Sbornik: *Mathematics* **192**:7 933–949

Matematicheskiĭ Sbornik **192**:7 3–20
DOI 10.1070/SM2001v192n07ABEH000576

Averaging in a perforated domain with an oscillating third boundary condition

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Abstract. We study an example averaging problem for a second-order elliptic equation in a periodically perforated domain with a third boundary condition (Fourier condition) on the boundary of the holes. Under the assumption that the coefficients of the boundary operator are bounded and the corresponding averages are small we construct the leading terms of the asymptotic expansion of the solution and estimate the error.

Bibliography: 30 titles.

Introduction

The purpose of the present paper is to study an example problem for a second-order elliptic equation in a perforated domain with a third boundary condition on the boundary of the cavities. In contrast to the cases studied previously, in which the compactness of the family of solutions was guaranteed by the smallness of the corresponding coefficient in the third boundary condition, we do not require this coefficient to be small, nor do we require that the right-hand side in the boundary condition be small. We replace these conditions by the weaker condition that the corresponding averages over the surface of the inclusions be small.

At present there are many mathematical papers devoted to the asymptotic analysis of problems in perforated domains. Various results on averaging have been obtained for periodic, almost-periodic and random structures. A detailed bibliography can be found, for example, in [1]–[6]. In particular, problems with a Neumann condition on the boundary of the cavities were studied in [7], [8], and problems with a third boundary condition (Fourier condition) on the boundary of the cavities were studied in [9]–[12], as well as in [13]–[15]. Interesting cases were studied in [16]–[18], where the asymptotics of the problem with an oblique derivative on the surface of the cavities and problems of Steklov type were studied. An interesting case was also studied in [19]. Of particular interest are problems in which the coefficient of the third boundary condition is not small. In the special case when the problem has

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The work of the second and third authors was partially supported by the Russian Foundation for Basic Research (grant no. 98-01-00062). The work of G. A. Chechkin was supported by the Russian Foundation for Basic Research (grant nos. 99-01-00056 and 99-01-01143).

AMS 2000 *Mathematics Subject Classification*. Primary 35J25, 35B27, 35B40; Secondary 74510.

a dissipative character, which can be guaranteed by the correct choice of the sign of the corresponding coefficient in the boundary condition, weak convergence of the solutions of the periodic problem was studied in [13], [20], [21]. The paper [11] contains a study of the asymptotic behavior of the spectrum of the boundary-value problem with a third boundary condition on the boundary of the cavities, in which a large dissipation is compensated for by introducing an unboundedly increasing potential into the equation.

By applying the method of compensated compactness of [22], [23] or the method of two-scale convergence in [24], [25] (see also [26], where the method of two-scale convergence was adapted for perforated domains with a third boundary condition on the boundary of the cavities), one can construct a limiting problem and prove an averaging theorem; but these methods do not provide any estimates of the error. In the present paper we shall use the technique of asymptotic expansions of [27], [28] (see also [29]), which requires some regularity of the data and coefficients, but makes it possible to estimate the rate of convergence. For simplicity we assume that the perforation has a purely periodic structure, although the technique developed in the paper makes it possible to obtain analogous results in the locally periodic case as well. We also assume that the perforation does not intersect the outer boundary of the domain.

The statement of the third boundary condition (the Fourier condition) on the boundary of the cavities involves a non-trivial potential in the limiting equation; in the periodic case this potential is a constant. We emphasize that there is a great difference between the case of a degenerate coefficient in the third boundary condition (the presence of a small parameter as a factor in the coefficient) and the case of a coefficient of order one. In the first case the limiting operator contains only the average of the coefficient over the surface of the hole, while the oscillation of the coefficient with respect to the mean makes no contribution to the limiting operator (see [14]). In the second case (which will be the subject of the present paper) the average is required to be zero (otherwise the solution degenerates rapidly or "blows up" and to compensate for this effect it is necessary to introduce a large parameter into the coefficients of the original operator), and a non-trivial potential arises in the limiting equation as a result of the oscillation. For that reason the problem including a coefficient with zero average differs essentially from the problem containing a positive coefficient in the third boundary condition.

The question of coerciveness of this family of operators is not elementary in the present case. As was shown in [16], the question of coerciveness of the original problem reduces to verifying the coerciveness of the formally averaged operator. In this connection estimates of the potential in the averaged operator are relevant. In this paper we propose a method of obtaining such estimates using an auxiliary problem of Steklov type. It is also interesting to note that this potential always has a "bad" sign, that is, it worsens the coercive properties of the problem.

One example leading to the equations studied in this paper is the problem of the distribution of a stationary temperature field in a porous medium (see Fig. 1).

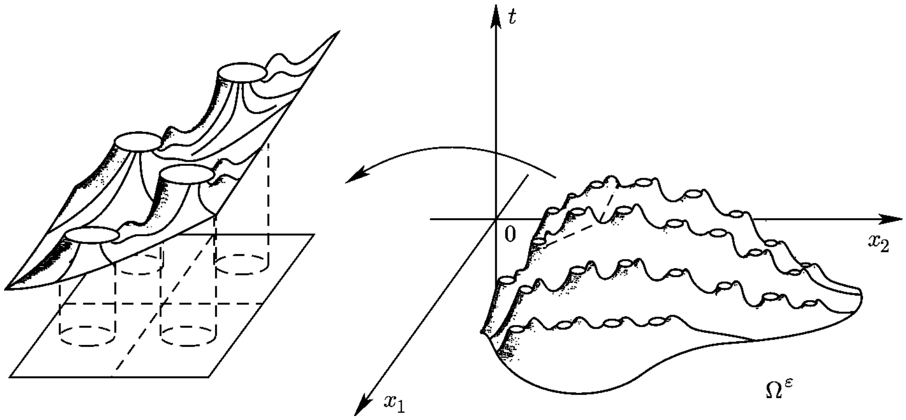

Figure 1. Temperature distribution in a perforated body

§ 1. Statement of the problem

Let Ω be a smooth bounded domain in $\mathbb{R}^d$, $d \geqslant 2$. We use the notation

$$J^\varepsilon = \bigl\{ j \in \mathbb{R}^d : \operatorname{dist}(\varepsilon j, \partial\Omega) \geqslant \varepsilon\sqrt{d}\,\bigr\}, \qquad \square \equiv \Bigl\{ \xi : -\frac{1}{2} < \xi_j < \frac{1}{2},\ j = 1, \dots, d \Bigr\}.$$

Introducing a smooth function $F(\xi)$ of period 1 in ξ and such that $F(\xi)\big|_{\xi\in\partial\square} \geqslant \mathrm{const} > 0$, $F(0) = -1$, $\nabla_\xi F \neq 0$ for $\xi \in \square \setminus \{0\}$, we define

$$Q_j^\varepsilon = \Bigl\{ x \in \varepsilon(\square + j) : F\Bigl(\frac{x}{\varepsilon}\Bigr) \leqslant 0 \Bigr\}$$

and we introduce the perforated domain as follows:

$$\Omega^\varepsilon = \Omega \setminus \bigcup_{j\in J^\varepsilon} Q_j^\varepsilon.$$

In accordance with the construction given above, the boundary $\partial\Omega^\varepsilon$ consists of $\partial\Omega$ and the boundaries of the inclusions $S_\varepsilon \subset \Omega$, $S_\varepsilon = (\partial\Omega^\varepsilon) \cap \Omega$.

We denote an inclusion by $Q = \{\xi : -\frac{1}{2} < \xi_j < \frac{1}{2},\ j = 1, \dots, d,\ F(\xi) \leqslant 0\}$, the boundary of the inclusion Q by $S = \{\xi : F(\xi) = 0\}$, and the outward normal vector to S in "stretched" coordinates by ν.

Here and below we shall assume summation over repeated indices. We consider the following problem:

$$\begin{gathered} -\mathcal{L}_\varepsilon u_\varepsilon := \frac{\partial}{\partial x_k}\Bigl(a_{kj}\Bigl(\frac{x}{\varepsilon}\Bigr) \frac{\partial u_\varepsilon}{\partial x_j} \Bigr) = f(x) \quad \text{in } \Omega^\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \gamma} + p\Bigl(\frac{x}{\varepsilon}\Bigr) u_\varepsilon + \varepsilon q\Bigl(\frac{x}{\varepsilon}\Bigr) u_\varepsilon = g\Bigl(\frac{x}{\varepsilon}\Bigr) \quad \text{on } S_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \ \partial\Omega, \end{gathered} \tag{1}$$

where $\dfrac{\partial u_\varepsilon}{\partial \gamma} := a_{kj} \dfrac{\partial u_\varepsilon}{\partial x_j} \nu_k^\varepsilon$ $\nu^\varepsilon = (\nu_1^\varepsilon, \dots, \nu_d^\varepsilon)$ is the unit outward normal vector to the boundary of the inclusions. It is assumed here that the matrix $(a_{kj}(\xi))$ is symmetric and positive-definite, that is, $\varkappa_1\eta^2 \leqslant a_{kj}\eta_k\eta_j \leqslant \varkappa_2\eta^2$ for any vector η, where $\varkappa_1$ and $\varkappa_2$ are positive constants, and that all the functions $a_{kj}(\xi)$, $p(\xi)$, $q(\xi)$ and $g(\xi)$ are of period 1 with respect to $\xi \in \mathbb{R}^d$. We further require that

$$\langle p(\xi)\rangle_S = \langle g(\xi)\rangle_S = 0, \tag{2}$$

where $\langle\,\cdot\,\rangle_S := \displaystyle\int_S \cdot\, d\sigma$. For convenience, from now on we shall denote the boundary-value problem (1) by the symbol A^ε.

§ 2. Formal asymptotic analysis

We shall seek a solution in the form of a formal asymptotic series

$$u_\varepsilon(x) \sim u_0(x) + \varepsilon u_1(x,\xi) + \varepsilon^2 u_2(x,\xi) + \cdots, \qquad \xi = \frac{x}{\varepsilon}, \tag{3}$$

where all the functions $u_i(x,\xi)$ are assumed periodic with respect to ξ. We introduce the notation (see [29])

$$-\mathfrak{L}_{\alpha\beta}\varphi(x,\xi) := \frac{\partial}{\partial\alpha_k}\left(a_{kj}(\xi)\frac{\partial\varphi(x,\xi)}{\partial\beta_j}\right), \quad \frac{\partial\varphi(x,\xi)}{\partial\gamma_\alpha} := a_{kj}(\xi)\frac{\partial\varphi(x,\xi)}{\partial\alpha_j}\,\nu_k;$$

here α and β assume the values x or ξ. Substituting the series (3) into the problem (1) and gathering terms of the same order in ε both in the equation and in the boundary condition, we obtain a recursive sequence of problems, the leading one of which has the form

$$\begin{gathered} \mathfrak{L}_{\xi\xi}u_1 + \mathfrak{L}_{\xi x}u_0 = 0 \quad \text{in } \square\setminus Q, \\ \frac{\partial u_1}{\partial\gamma_\xi} + \frac{\partial u_0}{\partial\gamma_x} + p(\xi)u_0 = g(\xi) \quad \text{on } S. \end{gathered} \tag{4}$$

The integral identity of the problem (4) looks as follows:

$$\int_{\square\setminus Q} a_{kj}\frac{\partial u_1}{\partial\xi_j}\,\frac{\partial v}{\partial\xi_k}\,d\xi + \int_{\square\setminus Q} a_{kj}\frac{\partial u_0}{\partial x_j}\,\frac{\partial v}{\partial\xi_k}\,d\xi + \int_S p(\xi)u_0 v\,d\sigma = \int_S g(\xi)v\,d\sigma, \tag{5}$$

where $v \in H^1_{\mathrm{per}}(\square\setminus Q)$. The form of the integral identity suggests the structure of the function $u_1(x,\xi)$:

$$u_1(x,\xi) = L(\xi) + M(\xi)u_0(x) + N_i(\xi)\frac{\partial u_0(x)}{\partial x_i}\,. \tag{6}$$

Substituting this expression into (5) and grouping the corresponding terms, we arrive at the following problems for the functions $N_i(\xi)$, $M(\xi)$, and $L(\xi)$:

$$\int_{\square\setminus Q} a_{kj}\frac{\partial N_i}{\partial\xi_j}\,\frac{\partial v}{\partial\xi_k}\,d\xi + \int_{\square\setminus Q} a_{ki}\frac{\partial v}{\partial\xi_k}\,d\xi = 0 \tag{7}$$

or, in classical form,

$$\mathcal{L}_{\xi\xi}(N_i(\xi)+\xi_i)=0 \quad \text{in } \square\setminus Q,$$
$$\frac{\partial N_i(\xi)}{\partial\gamma_\xi}=-a_{ki}(\xi)\nu_k \quad \text{on } S,$$

where $i=1,\dots,d$;

$$\int_{\square\setminus Q} a_{kj}\frac{\partial M}{\partial\xi_j}\frac{\partial v}{\partial\xi_k}\,d\xi+\int_S p(\xi)v\,d\sigma=0 \tag{8}$$

or

$$\mathcal{L}_{\xi\xi}M(\xi)=0 \quad \text{in } \square\setminus Q,$$
$$\frac{\partial M(\xi)}{\partial\gamma_\xi}=-p(\xi) \quad \text{on } S$$

and

$$\int_{\square\setminus Q} a_{kj}\frac{\partial L}{\partial\xi_j}\frac{\partial v}{\partial\xi_k}\,d\xi=\int_S g(\xi)v\,d\sigma \tag{9}$$

or

$$\mathcal{L}_{\xi\xi}L(\xi)=0 \quad \text{in } \square\setminus Q,$$
$$\frac{\partial L(\xi)}{\partial\gamma_\xi}=g(\xi) \quad \text{on } S.$$

The consistency condition is easily verified in the problem (7) using integration by parts, and it follows from (2) in problems (8) and (9). We remark that the functions $L(\xi)$, $M(\xi)$, and $N_i(\xi)$ are defined only up to an additive constant; the natural normalizing condition is

$$\langle L\rangle_{\square\setminus Q}=\langle M\rangle_{\square\setminus Q}=\langle N_i\rangle_{\square\setminus Q}=0 \qquad \forall\, i=1,\dots,d.$$

In what follows these conditions are assumed to hold.

The next degree ε gives us the problem of determining $u_2(x,\xi)$:

$$\begin{gathered}\mathcal{L}_{\xi\xi}u_2+\mathcal{L}_{x\xi}u_1+\mathcal{L}_{\xi x}u_1+\mathcal{L}_{xx}u_0=-f \quad \text{in } \square\setminus Q,\\ \frac{\partial u_2}{\partial\gamma_\xi}+\frac{\partial u_1}{\partial\gamma_x}+p(\xi)u_1+q(\xi)u_0=0 \quad \text{on } S.\end{gathered} \tag{10}$$

We shall need the following proposition.

Lemma 1. *The functions $M(\xi)$ and $N_k(\xi)$ are related by the following integral equality*:

$$\frac{\partial u_0(x)}{\partial x_k}\left(\int_{\square\setminus Q} a_{kj}\frac{\partial M}{\partial\xi_j}\,d\xi-\int_S pN_k\,d\sigma\right)=0.$$

Proof. Substituting $N_i(\xi)$ as a test function into the identity (8) we obtain

$$\int_{\square\setminus Q} a_{kj}\frac{\partial M}{\partial \xi_j}\,\frac{\partial N_i}{\partial \xi_k}\,d\xi + \int_S p(\xi)N_i\,d\sigma = 0.$$

Similarly, using $M(\xi)$ as a test function in the identity (7) we have

$$\int_{\square\setminus Q} a_{kj}\frac{\partial N_i}{\partial \xi_j}\,\frac{\partial M}{\partial \xi_k}\,d\xi + \int_{\square\setminus Q} a_{ki}\frac{\partial M}{\partial \xi_k}\,d\xi = 0.$$

It follows from the symmetry of the matrix $\{a_{kj}\}$ that

$$\int_{\square\setminus Q} a_{kj}\frac{\partial M}{\partial \xi_j}\,d\xi = \int_S pN_k\,d\sigma.$$

The lemma is now proved.

We need the integral identity of the problem (10)

$$\int_{\square\setminus Q} a_{kj}\frac{\partial u_2}{\partial \xi_j}\,\frac{\partial v}{\partial \xi_k}\,d\xi + \int_{\square\setminus Q} a_{kj}\frac{\partial u_1}{\partial x_j}\,\frac{\partial v}{\partial \xi_k}\,d\xi + \int_S p(\xi)u_1 v\,d\sigma + u_0(x)\int_S q(\xi)v\,d\sigma$$
$$-\int_{\square\setminus Q} a_{kj}\frac{\partial M}{\partial \xi_j}v\,d\xi\,\frac{\partial u_0}{\partial x_k} - \int_{\square\setminus Q}\left(a_{ij}\frac{\partial N_k}{\partial \xi_j} + a_{ik}\right)v\,d\xi\,\frac{\partial^2 u_0}{\partial x_i\partial x_k} + |\square\setminus Q|f = 0.$$

The condition for solubility of the problem (10) leads to an equation for the function $u_0(x)$, which is the required formally averaged (limiting) equation. Applying Lemma 1 we rewrite the equation as follows:

$$\begin{aligned}\widehat{a}_{kj}\frac{\partial^2 u_0(x)}{\partial x_k\partial x_j} - u_0(x)\left(\int_S q(\xi)\,d\sigma + \int_S p(\xi)M(\xi)\,d\sigma\right)&\\ = |\square\setminus Q|f(x) + \int_S g(\xi)M(\xi)\,d\sigma,&\end{aligned}\tag{11}$$

where

$$\widehat{a}_{ik} := \int_{\square\setminus Q}\left(a_{ij}(\xi)\frac{\partial N_k(\xi)}{\partial \xi_j} + a_{ik}(\xi)\right)d\xi.$$

Thus, the averaged problem has the form

$$\begin{gathered}\widehat{a}_{kj}\frac{\partial^2 u_0(x)}{\partial x_k\partial x_j} - \langle q\rangle_S u_0(x) + mu_0(x) = |\square\setminus Q|f(x) + l \quad \text{in } \Omega,\\ u_0(x) = 0 \quad \text{on } \partial\Omega,\end{gathered}\tag{12}$$

where $m := -\langle pM\rangle_S$, $l := \langle pL\rangle_S = -\langle gM\rangle_S$. The symbol $\widehat{A}$ denotes the operator of the boundary-value problem (12).

Remark 1. It should be noted that the coerciveness of the limiting operator (12) is a delicate problem, since the constant m, as will be shown below, is always positive. In particular, the well-posedness of the problem (12), which is connected with the coerciveness of the operator, is guaranteed by the inequality $m - \langle q\rangle_S < \lambda_0$, where λ_0 is the first eigenvalue of the operator $\widehat{a}_{ij}\dfrac{\partial}{\partial x_i}\,\dfrac{\partial}{\partial x_j}$ in the space $\overset{\circ}{H}{}^1(\Omega)$.

§ 3. The basic propositions and estimates

In this section we obtain estimates for the averaged potential in (12), after which we state the main result of the paper.

Consider the following auxiliary spectral problem of Steklov type:

$$\begin{aligned} \frac{\partial}{\partial\xi_k}\left(a_{kj}(\xi)\frac{\partial\theta}{\partial\xi_j}\right) &= 0 \quad \text{in } \square\setminus Q, \\ \frac{\partial\theta}{\partial\gamma} &= \Upsilon\theta \quad \text{on } S, \\ \langle\theta\rangle_S &= 0, \end{aligned} \tag{13}$$

posed in the space of functions of period 1 with respect to ξ. And let Υ_1 be the first eigenvalue of this problem, which can be found using the formula

$$\Upsilon_1 = \inf_{\substack{\psi\in H^1_{\mathrm{per}}(\square)\setminus\{0\}\\ \langle\psi\rangle_S=0}} \frac{a(\psi,\psi)}{\langle\psi^2\rangle_S},$$

where $a(u,v) := \displaystyle\int_{\square\setminus Q} a_{kj}\frac{\partial u}{\partial\xi_j}\,\frac{\partial v}{\partial\xi_k}\,d\xi$.

The following lemma holds.

Lemma 2. *The constant m is positive. Moreover the following estimate holds*:

$$\langle p^2\rangle_S\,\frac{\langle p^2\rangle_S}{a(p,p)} \leqslant m \leqslant \frac{\langle p^2\rangle_S}{\Upsilon_1}\,. \tag{14}$$

Remark 2. We note that the equality in the expression (14) is attained at the functions $p(\xi)$, which belong to the eigenspace of the problem (13) corresponding to the first eigenvalue Υ_1.

Proof. Choosing $M(\xi)$ as a test function in the problem (8), we have

$$\int_{\square\setminus Q} a_{kj}\frac{\partial M}{\partial\xi_j}\,\frac{\partial M}{\partial\xi_k}\,d\xi + \int_S p(\xi)M\,d\sigma = 0.$$

Therefore,

$$m = -\langle pM\rangle_S = \left\langle a_{kj}\frac{\partial M}{\partial\xi_j}\,\frac{\partial M}{\partial\xi_k}\right\rangle_{\square\setminus Q} > 0,$$

if $M \neq 0$. It should be noted that M is identically zero provided that p is identically zero.

Consider the variational problem

$$\inf_{\psi\in H^1_{\mathrm{per}}(\square)} H(\psi) \equiv \inf_{\psi\in H^1_{\mathrm{per}}(\square)} \{a(\psi,\psi) + 2\langle p\psi\rangle_S\}. \tag{15}$$

It follows from the integral identity (8) that the infimum in (15) is attained at the function M. Hence it follows that

$$-\inf_{\psi\in H^1_{\mathrm{per}}(\square)} \{a(\psi,\psi) + 2\langle p\psi\rangle_S\} = -a(M,M) - 2\langle pM\rangle_S = -\langle pM\rangle_S = m.$$

Substituting $\psi = -tp$ into the functional $H(\psi)$ we obtain

$$H(-tp) = t^2 a(p,p) - 2t\langle p^2\rangle_S.$$

To find the minimum over t of the function $H(-tp)$, we solve the equation

$$0 = H'_t(-t_0 p) = 2t_0 a(p,p) - 2\langle p^2\rangle_S.$$

The result is

$$t_0 = \frac{\langle p^2\rangle_S}{a(p,p)}$$

and, consequently,

$$H(-t_0 p) = \frac{(\langle p^2\rangle_S)^2}{a(p,p)} - 2\frac{(\langle p^2\rangle_S)^2}{a(p,p)} = -\frac{(\langle p^2\rangle_S)^2}{a(p,p)}\,.$$

Thus the first of inequalities (14) has been proved.

Substituting $\psi = -t\varphi$ and using a similar procedure we obtain

$$H(-t_0\varphi) = -\frac{(\langle p\varphi\rangle_S)^2}{a(\varphi,\varphi)}\,.$$

Since

$$m = \sup_{\varphi\in H^1_{\mathrm{per}}(\square)} \frac{(\langle p\varphi\rangle_S)^2}{a(\varphi,\varphi)} \tag{16}$$

and the supremum is attained at $\varphi = M$, it follows that for an arbitrary φ we obtain $m \geqslant \dfrac{(\langle p\varphi\rangle_S)^2}{a(\varphi,\varphi)}$. It follows from (16) that

$$\frac{1}{m} = \inf_{\varphi\in H^1_{\mathrm{per}}(\square)\setminus\{0\}} \frac{a(\varphi,\varphi)}{(\langle p\varphi\rangle_S)^2} \geqslant \inf_{\substack{\varphi\in H^1_{\mathrm{per}}(\square)\setminus\{0\}\\ \langle\varphi\rangle_S=0}} \frac{a(\varphi,\varphi)}{\langle p^2\rangle_S\langle\varphi^2\rangle_S} = \frac{\Upsilon_1}{\langle p^2\rangle_S}\,.$$

Finally,

$$m \leqslant \frac{\langle p^2\rangle_S}{\Upsilon_1}\,,$$

where Υ_1 is the first eigenvalue of the problem (13). Thus the second inequality of (14) has been proved. The lemma is now proved.

Remark 3. To give a simple explanation of the positivity of the coefficient m in the equation (12), we modify the problem (1) by substituting a Neumann boundary condition on the outer boundary, and for simplicity we set $q \equiv 0$. Then $-m$ is the first eigenvalue of the limiting problem, which by the variational principle coincides with the energy of the ground state.

Thus, keeping in mind the convergence of the spectra and the energies of the prelimiting problem to the spectrum and energy of the averaged problem, it suffices to verify that the energy of the ground state in the prelimiting problem is negative. Substituting the constant into the variational formula yields zero, so that the corresponding infimum is negative, which implies that $-m$ is negative also.

The following theorem justifies the asymptotics constructed for the solution of the problem (1) and gives an estimate of the remainder term.

Theorem 1. *Let $f(x) \in H^1(\Omega)$, and let $p(\xi)$, $q(\xi)$, and $g(\xi)$ be C^1-functions of period* 1. *Assume further that*

$$m < \lambda_0 + \langle q \rangle_S, \tag{17}$$

where λ_0 is defined in Remark 1.

Then for all sufficiently small ε the problem (1) *has a unique solution and the following estimate holds*:

$$\|u_0 + \varepsilon u_1 - u_\varepsilon\|_{H^1(\Omega^\varepsilon)} \leqslant K_1 \sqrt{\varepsilon} : \tag{18}$$

with a constant $K_1 > 0$ independent of ε. Here u_0 is the solution of the problem (12), *and u_1 has the form* (6) *with the functions $N_i(\xi)$, $M(\xi)$, and $L(\xi)$, constructed in* (7), (8), *and $L(\xi)$* (9) *respectively.*

§ 4. Preliminary lemmas

This section is devoted to various technical propositions that will be used in the subsequent analysis. The proofs of the first two lemmas can be found in [14], [30], so that only their statements will be given here.

Lemma 3. *Let $\zeta(x,\xi)$ be a sufficiently smooth function of period* 1 *in ξ and let*

$$Z(x) \equiv \int_S \zeta(x,\xi)\, d\sigma. \tag{19}$$

Then the following estimate holds:

$$\left| \frac{1}{|\square \setminus Q|} \int_{\Omega^\varepsilon} Z(x)u(x)v(x)\, dx - \varepsilon \int_{S_\varepsilon} \zeta\left(x, \frac{x}{\varepsilon}\right) u(x)v(x)\, ds \right| \leqslant C_2 \varepsilon \|u\|_{H^1(\Omega^\varepsilon)} \|v\|_{H^1(\Omega^\varepsilon)}, \tag{20}$$

for any $u(x), v(x) \in H^1(\Omega^\varepsilon)$. The constant C_2 is independent of ε.

Remark 4. Similar estimates were obtained in [11], [12], [16].

The proposition given below is essentially a modification of the preceding lemma.

Lemma 4. *Let $\zeta(x,\xi)$ be a sufficiently smooth function of period* 1 *and assume that*

$$\int_{\square \setminus Q} \zeta(x,\xi)\, d\xi \equiv 0. \tag{21}$$

Then the following estimate holds:

$$\left| \frac{1}{|\square \setminus Q|} \int_{\Omega^\varepsilon} \zeta\left(x, \frac{x}{\varepsilon}\right) u(x)v(x)\, dx \right| \leqslant C_3 \varepsilon \|u\|_{H^1(\Omega^\varepsilon)} \|v\|_{H^1(\Omega^\varepsilon)}$$

for any $u(x), v(x) \in H^1(\Omega^\varepsilon)$. The constant C_3 is independent of ε.

Let λ_0 be defined as in Remark 1. The following lemma holds.

Lemma 5. *If $m < \lambda_0 + \langle q\rangle_S$, then the problem* (12) *is coercive.*

Proof. Using the variational properties of the operator $\widehat{A}$ we arrive at the following relation:

$$\begin{aligned}\inf_{\substack{v\in \overset{\circ}{H}{}^1(\Omega)\\ \|v\|_{L_2(\Omega)}=1}} (-\widehat{A}v, v)_{L_2(\Omega)} &= \inf_{\substack{v\in \overset{\circ}{H}{}^1(\Omega)\\ \|v\|_{L_2(\Omega)}=1}} \int_\Omega \Bigl(\widehat{a}_{ij}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} + (\langle q\rangle_S - m)v^2\Bigr)\,dx \\ &= \inf_{\substack{v\in \overset{\circ}{H}{}^1(\Omega)\\ \|v\|_{L_2(\Omega)}=1}} \int_\Omega \widehat{a}_{ij}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx + (\langle q\rangle_S - m) \\ &= \lambda_0 + \langle q\rangle_S - m.\end{aligned}$$

Thus under the hypotheses of the lemma,

$$(-\widehat{A}v, v)_{L_2(\Omega)} \geqslant C_4\|v\|^2_{L_2(\Omega)}, \qquad C_4 > 0,$$

which completes the proof.

The following proposition is really a modified version of Lemma 3. Here we do not require that the functions u and v be periodic, and we assume that $\square$ is any one of the periodicity cells of the function $p(\xi)$.

Lemma 6. *If $\langle p\rangle_S = 0$, then the following inequality holds*:

$$\Bigl|\int_S p(\xi)u(\xi)v(\xi)\,d\sigma\Bigr| \leqslant C_5\bigl(\|\nabla u\|_{L_2(\square)}\|v\|_{L_2(\square)} + \|u\|_{L_2(\square)}\|\nabla v\|_{L_2(\square)}\bigr), \tag{22}$$

for any $u(\xi), v(\xi) \in H^1(\square)$. The constant C_5 is independent of ε.

Proof. It follows from the hypothesis of the lemma that the problem

$$\begin{aligned}\Delta_\xi\Psi(\xi) &= 0 \quad \text{in } \square\setminus Q,\\ \frac{\partial\Psi}{\partial n} &= p(\xi) \quad \text{on } S,\\ \frac{\partial\Psi}{\partial n} &= 0 \quad \text{on } \partial\square\end{aligned}\tag{23}$$

is soluble. Moreover, this solution is unique up to an additive constant.

We multiply the equation of the problem (23) by the function uv, where $u(\xi), v(\xi)$ lie in $\in H^1(\square)$, and we integrate over the domain $\square\setminus Q$. Integration by parts yields

$$\begin{aligned}\Bigl|\int_S p(\xi)u(\xi)v(\xi)\,d\sigma\Bigr| &= \Bigl|\int_{\square\setminus Q}\Delta_\xi\Psi(\xi)u(\xi)v(\xi)\,d\xi - \int_S p(\xi)u(\xi)v(\xi)\,d\sigma\Bigr| \\ &\leqslant \Bigl|\int_{\square\setminus Q}\bigl((\nabla_\xi\Psi(\xi)), \nabla_\xi(u(\xi)v(\xi))\bigr)\,d\xi\Bigr| \\ &\leqslant C_5\bigl(\|\nabla u\|_{L_2(\square)}\|v\|_{L_2(\square)} + \|u\|_{L_2(\square)}\|\nabla v\|_{L_2(\square)}\bigr).\end{aligned}\tag{24}$$

The lemma is now proved.

The uniform coerciveness of the bilinear form in the integral identity of the problem (1) with respect to ε is the subject of Lemma 7, from which it follows in particular that the problem (1) is well posed.

Lemma 7. *The coerciveness of the averaged problem* (12) *implies the coerciveness of the original problem* (1) *for all sufficiently small* ε.

Proof. We begin by showing that

$$\int_{S_\varepsilon} p\left(\frac{x}{\varepsilon}\right) u^2(x)\,ds \leqslant C_6\left(\alpha \int_{\Omega^\varepsilon} |\nabla u|^2\,dx + \frac{1}{\alpha}\int_{\Omega^\varepsilon} u^2\,dx\right) \tag{25}$$

for any $\alpha > 0$. Indeed, using Lemma 6, we have

$$\left|\int_S p(\xi)u^2(\xi)\,d\sigma\right| \leqslant 2C_5\|\nabla u\|_{L_2(\square)}\|u\|_{L_2(\square)} \leqslant C_7\left(\frac{\alpha}{\varepsilon}\int_\square |\nabla u|^2\,d\xi + \frac{\varepsilon}{\alpha}\int_\square u^2\,d\xi\right).$$

Now, passing to coordinates $x = \varepsilon\xi$ and summing over all cells, we obtain the required inequality (see also [11], [12]).

We shall prove that there exists a sufficiently large Λ independent of ε such that the operator of the boundary-value problem

$$\begin{gathered} \mathcal{L}_\varepsilon u_\varepsilon + \Lambda u_\varepsilon = -f(x) \quad \text{in } \Omega^\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \gamma} + p\left(\frac{x}{\varepsilon}\right) u_\varepsilon + \varepsilon q\left(\frac{x}{\varepsilon}\right) u_\varepsilon = g\left(\frac{x}{\varepsilon}\right) \quad \text{on } S_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{26}$$

is coercive for any ε.

Using Lemmas 3 and 6, we deduce that

$$\begin{aligned} &\int_{\Omega^\varepsilon} a_{ik}\left(\frac{x}{\varepsilon}\right)\frac{\partial v}{\partial x_k}\frac{\partial v}{\partial x_i}\,dx + \int_{S_\varepsilon} p\left(\frac{x}{\varepsilon}\right) v^2\,ds + \varepsilon\int_{S_\varepsilon} q\left(\frac{x}{\varepsilon}\right) v^2\,ds + \int_{\Omega^\varepsilon} \Lambda v^2\,dx \\ &\qquad \geqslant \varkappa_1\|\nabla v\|^2_{L_2(\Omega^\varepsilon)} - (C_6\alpha + O(\varepsilon))\|\nabla u\|^2_{L_2(\Omega^\varepsilon)} + \left(\langle q\rangle_S - \frac{C_6}{\alpha} + \Lambda\right)\|v\|^2_{L_2(\Omega^\varepsilon)}. \end{aligned} \tag{27}$$

Choosing a sufficiently small α and then a sufficiently large Λ, we get the quadratic form on the right-hand side of inequality (27) to be positive-definite, and hence we also get the coercivity.

Consider the following spectral problems:

$$(-A^\varepsilon + \Lambda)^{-1}u^k_\varepsilon = \lambda^k_\varepsilon u^k_\varepsilon, \tag{28}$$

$$(-\widehat{A} + \Lambda)^{-1}u^k = \lambda^k u^k, \tag{29}$$

where A^ε is the operator of the boundary-value problem (1), and $\widehat{A}$ is the operator for the averaged problem (12).

Keeping in mind the coercivity shown above, one can easily verify that the operators $(-A^\varepsilon + \Lambda)^{-1}$ and $(-\widehat{A} + \Lambda)^{-1}$ satisfy conditions $C1$–$C4$ of the spectral convergence theorem for families of operators defined in different spaces (see [4], Ch. III, Theorem 1.9). It follows in particular from this theorem that $\lambda^0_\varepsilon \to \lambda^0$ as $\varepsilon \to 0$. We denote by μ^0_ε and μ^0 the first eigenvalues of the operators $-A^\varepsilon$ and $-\widehat{A}$, respectively. Then $\mu^0_\varepsilon \equiv -\Lambda + 1/\lambda^0_\varepsilon \to \mu^0 \equiv -\Lambda + 1/\lambda^0$ as $\varepsilon \to 0$.

From this it follows by use of the variational principle that the positive-definiteness of the operator $-\widehat{A}$ implies the positive-definiteness of the operator $-A^\varepsilon$ for all sufficiently small ε. The lemma is now proved.

Remark 5. The device connected with passage to the auxiliary problem by a spectral shift plays a role in the analogous situation in [16], Lemma 3, and the coercivity of the problem (26) essentially follows from the results of that paper.

§ 5. Justification of the asymptotics

Proof of Theorem 1. We need to estimate the H^1-norm of the remainder:

$$\|u_0 + \varepsilon u_1 - u_\varepsilon\|_{H^1(\Omega^\varepsilon)}.$$

To do this, we substitute the expression

$$z_\varepsilon\left(x, \frac{x}{\varepsilon}\right) = u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) - u_\varepsilon(x)$$

into equation (1). This yields the equality

$$\begin{aligned}\mathcal{L}_\varepsilon\left(z_\varepsilon\left(x, \frac{x}{\varepsilon}\right)\right) &= \frac{1}{\varepsilon}\mathcal{L}_{\xi x} u_0(x)\Big|_{\xi=\frac{x}{\varepsilon}} + \mathcal{L}_\varepsilon u_0(x) + \varepsilon \mathcal{L}_{xx} u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}} \\ &+ \mathcal{L}_{\xi x} u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}} + \mathcal{L}_{x\xi} u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}} + \frac{1}{\varepsilon}\mathcal{L}_{\xi\xi} u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}} - \mathcal{L}_\varepsilon u_\varepsilon(x).\end{aligned} \tag{30}$$

Keeping in mind the relations

$$\begin{gathered}\mathcal{L}_{\xi\xi} u_1(x,\xi) = -\mathcal{L}_{\xi x} u_0(x), \qquad \mathcal{L}_\varepsilon u_\varepsilon(x) = -f(x), \\ -\mathcal{L}_{\xi x} u_1(x,\xi) = \frac{\partial}{\partial \xi_i}\left(a_{ij}(\xi)\frac{\partial u_0(x)}{\partial x_j} M(\xi)\right) + \frac{\partial}{\partial \xi_i}\left(a_{ij}(\xi)\frac{\partial^2 u_0(x)}{\partial x_j \partial x_k} N_k(\xi)\right), \\ -\mathcal{L}_{x\xi} u_1(x,\xi) = a_{ij}(\xi)\frac{\partial u_0(x)}{\partial x_i}\,\frac{\partial M(\xi)}{\partial \xi_j} + a_{ij}(\xi)\frac{\partial^2 u_0(x)}{\partial x_i \partial x_k}\,\frac{\partial N_k(\xi)}{\partial \xi_j}\end{gathered} \tag{31}$$

and

$$\widehat{a}_{kj}\frac{\partial^2 u_0(x)}{\partial x_k \partial x_j} - \langle q\rangle_S u_0(x) + m u_0(x) = |\square \setminus Q| f(x) - l \quad \text{in } \Omega, \tag{32}$$

we can rewrite (30) in the domain Ω^ε as follows:

$$\begin{aligned}-\mathcal{L}_\varepsilon\left(z_\varepsilon\left(x, \frac{x}{\varepsilon}\right)\right) &= -\varepsilon\mathcal{L}_{xx} u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}} \\ &+ \frac{\partial}{\partial \xi_i}\left(a_{ij}(\xi)\frac{\partial u_0(x)}{\partial x_j} M(\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} + \frac{\partial}{\partial \xi_i}\left(a_{ij}(\xi)\frac{\partial^2 u_0(x)}{\partial x_j \partial x_k} N_k(\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} \\ &+ a_{ij}(\xi)\frac{\partial u_0(x)}{\partial x_i}\,\frac{\partial M(\xi)}{\partial \xi_j}\Big|_{\xi=\frac{x}{\varepsilon}} + a_{ij}(\xi)\frac{\partial^2 u_0(x)}{\partial x_i \partial x_k}\,\frac{\partial N_k(\xi)}{\partial \xi_j}\Big|_{\xi=\frac{x}{\varepsilon}} - \mathcal{L}_\varepsilon u_0(x) \\ &- \frac{1}{|\square \setminus Q|}\widehat{a}_{kj}\frac{\partial^2 u_0(x)}{\partial x_k \partial x_j} + \frac{\langle q\rangle_S - m}{|\square \setminus Q|} u_0(x) + \frac{l}{|\square \setminus Q|}\,.\end{aligned} \tag{33}$$

Similarly, on S_ε we have

$$\begin{aligned}\frac{\partial z_\varepsilon(x,\frac{x}{\varepsilon})}{\partial\gamma} &= -\frac{\partial u_\varepsilon(x)}{\partial\gamma_x}+\frac{\partial u_0(x)}{\partial\gamma_x}+\varepsilon\frac{\partial u_1(x,\xi)}{\partial\gamma_x}\Big|_{\xi=\frac{x}{\varepsilon}}+\frac{\partial u_1(x,\xi)}{\partial\gamma_\xi}\Big|_{\xi=\frac{x}{\varepsilon}}\\ &= p\left(\frac{x}{\varepsilon}\right)u_\varepsilon(x)+\varepsilon q\left(\frac{x}{\varepsilon}\right)u_\varepsilon(x)-g\left(\frac{x}{\varepsilon}\right)+\frac{\partial u_0(x)}{\partial\gamma_x}+\varepsilon\frac{\partial u_1(x,\xi)}{\partial\gamma_x}\Big|_{\xi=\frac{x}{\varepsilon}}\\ &\quad+\frac{\partial L(\xi)}{\partial\gamma_\xi}\Big|_{\xi=\frac{x}{\varepsilon}}+u_0(x)\frac{\partial M(\xi)}{\partial\gamma_\xi}\Big|_{\xi=\frac{x}{\varepsilon}}+\frac{\partial u_0(x)}{\partial x_i}\frac{\partial N_i(\xi)}{\partial\gamma_\xi}\Big|_{\xi=\frac{x}{\varepsilon}}.\end{aligned}$$

Now multiplying equation (33) by $v(x)\in\overset{\circ}{H}{}^1(\Omega)$ and integrating over Ω^ε we obtain

$$\begin{aligned}-\int_{\Omega^\varepsilon}\mathcal{L}_\varepsilon\left(z_\varepsilon\left(x,\frac{x}{\varepsilon}\right)\right)v(x)\,dx &= -\varepsilon\int_{\Omega^\varepsilon}\mathcal{L}_{xx}u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,dx\\ &\quad+\int_{\Omega^\varepsilon}\frac{\partial}{\partial\xi_i}\left(a_{ij}(\xi)\frac{\partial u_0(x)}{\partial x_j}M(\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,dx\\ &\quad+\int_{\Omega^\varepsilon}\frac{\partial}{\partial\xi_i}\left(a_{ij}(\xi)\frac{\partial^2u_0(x)}{\partial x_j\partial x_k}N_k(\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,dx\\ &\quad+\int_{\Omega^\varepsilon}a_{ij}(\xi)\frac{\partial u_0(x)}{\partial x_i}\frac{\partial M(\xi)}{\partial\xi_j}\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,dx\\ &\quad+\int_{\Omega^\varepsilon}a_{ij}(\xi)\frac{\partial^2u_0(x)}{\partial x_i\partial x_k}\frac{\partial N_k(\xi)}{\partial\xi_j}\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,dx-\int_{\Omega^\varepsilon}\mathcal{L}_\varepsilon u_0(x)v(x)\,dx\\ &\quad-\frac{1}{|\Box\setminus Q|}\int_{\Omega^\varepsilon}\widehat{a}_{kj}\frac{\partial^2u_0(x)}{\partial x_k\partial x_j}v(x)\,dx+\frac{1}{|\Box\setminus Q|}\int_{\Omega^\varepsilon}(\langle q\rangle_S-m)u_0(x)v(x)\,dx\\ &\quad+\frac{1}{|\Box\setminus Q|}\int_{\Omega^\varepsilon}lv(x)\,dx.\end{aligned}\tag{34}$$

On the other hand, using Green's formula one can transform the left-hand side of equation (34) as follows:

$$\begin{aligned}-\int_{\Omega^\varepsilon}\mathcal{L}_\varepsilon\left(z_\varepsilon\left(x,\frac{x}{\varepsilon}\right)\right)v(x)\,dx &= \int_{S_\varepsilon}\frac{\partial z_\varepsilon}{\partial\gamma}v(x)\,ds-\int_{\Omega^\varepsilon}\nabla z_\varepsilon\nabla v(x)\,dx\\ &= \int_{S_\varepsilon}p\left(\frac{x}{\varepsilon}\right)u_\varepsilon(x)v(x)\,ds+\varepsilon\int_{S_\varepsilon}q\left(\frac{x}{\varepsilon}\right)u_\varepsilon(x)v(x)\,ds-\int_{S_\varepsilon}g\left(\frac{x}{\varepsilon}\right)v(x)\,ds\\ &\quad+\int_{S_\varepsilon}\frac{\partial u_0(x)}{\partial\gamma_x}v(x)\,ds+\varepsilon\int_{S_\varepsilon}\frac{\partial u_1(x,\xi)}{\partial\gamma_x}\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,ds\\ &\quad+\int_{S_\varepsilon}\left(\frac{\partial L(\xi)}{\partial\gamma_\xi}+u_0(x)\frac{\partial M(\xi)}{\partial\gamma_\xi}+\frac{\partial u_0(x)}{\partial x_i}\frac{\partial N_i(\xi)}{\partial\gamma_\xi}\right)\Big|_{\xi=\frac{x}{\varepsilon}}v(x)\,ds\\ &\quad-\int_{\Omega^\varepsilon}\nabla z_\varepsilon\left(x,\frac{x}{\varepsilon}\right)\nabla v(x)\,dx.\end{aligned}\tag{35}$$

From (34) and (35) we deduce

$$\begin{aligned}
\int_{\Omega^\varepsilon} \nabla z_\varepsilon\left(x, \frac{x}{\varepsilon}\right) \nabla v(x)\, dx &= \int_{S_\varepsilon} p\left(\frac{x}{\varepsilon}\right) u_\varepsilon(x) v(x)\, ds + \varepsilon \int_{S_\varepsilon} q\left(\frac{x}{\varepsilon}\right) u_\varepsilon(x) v(x)\, ds \\
&- \int_{S_\varepsilon} g\left(\frac{x}{\varepsilon}\right) v(x)\, ds + \int_{S_\varepsilon} \frac{\partial u_0(x)}{\partial \gamma_x} v(x)\, ds + \varepsilon \int_{S_\varepsilon} \frac{\partial u_1(x,\xi)}{\partial \gamma_x}\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, ds \\
&+ \int_{S_\varepsilon} \left(\frac{\partial L(\xi)}{\partial \gamma_\xi} + u_0(x) \frac{\partial M(\xi)}{\partial \gamma_\xi} + \frac{\partial u_0(x)}{\partial x_i} \frac{\partial N_i(\xi)}{\partial \gamma_\xi}\right)\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, ds \\
&- \int_{\Omega^\varepsilon} \frac{\partial}{\partial \xi_i}\left(a_{ij}(\xi) \frac{\partial u_0(x)}{\partial x_j} M(\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx \\
&- \int_{\Omega^\varepsilon} \frac{\partial}{\partial \xi_i}\left(a_{ij}(\xi) \frac{\partial^2 u_0(x)}{\partial x_j \partial x_k} N_k(\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx \\
&- \int_{\Omega^\varepsilon} a_{ij}(\xi) \frac{\partial u_0(x)}{\partial x_i} \frac{\partial M(\xi)}{\partial \xi_j}\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx + \varepsilon \int_{\Omega^\varepsilon} \mathcal{L}_{xx} u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx \\
&- \int_{\Omega^\varepsilon} a_{ij}(\xi) \frac{\partial^2 u_0(x)}{\partial x_i \partial x_k} \frac{\partial N_k(\xi)}{\partial \xi_j}\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx + \int_{\Omega^\varepsilon} \mathcal{L}_\varepsilon u_0(x) v(x)\, dx \\
&+ \frac{1}{|\square \setminus Q|} \int_{\Omega^\varepsilon} \widehat{a}_{kj} \frac{\partial^2 u_0(x)}{\partial x_k \partial x_j} v(x)\, dx \\
&- \frac{1}{|\square \setminus Q|} \int_{\Omega^\varepsilon} (\langle q \rangle_S - m) u_0(x) v(x)\, dx - \frac{1}{|\square \setminus Q|} \int_{\Omega^\varepsilon} l v(x)\, dx.
\end{aligned} \tag{36}$$

In view of the obvious relation

$$\begin{aligned}
\frac{\partial}{\partial \xi_i}\left(a_{ij}(\xi) \frac{\partial^2 u_0(x)}{\partial x_j \partial x_k} N_k(\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} &= \varepsilon \frac{\partial}{\partial x_i}\left(a_{ij}(\xi) \frac{\partial^2 u_0(x)}{\partial x_j \partial x_k} N_k(\xi)\Big|_{\xi=\frac{x}{\varepsilon}}\right) \\
&\quad - \varepsilon \frac{\partial}{\partial x_i}\left(a_{ij}(\xi) \frac{\partial^2 u_0(x)}{\partial x_j \partial x_k} N_k(\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}}
\end{aligned}$$

Stokes's theorem yields

$$\begin{aligned}
&\int_{\Omega^\varepsilon} \frac{\partial}{\partial \xi_i}\left(a_{ij}(\xi) \frac{\partial^2 u_0(x)}{\partial x_j \partial x_k} N_k(\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx \\
&\quad + \int_{\Omega^\varepsilon} \frac{\partial}{\partial \xi_i}\left(a_{ij}(\xi) \frac{\partial u_0(x)}{\partial x_j} M(\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, dx \\
&\qquad = \varepsilon \int_{S_\varepsilon} \frac{\partial u_1(x,\xi)}{\partial \gamma_x}\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\, ds + O(\varepsilon) \|v\|_{H^1(\Omega^\varepsilon)}.
\end{aligned} \tag{37}$$

Using (36) and the boundary condition in (16), we evaluate the expression

$$\begin{aligned}
&\left|\int_{\Omega^\varepsilon} \nabla z_\varepsilon\left(x,\frac{x}{\varepsilon}\right)\nabla v(x)\,dx + \int_{S_\varepsilon}\left(p\left(\frac{x}{\varepsilon}\right)+\varepsilon q\left(\frac{x}{\varepsilon}\right)\right)z_\varepsilon\left(x,\frac{x}{\varepsilon}\right)v(x)\,ds\right| \\
&\quad\leqslant \varepsilon\left|\varepsilon\int_{S_\varepsilon} q\left(\frac{x}{\varepsilon}\right)u_1\left(x,\frac{x}{\varepsilon}\right)v(x)\,ds\right| + \left|\int_{S_\varepsilon}\frac{\partial L}{\partial\gamma_\xi}\bigg|_{\xi=\frac{x}{\varepsilon}} v\,ds - \int_{S_\varepsilon} gv\,ds\right| \\
&\qquad + \left|\int_{S_\varepsilon} u_0\frac{\partial M}{\partial \gamma_\xi}\bigg|_{\xi=\frac{x}{\varepsilon}} v\,ds + \int_{S_\varepsilon} pu_0 v\,ds\right| \\
&\qquad + \left|\varepsilon\int_{S_\varepsilon} q\left(\frac{x}{\varepsilon}\right)u_0(x)v(x)\,ds - \frac{1}{|\square\setminus Q|}\int_{\Omega^\varepsilon}\langle q\rangle_S u_0(x)v(x)\,dx\right| \\
&\qquad + \left|\varepsilon\int_{\Omega^\varepsilon}\mathcal{L}_{xx}u_1(x,\xi)\Big|_{\xi=\frac{x}{\varepsilon}} v(x)\,dx + O(\varepsilon)\|v\|_{H^1(\Omega^\varepsilon)}\right| \\
&\qquad + \left|\int_{S_\varepsilon}\left(\frac{\partial u_0(x)}{\partial\gamma_x} + \frac{\partial u_0(x)}{\partial x_i}\,\frac{\partial N_i(x,\xi)}{\partial\gamma_\xi}\bigg|_{\xi=\frac{x}{\varepsilon}}\right)v(x)\,ds\right| \\
&\qquad + \left|\int_{S_\varepsilon}\varepsilon M\left(\frac{x}{\varepsilon}\right)p\left(\frac{x}{\varepsilon}\right)u_0(x)v(x)\,ds + \frac{1}{|\square\setminus Q|}\int_{\Omega^\varepsilon} mu_0(x)v(x)\,dx\right| \\
&\qquad + \left|\frac{1}{|\square\setminus Q|}\int_{\Omega^\varepsilon} lv(x)\,dx + \int_{S_\varepsilon}\varepsilon p\left(\frac{x}{\varepsilon}\right)L\left(\frac{x}{\varepsilon}\right)v(x)\,ds\right| \\
&\qquad + \left|\int_{S_\varepsilon}\varepsilon p\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0}{\partial x_k}N_k\left(\frac{x}{\varepsilon}\right)v(x)\,ds - \int_{\Omega^\varepsilon} a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0}{\partial x_i}\,\frac{\partial M(\xi)}{\partial\xi_j}\bigg|_{\xi=\frac{x}{\varepsilon}} v(x)\,dx\right| \\
&\qquad + \left|\int_{\Omega^\varepsilon}\left(\frac{\widehat{a}_{kj}}{|\square\setminus Q|}\,\frac{\partial^2 u_0(x)}{\partial x_k\partial x_j}v(x) - a_{ij}(\xi)\frac{\partial^2 u_0(x)}{\partial x_i\partial x_k}\,\frac{\partial N_k(\xi)}{\partial \xi_j}\bigg|_{\xi=\frac{x}{\varepsilon}} v(x)\right.\right. \\
&\qquad \left.\left. + \mathcal{L}_\varepsilon u_0(x)v(x)\right)dx\right| \\
&\quad = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10}.
\end{aligned} \tag{38}$$

Let us estimate the term I_4. According to Lemma 3,

$$\begin{aligned}
I_4 &= \left|\varepsilon\int_{S_\varepsilon} q\left(\frac{x}{\varepsilon}\right)u_0(x)v(x)\,ds - \frac{1}{|\square\setminus Q|}\int_{\Omega_\varepsilon}\langle q\rangle_S u_0(x)v(x)\,dx\right| \\
&\leqslant C_2\varepsilon\|u_0\|_{H^1(\Omega^\varepsilon)}\|v\|_{H^1(\Omega^\varepsilon)}.
\end{aligned}$$

Similarly, using Lemma 3, we can estimate I_7 and I_8:

$$I_7 \leqslant C_8\varepsilon\|v\|_{H^1(\Omega^\varepsilon)}, \qquad I_8 \leqslant C_9\varepsilon\|v\|_{H^1(\Omega^\varepsilon)},$$

and using Lemma 4 we can estimate the expression I_{10}:

$$I_{10} \leqslant C_{10}\varepsilon\|v\|_{H^1(\Omega^\varepsilon)}.$$

It is clear that the terms I_1 and I_5 admit the following estimate:

$$|I_1| + |I_5| \leqslant C_{11}\varepsilon\|v\|_{H^1(\Omega^\varepsilon)}.$$

The identities $I_2 \equiv 0$, $I_3 \equiv 0$, and $I_6 \equiv 0$ follow from relations (7)–(9). It follows from Lemma 1 that $I_9 \equiv 0$.

The function z_ε does not vanish on the boundary $\partial\Omega$ thanks to the presence of the corrector u_1. Introducing the standard truncation $\chi_\varepsilon(x)$ in the ε-neighbourhood of the outer boundary, we consider the test function

$$v = u_0 + \varepsilon\chi_\varepsilon(x)u_1 - u_\varepsilon.$$

Here

$$\|\varepsilon u_1(1-\chi_\varepsilon)\|_{H^1(\Omega^\varepsilon)} \leqslant C_{12}\sqrt{\varepsilon}.$$

Substituting the function v into (38) and keeping in mind all the preceding estimates, we arrive at the required inequality (18). The theorem is now proved.

The idea of writing this paper arose as the result of interesting conversations with colleagues from the Univerisity of Aizu in the Fukushima Prefecture of Japan, where G. A. Chechkin spent the month of February 1999. The final version of the paper was written at the Technological Institute of the city of Narvik (Norway) during the autumn of 1999. The authors are very grateful for the support of these organizations.

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Received 22/NOV/99 and 25/SEP/00
Translated by R. COOKE

Typeset by $\mathcal{AMS}$-TEX