^0 is the root of (19). Substituting (20) into (18), we find α_j:

$$\alpha_1 = -(\partial S_1/\partial x_2)(\partial^2 S_0/\partial x_2^2)^{-1}, \quad \text{etc.}$$

Applying now the method of stationary phase with a parameter (see [1] and [8]), we obtain

$$I \sim e^{ik(x_2^\varepsilon p_2 - S_0(x_1, x_2^\varepsilon, y) - \cdots)}\left(\varphi_0^0(x_1, x_2^\varepsilon, y) + \cdots\right)$$
$$\times e^{-i\pi/2 \,\text{inerdex}\, \partial^2 S_0/\partial x_2^2} \left| \frac{\partial^2}{\partial x_2^2}\left(x_2 p_2 - S_0(x, y) - \cdots\right)\Big|_{x_2 = x_2^\varepsilon}\right|^{-1/2}.$$

Substituting the expression (20) for x_2^ε and expanding S_j and φ_j^ε in Taylor series, we obtain the assertion of the theorem.

On the basis of Theorems 2 and 3 it is possible to extend the solution into a singular chart. In a singular chart the equation for k^2 and ε^0 has the form

$$\left(\frac{\partial \tilde{S}_0}{\partial x_1} + \frac{\partial \tilde{S}_1}{\partial y}\right)^2 + p_2^2 = q\left(x_1, -\frac{\partial \tilde{S}_0}{\partial p_2}, y\right)$$

or

$$\frac{\partial \tilde{S}_0}{\partial x_1} = M\left\{Q\left(x_1, -\frac{\partial \tilde{S}_0}{\partial p_2}, y, p_2\right)\right\}.$$

From this we find that $-\tilde{S}_0$ is the Legendre transform of the function S_0 (see [1] and [6]), and hence S_0 and $\tilde{S}_0$ give the phase of the averaged equation in different representations:

$$\tilde{S}_0(x_1, p_2) = S_0(x) - x_2 p_2.$$

Further,

$$\frac{\partial \tilde{S}_1}{\partial y} = Q(x, y, p_2) - M\{Q(x, y, p_2)\}.$$

Thus, the functions $\mathring{\tilde{S}}_1$ and $\mathring{S}_1$ coincide on the manifold Λ^2.

We consider the equation for k^2 and ε^1:

$$2\left(\frac{\partial \tilde{S}_0}{\partial x_1} + \frac{\partial \tilde{S}_1}{\partial y}\right)\left(\frac{\partial \tilde{S}_1}{\partial x_1} + \frac{\partial \tilde{S}_2}{\partial y}\right) + \frac{\partial}{\partial x_2} q(x, y)\,|_{x_2 = -\partial \tilde{S}_0/\partial p_2} \frac{\partial \tilde{S}_1}{\partial p_2} = 0$$

or

$$\frac{d}{dt}\tilde{\mu}_1 = -\frac{1}{2} M\left\{\left(\frac{\partial}{\partial x_2} q(x, y)\right) \frac{\partial \mathring{\tilde{S}}_1}{\partial p_2} (Q(x, y, p_2))^{-1}\right\}. \tag{21}$$

We shall show that along a trajectory this equation coincides with the equation for μ_1. For this we simplify the right sides of both equations:

$$M\left\{\frac{p_2 \dfrac{\partial \mathring{S}_1}{\partial x_2}}{Q(x, y, p_2)}\right\}$$

$$= M\left\{\frac{p_2}{Q(x, y, p_2)}\left[\int_{y_0}^{y}\left(\frac{p_2 \dfrac{\partial^2 S_0}{\partial x_2^2}}{Q(x, z, p_2)} - M\left\{\frac{p_2 \dfrac{\partial^2 S_0}{\partial x_2^2}}{Q(x, y, p_2)}\right\}\right) dz\right.\right.$$

$$-M\left\{\int_{y_0}^{y}\left(\frac{p_2 \dfrac{\partial^2 S_0}{\partial x_2^2}}{Q(x, z, p_2)} - M\left\{\frac{p_2 \dfrac{\partial^2 S_0}{\partial x_2^2}}{Q(x, y, p_2)}\right\}\right) dz\right\}$$

$$+\frac{1}{2}\int_{y_0}^{y}\left(\frac{\dfrac{\partial q}{\partial x_2}(x, z)}{Q(x, z, p_2)} - M\left\{\frac{\dfrac{\partial q}{\partial x_2}(x, y)}{Q(x, y, p_2)}\right\}\right) dz$$

$$\left.\left.-M\left\{\int_{y_0}^{y}\left(\frac{\dfrac{\partial q}{\partial x_2}(x, z)}{Q(x, z, p_2)} - M\left\{\frac{\dfrac{\partial q}{\partial x_2}(x, y)}{Q(x, y, p_2)}\right\}\right) dz\right\}\right]\right\}$$

$$= \frac{1}{2} M\left\{\frac{p_2}{Q(x, z, p_2)}\left[\int_{y_0}^{y}\left(\frac{\dfrac{\partial q}{\partial x_2}(x, z)}{Q(x, z, p_2)} - M\left\{\frac{\dfrac{\partial q}{\partial x_2}(x, y)}{Q(x, y, p_2)}\right\}\right) dz\right.\right.$$

$$\left.\left.-M\left\{\int_{y_0}^{y}\left(\frac{\dfrac{\partial q}{\partial x_2}(x, z)}{Q(x, z, p_2)} - M\left\{\frac{\dfrac{\partial q}{\partial x_2}(x, y)}{Q(x, y, p_2)}\right\}\right) dz\right\}\right]\right\}.$$

Similarly, transforming the right side of (21), we find that

$$M\left\{\frac{\frac{\partial q}{\partial x_2}(x, y)}{Q(x, y, p_2)}\frac{\partial \mathring{\tilde{S}}_1}{\partial p_2}\right\}$$

$$= M\left\{\frac{1}{2}\frac{\frac{\partial q}{\partial x_2}(x, y)}{Q(x, y, p_2)}\left[\int_{y_0}^{y}\left(\frac{p_2}{Q(x, z, p_2)} - M\left\{\frac{p_2}{Q(x, y, p_2)}\right\}\right) dz\right.\right.$$

$$\left.\left. - M\left\{\int_{y_0}^{y}\left(\frac{p_2}{Q(x, z, p_2)} - M\left\{\frac{p_2}{Q(x, y, p_2)}\right\}\right) dz\right\}\right]\right\}.$$

By integration by parts it is not hard to verify that the right sides in the last two formulas coincide, whence we obtain the equality $\tilde{\mu}_1 = \mu_1$ and hence also $\tilde{S}_1 = S_1$ and also the fact that these functions have no points of discontinuity on Λ^2.

The functions S_2 and $\tilde{S}_2$ do not coincide, but their difference is a known function depending only on S_0 and S_1 (or $\tilde{S}_0$ and $\tilde{S}_1$). One of the functions S_2 or $\tilde{S}_2$ (but not both at once) may become infinite. The same holds for all the remaining S_j.

In the case of higher dimension we proceed similarly; namely, to each point of the manifold Λ^n we assign not one function S_j but rather a collection of 2^{n-1} functions $S_j(x, y), \ldots, \tilde{S}_j(x_1, p_2, \ldots, p_n, y)$. Several elements of this collection, but not all simultaneously, may become infinite. By means of Theorems 2 and 3 it is easy to verify the following assertions.

1. *The collections do not depend on the decomposition of Λ^n into charts or on the choice of local coordinates in them, but depend only on the point of the manifold.*

2. *The elements of this collection are connected by a known additive relation.*

We return to the two-dimensional case. We find the leading term of the amplitude $\tilde{\varphi}_0^0$ from the equation for k and ε^{-1}:

$$2\left(\frac{\partial}{\partial y}\tilde{\varphi}_0^0(x_1, p_2, y)\right)\left(\frac{\partial \tilde{S}_0}{\partial x_1} + \frac{\partial \tilde{S}_1}{\partial y}\right) + \tilde{\varphi}_0^0(x_1, p_2, y)\frac{\partial^2 \tilde{S}_1}{\partial y^2} = 0.$$

Solving this equation, we obtain

$$\tilde{\varphi}_0^0 = \tilde{c}(x_1, p_2)\left(\frac{\partial \tilde{S}_0}{\partial x_1} + \frac{\partial \tilde{S}_1}{\partial y}\right)^{-1/2}.$$

The equation for k^1 and ε^0 can be written in the form

$$2\left(\frac{\partial}{\partial x_1}\tilde{\varphi}_0^0\right)\left(\frac{\partial \tilde{S}_0}{\partial x_1} + \frac{\partial \tilde{S}_1}{\partial y}\right) + 2\left(\frac{\partial}{\partial y}\tilde{\varphi}_1^0\right)\left(\frac{\partial \tilde{S}_0}{\partial x_1} + \frac{\partial \tilde{S}_1}{\partial y}\right) + 2\left(\frac{\partial}{\partial y}\tilde{\varphi}_0^0\right)\left(\frac{\partial \tilde{S}_1}{\partial x_1} + \frac{\partial \tilde{S}_2}{\partial y}\right)$$

$$+\tilde{\varphi}_0^0\left(\frac{\partial^2 \tilde{S}_0}{\partial x_1^2} + 2\frac{\partial^2 \tilde{S}_1}{\partial x_1 \partial y} + \frac{\partial^2 \tilde{S}_2}{\partial y^2}\right) + \tilde{\varphi}_1^0\frac{\partial^2 \tilde{S}_1}{\partial y^2}$$

$$+ \left(\frac{\partial}{\partial p_2}\tilde{\varphi}_0^0\right)\frac{\partial}{\partial x_2}q(x, y)\big|_{x_2 = -\partial \tilde{S}_0/\partial p_2}$$

$$+\frac{1}{2}\tilde{\varphi}_0^0\frac{\partial^2 \tilde{S}_0}{\partial p_2^2}\frac{\partial^2}{\partial x_2^2}q(x, y)\big|_{x_2 = -\partial \tilde{S}_0/\partial p_2}.$$

Transforming this equation in analogy to (12), we obtain the equation along trajectories

$$\frac{1}{\sqrt{\tilde{J}}}\frac{d}{dt}\left(\tilde{c}(x_1, p_2)\sqrt{\tilde{J}}\right) = 0,$$

where $\tilde{J}$ denotes the determinant

$$\tilde{J} = \det \frac{\partial(x_1, p_2)}{\partial(t, \xi)}.$$

From this we find $\tilde{c}(x_1, p_2)$. We note that $c(x)$ and $\tilde{c}(x_1, p_2)$ are connected by the relation

$$\tilde{c}(x_1, p_2) = \left|\frac{\partial p_2}{\partial x_2}\right|^{-1/2} e^{-i\pi/2\,\text{inerdex}\,\partial p_2/\partial x_2} c(x).$$

We therefore introduce the Maslov index on the manifold Λ^n in the usual way, and we construct the following analogue of the canonical operator.

Suppose that on Λ^n there is given a smooth function $\kappa(r)$, where r is a point of Λ^n, and suppose that on Λ^n a canonical atlas and a subordinate partition of unity are fixed. We define an operator mapping functions on Λ^n into functions on $\mathbf{R}^n$ as follows.

1. In a nonsingular chart Ω_j

$$K^{\varepsilon}_{\Lambda^n}\big(l_j(s)\kappa(r)\big) = e^{ik(S_0(x)+\cdots+\varepsilon^N S_N(x,x_1/\varepsilon))}\left|\det \frac{\partial(t,\xi)}{\partial x}\right|^{1/2}$$
$$\times \big(l_j(r)\kappa(r)\big)\left(q\left(x, \frac{x_1}{\varepsilon}\right) - |p'|^2\right)^{-1/4} e^{i\pi/2\,\text{ind}\,\gamma(r^0,r)}.$$

2. In a singular chart Ω_j with coordinates $x_I, p_{\bar{I}}$

$$K^{\varepsilon}_{\Lambda^n}\big(l_j(r)\kappa(r)\big)$$
$$= F^{-1}_{k,p_{\bar{I}}\to x_{\bar{I}}}\Bigg(e^{ik(\tilde{S}_0(x_I,p_{\bar{I}})+\cdots+\varepsilon^N \tilde{S}_N(x_I,p_{\bar{I}},x_1/\varepsilon))}$$
$$\times\left|\det \frac{\partial(t,\xi)}{\partial(x_I, p_{\bar{I}})}\right|^{1/2}\big(l_j(r)\kappa(r)\big)\left(q\left(x, \frac{x_1}{\varepsilon}\right) - |p'|^2\right)^{-1/4} e^{i\pi/2\,\text{ind}\,\gamma(r^0,r)}\Bigg).$$

Here l_j are the elements of the partition of unity, supp $l_j \subset \Omega_j$, and r^0 is a fixed point of Λ^n satisfying the condition $x_1(r^0) < -a$. We now define the operator $K^{\varepsilon}_{\Lambda^n}$ by means of a partition of unity:

$$K^{\varepsilon}_{\Lambda^n}\kappa = \sum_j K^{\varepsilon}_{\Lambda^n}(l_j\kappa).$$

It can now be verified that, up to lower order terms, the operator we have constructed does not depend either on the atlas or on the partition of unity. If for κ_0 we take the leading term of the amplitude of the averaged equation, then on substituting $K^{\varepsilon}_{\Lambda^n}\kappa_0$ into (1) we obtain a function of order $O(k^{1-\delta})$. It is possible to refine the construction of the operator $K^{\varepsilon}_{\Lambda^n}$ and to construct asymptotics satisfying (1) to accuracy $O(k^{-N})$, where N is any number. This can be done in analogy to [1] and [6]. However, in this case the atlas and the partition of unity must be fixed, and the construction of subsequent approximations depends on them. For example, it is convenient to choose the atlas and the partition of unity as in [9]. To construct subsequent terms of the expansion it is necessary to seek $\kappa(r)$ as an asymptotic series

$$\kappa(r) \sim \left(\kappa_0(r) + \varepsilon\kappa_0^1\left(r, \frac{x_1}{\varepsilon}\right) + \cdots\right) + (\varepsilon k)^{-1}\left(\kappa_1^0\left(r, \frac{x_1}{\varepsilon}\right) + \varepsilon\kappa_1^1\left(r, \frac{x_1}{\varepsilon}\right) + \cdots\right) + \cdots.$$

The next terms of the expansion are constructed as in [1] and [6].

§3. The Schrödinger equation

In this section we consider the equation

$$ih\frac{\partial\psi}{\partial t} = -\frac{h^2}{2m}\Delta\psi + v\left(x, \frac{x_1}{\varepsilon}\right)\psi$$

with the initial conditions

$$\psi(x, t)|_{t=0} = \psi_0(x)e^{iS_0(x)/h}.$$

Of the potential $v(x, x_1/\varepsilon)$ we require that the following conditions be satisfied: $v(x, y)$ is a smooth function periodic in y, and $v(x, y)$ does not depend on y for x in $\mathbf{R}^n \setminus \Omega$, where Ω is a bounded domain. For each fixed ε the function $v(x, x_1/\varepsilon)$ lies in the Schwartz class $S(\mathbf{R}^n)$. Suppose, finally, that $\operatorname{supp} x_0 \cap \Omega = \varnothing$. For this problem it is possible to construct an asymptotic expansion just as in §§1 and 2.

We consider the Hamiltonian equations with Hamiltonian

$$L(x, p, t, E) = p_1 - M\left\{\sqrt{E - |p'|^2 - v(x, y)}\right\}.$$

THEOREM 4. *Suppose that for $t < T$ there exist solutions of the indicated Hamiltonian system with initial conditions*

$$x|_{\tau=0} \in \operatorname{supp}\psi_0, \quad p|_{\tau=0} = \frac{\partial S_0}{\partial x}, \quad t|_{\tau=0} = 0, \quad E|_{\tau=0} = \left|\frac{\partial S_0}{\partial x}\right|^2 + v(x).$$

Then for $t < T$ there is the asymptotic formula $(h \to 0)$

$$\psi(x, t) \sim K^{\varepsilon}_{\Lambda^{n+1}}\left(\chi_0(r) + \varepsilon\chi_1\left(r, \frac{x_1}{\varepsilon}\right) + \cdots\right).$$

For the proof it is necessary to repeat the arguments of the preceding section.

The asymptotics found in the present problem can be justified. Using Theorem 10.6 of [1], for the remainder term of the expansion we get

$$\sup_{0 \leqslant t < T} \|\psi(x, t) - \psi_{d(N)}(x, t)\|_{L^2(\mathbf{R}^n)} \leqslant ch^{-N}.$$

In conclusion the author expresses his gratitude to M. I. Višik for his attention to the work, and to S. M. Kozlov for assistance in formulation of the problem and discussion of the results.

Received 22/MAY/79

BIBLIOGRAPHY

1. V. P. Maslov and M. V. Fedorjuk, *Quasiclassical approximation for the equations of quantum mechanics*, "Nauka", Moscow, 1976. (Russian)

2. B. R. Vaĭnberg, *Quasiclassical approximation in stationary scattering problems*, Funkcional. Anal. i Priložen. **11** (1977), no. 4, 6–18; English transl. in Functional Anal. Appl. **11** (1977).

3. ______, *The short-wave asymptotic behavior of the solutions of stationary problems, and the asymptotic behavior as $t \to \infty$ of the solutions of nonstationary problems*, Uspehi Mat. Nauk **30** (1975), no. 2(182), 3–55; English transl. in Russian Math. Surveys **30** (1975).

4. ______, *Stationary problems of scattering theory and the behavior of solutions of nonstationary problems for large values of time*, Proc. All-Union Conf. Partial Differential Equations (Moscow, 1976), (Commemorating the 75th Anniversary of the Birth of I. G. Petrovskiĭ), Moskov. Gos. Univ. Meh.-Mat. Fak., Moscow, 1976, pp. 58–62. (Russian)

5. L. M. Brehovskih, *Waves in layered media*, Izdat. Akad. Nauk SSSR, Moscow, 1957; English transl., Academic Press, 1960.

6. A. S. Miščenko, B. Ju. Sternin and V. E. Šatalov, *Lagrangian manifolds and the method of the canonical operators*, "Nauka", Moscow, 1978. (Russian)

7. A. G. Kuroš, *A course of higher algebra*, GITTL, Moscow, 1942; English transl. of 10th (1971) ed., "Mir", Moscow, 1975.

8. M. V. Fedorjuk, *The saddle-point method*, "Nauka", Moscow, 1977. (Russian)

9. V. V. Kučerenko, *Asymptotic solutions of equations with complex characteristics*, Mat. Sb. **95 (137)** (1974), 163–213; English transl. in Math. USSR Sb. **24** (1974).

10. N. S. Bahvalov, *Averaging of partial differential equations with rapidly oscillating coefficients*, Dokl. Akad. Nauk SSSR **221** (1975), 516–519; English transl. in Soviet Math. Dokl. **16** (1975).

Translated by J. R. SCHULENBERGER

ABCDEFGHIJ–AMS–898765432