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HOMOGENIZATION OF A SINGLE PHASE FLOW THROUGH A POROUS MEDIUM IN A THIN LAYER

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The paper deals with homogenization of stationary and non-stationary high contrast periodic double porosity type problem stated in a porous medium containing a 2D or 3D thin layer. We consider two different types of high contrast medium. The medium of the first type is characterized by the asymptotically vanishing volume fraction of fractures (highly permeable part). The medium of the second type has uniformly positive volume fraction of fracture part. In both cases we construct the homogenized models and prove the convergence results. The techniques used in this work are based on a special version of the two-scale convergence method adapted to thin structures. The resulting homogenized problems are dual-porosity type models that contain terms representing memory effects.

Keywords: Homogenization; double porosity; thin porous structure; single phase flow.

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1. Introduction

Modeling of flow in fractured media is a subject of intensive research in many engineering disciplines, such as petroleum engineering, water resources management, civil engineering. A fissured medium is a structure consisting of a porous and permeable matrix which is interlaced on a fine scale by a system of highly permeable fissures. The majority of fluid transport will occur along flow paths through the fissure system, and the relative volume and storage capacity of the porous matrix is much larger than that of the fissure system. When the system of fissures is so well developed that the matrix is broken into individual blocks or cells that are isolated from each other, there is consequently no flow directly from cell to cell, but only an exchange of fluid between each cell and the surrounding fissure system. The large-scale description will have to incorporate the two different flow mechanisms. For some permeability ratios and some fissures width, the large-scale description is achieved by introducing the so-called double porosity model. It was introduced first for describing the global behavior of fractured porous media by Barenblatt *et al.*[6] and it has been since used in a wide range of engineering specialties related to geohydrology, petroleum reservoir engineering, civil engineering or soil science. More recently, fractured rock domains corresponding to the so-called Excavation Damaged Zone (EDZ) received an increasing attention in connection with the behavior of geological isolation of radioactive waste after the drilling of the wells or shafts (see, e.g., Ref. 12).

The usual double porosity model is to assume that the width of the fractures containing highly permeable porous media is of the same order as the size of the blocks. The related homogenization problem was studied in Ref. 4, and was then revisited in the mathematical literature by many other authors (see, e.g., Refs. 8, 15, 18, 21 and Refs. 22, 7 and 17 and the references therein). The double porosity type problems in the case when the volume of the fracture part is small with respect to the volume of the original domain were studied either by the method involving only one small parameter in Refs. 19, 20 or by the method with two small parameters in Refs. 2, 3 and 10. The singular double porosity model was studied in Ref. 9. Notice that in all these papers it was assumed that the porous reservoir was not very thin.

As was underlined above, the geometry of the nuclear waste depository leads to models stated in a porous domain having a singular geometry (see for instance Ref. 11). Mathematically this results in a double-porosity type problem defined in a thin layer or plate. It is known in the geology that both the fissure part and the matrix system are porous media crossed by many small fissures. The permeability of the matrix is much less than that of the fissure part, thus if we set the permeability of the fissure part to be of order 1, then the permeability of the matrix is very small. In the model problem studied in this paper, the matrix part is made of cubic porous blocks situated periodically along a hyperplane. The complement to the union of blocks, i.e. the fissure part, is a connected set. There are two small parameters in our model. The first one, ε, characterizes the typical size of inhomogeneity and the thickness of the domain. Another parameter δ is responsible for volume fraction of the fissure part.

We consider a single phase flow of a slightly compressible fluid in thin periodic fractured-porous media made of a set of porous blocks with permeability of order $(\varepsilon\delta)^2$, where $0 < \varepsilon \ll \delta \ll 1$; these porous blocks are surrounded by a system of connected fissures. The model is described by a linear parabolic equation stated in a thin domain depending on the parameter ε such that the measure of the domain vanishes as $\varepsilon \to 0$. Our homogenization process consists of two main steps. In the first step we apply the Laplace transform to the studied initial–boundary problem in order to reduce it to a stationary elliptic problem. For each fixed $\delta > 0$ we then homogenize this elliptic problem, i.e. pass to the limit, as ε tends to zero. At this step we face some difficulties with using the two-scale convergence method because the standard two-scale convergence technique applies to a bulk distributed structure while in our case the structure is situated in a small neighborhood of a hyperplane and has an asymptotically vanishing measure. In this connection we use the two-scale convergence method in a tricky way. Namely, we make an anisotropy scaling of the domain in such a way that its thickness is getting uniformly positive. This leads, however, to high anisotropy of the coefficients of the studied operator and, as a result, to highly anisotropic *a priori* estimates. The derivatives with respect to slow and fast variables are then mixed up in the limit equations, and a special analysis is required in order to separate the slow and fast variables in the homogenized problem and to identify the limit. This is the subject of Theorem 3.1 and Corollary 3.1 in Sec. 3. The homogenized problem obtained at the first step, is called the δ-model. Its coefficients still depend on the parameter δ.

In the second step we pass to the limit, as δ tends to zero, and obtain the final stationary homogenized model with no dependence on ε or on δ. It should be noted that the method of two small parameters was widely used in the homogenization theory for modeling various reticulated structures (see, e.g., Refs. 5 and 14 and the references herein).

The homogenized nonstationary model is then obtained by means of the inverse Laplace transform. The corresponding convergence results are given by Theorem 6.1 for the δ-model and by Theorem 2.1 for the fully homogenized problem.

The structure of the paper is as follows. In Sec. 2 we state the 2D version of the problem and formulate the convergence results for the nonstationary model.

In Sec. 3 we apply the Laplace transform to the original problem and then study the obtained stationary problem which is posed in a thin layer (strip). For each fixed $\delta > 0$ we pass to the limit as $\varepsilon \to 0$ and derive the homogenized model (the so-called δ-model). The proof of the convergence result relies on the two-scale convergence method appropriately adapted to thin domains.

In Sec. 4 we pass to the limit in the δ-model, as $\delta \to 0$, and obtain the stationary limit problem.

In Sec. 5 we prove the convergence result for the original nonstationary problem. The resulting homogenized problem is a dual-porosity type model that contains a nonlocal in temporal variable term representing memory effects. The nonstationary effective δ-model is obtained in Sec. 6, its derivation is based on the results of Sec. 3.

Finally, in Sec. 7 we extend the results of the previous sections to the case of 3D porous medium occupying a thin layer (plate). The technique is essentially the same as in the 2D case, the minor modifications required are listed in this section.

2. Statement of the Problem and Main Result

Let Ω^ε be a rectangle in $\mathbb{R}^2$,

$$\Omega^\varepsilon = \left(-\frac{\varepsilon}{2}, +\frac{\varepsilon}{2}\right) \times (0, L).$$

We introduce a periodic structure in Ω^ε as follows. Denote by $\mathcal{Y}$ the reference cell

$$\mathcal{Y} = \left(-\frac{1}{2}, +\frac{1}{2}\right) \times (0, 1)$$

and by $\mathcal{F}^\delta$ the reference fracture part $\mathcal{F}^\delta = \{y \in \mathcal{Y},\ \text{dist}\,(y, \partial\mathcal{Y}) < \frac{\delta}{2}\}$. The reference matrix block is then defined by $\mathcal{M}^\delta = \mathcal{Y}\backslash\overline{\mathcal{F}^\delta}$. Assuming that L is an integer multiplier of ε: $L = N\varepsilon$, $N \in \mathbb{N}$, we define

$$\Omega_f^{\varepsilon,\delta} = \bigcup_{j=0}^{N-1} \varepsilon(\mathcal{F}^\delta + (0, j)), \quad \Omega_m^{\varepsilon,\delta} = \bigcup_{j=0}^{N-1} \varepsilon(\mathcal{M}^\delta + (0, j)).$$

The flow in the matrix-fracture medium Ω^ε is described by the equation:

$$\textit{Micromodel}: \begin{cases} \omega^{\varepsilon,\delta}(x)u_t^{\varepsilon,\delta} - \text{div}\,(k^{\varepsilon,\delta}(x)\nabla u^{\varepsilon,\delta}) = G^{\varepsilon,\delta}(x) & \text{in } (0,T) \times \Omega^\varepsilon; \\ \nabla u^{\varepsilon,\delta} \cdot \boldsymbol{\nu} = 0 & \text{on } (0,T) \times \partial\Omega^\varepsilon; \\ u^{\varepsilon,\delta}(0, x) = 0 & \text{in } \Omega^\varepsilon \end{cases} \tag{2.1}$$

where

$$\omega^{\varepsilon,\delta}(x) = \begin{cases} \omega_f & \text{in } \Omega_f^{\varepsilon,\delta}; \\ \omega_m & \text{in } \Omega_m^{\varepsilon,\delta}; \end{cases} \quad k^{\varepsilon,\delta}(x) = \begin{cases} k_f & \text{in } \Omega_f^{\varepsilon,\delta}; \\ k_m(\varepsilon\delta)^2 & \text{in } \Omega_m^{\varepsilon,\delta}; \end{cases}$$

$$G^{\varepsilon,\delta}(x) = \begin{cases} (g+h)(x) & \text{in } \Omega_f^{\varepsilon,\delta}; \\ h(x) & \text{in } \Omega_m^{\varepsilon,\delta}. \end{cases}$$

Here $\omega_f, \omega_m, k_f, k_m$ are positive constants and $g, h \in C^1(\mathbb{R}^2)$. It is convenient to introduce the notation:

$$u^{\varepsilon,\delta} = \begin{cases} \rho^{\varepsilon,\delta} & \text{in } \Omega_f^{\varepsilon,\delta}; \\ \sigma^{\varepsilon,\delta} & \text{in } \Omega_m^{\varepsilon,\delta} \end{cases}$$

and to rewrite problem (2.1) separately in the fracture and matrix parts with the appropriate interface conditions. Namely, in the fracture domain Eq. (2.1) reads

$$\begin{cases} \omega_f \rho_t^{\varepsilon,\delta} - \operatorname{div}(k_f \nabla \rho^{\varepsilon,\delta}) = (g+h)(x) & \text{in } (0,T) \times \Omega_f^{\varepsilon,\delta}; \\ k_f \nabla \rho^{\varepsilon,\delta} \cdot \boldsymbol{\nu} = k_m (\varepsilon\delta)^2 \nabla \sigma^{\varepsilon,\delta} \cdot \boldsymbol{\nu} & \text{on } (0,T) \times \gamma_{mf}^{\varepsilon,\delta}; \\ \nabla \rho^{\varepsilon,\delta} \cdot \boldsymbol{\nu} = 0 & \text{on } (0,T) \times \partial\Omega^{\varepsilon}; \\ \rho^{\varepsilon,\delta}(0,x) = 0 & \text{in } \Omega_f^{\varepsilon,\delta}, \end{cases} \tag{2.2}$$

where $\gamma_{mf}^{\varepsilon,\delta}$ denotes the matrix-fracture interface. The flow in the matrix domain is controlled by

$$\begin{cases} \omega_m \sigma_t^{\varepsilon,\delta} - \operatorname{div}(k_m (\varepsilon\delta)^2 \nabla \sigma^{\varepsilon,\delta}) = h(x) & \text{in } (0,T) \times \Omega_m^{\varepsilon,\delta}; \\ \sigma^{\varepsilon,\delta} = \rho^{\varepsilon,\delta} & \text{on } (0,T) \times \gamma_{mf}^{\varepsilon,\delta}; \\ \sigma^{\varepsilon,\delta}(0,x) = 0 & \text{in } \Omega_m^{\varepsilon,\delta}. \end{cases} \tag{2.3}$$

It is well known that, for any $\varepsilon, \delta > 0$, there exists a unique solution $u^{\varepsilon,\delta} = \langle \rho^{\varepsilon,\delta}, \sigma^{\varepsilon,\delta} \rangle$ of the boundary value problem (2.1) (or of the equivalent system (2.2)–(2.3)) in the space $C(0,T; H^1(\Omega^\varepsilon))$.

The goal of this work is to study the asymptotic behavior of $u^{\varepsilon,\delta}$ as $\varepsilon, \delta \to 0$. We are going to show that for any fixed δ problem (2.1) admits homogenization (as $\varepsilon \to 0$) and that the homogenized solution converges, as $\delta \to 0$, to a solution of the effective problem:

$$\textit{Macromodel}: \begin{cases} \omega_f \rho_t^* - \dfrac{1}{2} k_f \dfrac{\partial^2 \rho^*}{\partial \xi^2} = G(\xi) + \mathbf{S}(\rho^*) & \text{in } (0,T) \times (0,L); \\ \dfrac{\partial \rho^*}{\partial \xi}(t,0) = \dfrac{\partial \rho^*}{\partial \xi}(t,L) = 0 & \text{on } (0,T); \\ \rho^*(0,\xi) = 0 & \text{in } (0,L) \end{cases} \tag{2.4}$$

with $G(\xi) = (g+h)(0,\xi)$ and the additional source term

$$\mathbf{S}(\rho^*) = -\frac{2\sqrt{k_m \omega_m}}{\sqrt{\pi}} \int_0^t \frac{\rho_t^*(\tau,\xi)}{\sqrt{t-\tau}} d\tau + 2h(0,\xi) \sqrt{\frac{t\, k_m}{\pi \omega_m}}. \tag{2.5}$$

Here and in what follows we identify the variables x_2 and ξ, as well as functions of x which do not depend on x_1, with the corresponding functions of ξ.

According to Ref. 16 problem (2.4)–(2.5) is well-posed and, under our standing assumptions, it has a unique solution $\rho^* \in L^2(0,T; H^1(0,L))$.

Notice that although the term $S(\rho^*)$ does depend on the unknown function ρ^*, traditionally in the mechanics of porous media it is called "additional source term". In this paper we keep this convention.

Let us emphasize that the presence of the convolution term in the limit equation (2.4) represents the memory effect in the limit dynamics.

The following result describes the limit behavior of u^ε, as $\varepsilon \to 0$. In the matrix part of the domain the diffusion is asymptotically negligible so that for a time-independent right-hand side $h(x)$ the corresponding solution is getting linear in time (see formula (2.6) below). Also, since the volume fraction of the fractured part of the domain is vanishing, as $\delta \to 0$, the first relation in (2.6) holds.

Theorem 2.1. *Let* $u^{\varepsilon,\delta} = \langle \rho^{\varepsilon,\delta}, \sigma^{\varepsilon,\delta} \rangle$ *be the solution of* (2.1). *Then, for any* $t \in (0,T)$,

(I) *the function* $\sigma^{\varepsilon,\delta}$, *as well as the function* $u^{\varepsilon,\delta}$, *converges to* $(th(x))$, *namely*:

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{1}{|\Omega^\varepsilon|} \left\| \omega^{\varepsilon,\delta}\sigma^{\varepsilon,\delta} - th \right\|^2_{L^2(\Omega_m^{\varepsilon,\delta})} = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{1}{|\Omega^\varepsilon|} \left\| \omega^{\varepsilon,\delta} u^{\varepsilon,\delta} - th \right\|^2_{L^2(\Omega^\varepsilon)} = 0; \tag{2.6}$$

(II) *the function* $\rho^{\varepsilon,\delta}$ *satisfies the limit relation*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{1}{|\Omega_f^{\varepsilon,\delta}|} \left\| \rho^{\varepsilon,\delta} - \rho^* \right\|^2_{L^2(\Omega_f^{\varepsilon,\delta})} = 0, \tag{2.7}$$

where $\rho^* = \rho^*(t,\xi)$ *is a solution of* (2.4)–(2.5).

(III) *For any* $t \in (0,T)$, *and any function* $\phi = \phi(x)$ *continuous in the vicinity of the segment* $\{x \in \mathbb{R}^2 : \; x_1 = 0; \; 0 \le x_2 \le L\}$, *it holds*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{L}{|\Omega_f^{\varepsilon,\delta}|} \int_{\Omega^\varepsilon} k^{\varepsilon,\delta}(x) \nabla u^{\varepsilon,\delta} \phi(x)\, dx = \frac{k_f}{2} \int_0^L \vec{\mathbf{R}}^*(t,\xi)\phi(0,\xi)\, d\xi \tag{2.8}$$

with

$$\vec{\mathbf{R}}^*(t,\xi) = \left(0, \frac{\partial \rho^*}{\partial \xi}(t,\xi)\right).$$

This paper also deals with the asymptotic behavior of the solution of problem (2.1), as $\varepsilon \to 0$, for a fixed positive δ. The corresponding homogenization result will be formulated and proved in Sec. 6, Theorem 6.1.

Remark 2.1. It is clear from (2.6) and (2.7) that the limit values of $u^{\varepsilon,\delta}$ on the matrix and fracture parts ($th(0,x_2)$ and $\rho^*(t,x_2)$, respectively) only depend on the slow variables x_2 and t and in general do not coincide. It contradicts our intuition because in the original problem (2.1) the solution $u^{\varepsilon,\delta}$ is continuous at the matrix-fracture interface. In order to explain this phenomenon we notice that for each fixed $\delta > 0$ the two-scale limit of $u^{\varepsilon,\delta}$ is continuous. However, as $\delta \to 0$, the two-scale limit function is getting closer to a constant everywhere in the matrix blocks except for a small neighborhood of the interface where the boundary layer type correctors arise. Since the result is given in terms of L^2-norms, we neglect these boundary layer functions (as $\delta \to 0$) and thus make the limit function discontinuous.

Remark 2.2. We assume in Theorem 2.1 that the right-hand side $h(x)$ does not depend on the temporal variable just for presentation simplicity. In general the right-hand side of the form $h(x,t)$ can be considered exactly in the same way. For a time-dependent $h(x,t)$ the relation (2.6) reads

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{|\Omega^\varepsilon|}\left\|\omega^{\varepsilon,\delta}u^{\varepsilon,\delta}-\int_0^t h(\cdot,s)ds\right\|^2_{L^2(\Omega^\varepsilon)}=0.$$

Theorem 2.1 will be proved in three steps. At the first step we apply the Laplace transform to problem (2.1) in the time variable and then study the asymptotic behavior of a solution of the corresponding stationary boundary value problem as $\varepsilon \to 0$, $\delta > 0$ being fixed. We then obtain a stationary homogenized problem stated on the interval $(0, L)$ with the coefficients depending on δ. At the second step we pass to the limit as $\delta \to 0$ and obtain a stationary limit problem, i.e. the problem independent of ε, δ. Finally, at the third step we make the inverse Laplace transform and prove the convergence for the original nonstationary problem.

3. Step 1. Homogenizing the Stationary Model

We begin by applying the Laplace transform to (2.1). This gives

$$\textit{Stationary micromodel}: \begin{cases}\lambda\omega^{\varepsilon,\delta}u_\lambda^{\varepsilon,\delta}-\operatorname{div}(k^{\varepsilon,\delta}\nabla u_\lambda^{\varepsilon,\delta})=G_\lambda^{\varepsilon,\delta} & \text{in } \Omega^\varepsilon;\\ \nabla u_\lambda^{\varepsilon,\delta}\cdot\boldsymbol{\nu}=0 & \text{on } \partial\Omega^\varepsilon,\end{cases} \tag{3.1}$$

where

$$G_\lambda^{\varepsilon,\delta}(x)=\begin{cases}(g_\lambda+h_\lambda)(x) & \text{in } \Omega_f^{\varepsilon,\delta};\\ h_\lambda(x) & \text{in } \Omega_m^{\varepsilon,\delta}\end{cases}$$

with $\lambda > 0$, $g_\lambda(x) = \lambda^{-1}g(x)$, $h_\lambda(x) = \lambda^{-1}h(x)$.

As in the previous section, we can rewrite (3.1) separately in the fracture and matrix parts. Namely,

$$\begin{cases}\lambda\omega_f\rho_\lambda^{\varepsilon,\delta}-\operatorname{div}(k_f\nabla\rho_\lambda^{\varepsilon,\delta})=(g_\lambda+h_\lambda)(x) & \text{in } \Omega_f^{\varepsilon,\delta};\\ k_f\nabla\rho_\lambda^{\varepsilon,\delta}\cdot\boldsymbol{\nu}=k_m(\varepsilon\delta)^2\nabla\sigma_\lambda^{\varepsilon,\delta}\cdot\boldsymbol{\nu} & \text{on } \gamma_{mf}^{\varepsilon,\delta};\\ \nabla\rho_\lambda^{\varepsilon,\delta}\cdot\boldsymbol{\nu}=0 & \text{on } \partial\Omega^\varepsilon\end{cases} \tag{3.2}$$

and

$$\begin{cases}\lambda\omega_m\sigma_\lambda^{\varepsilon,\delta}-\operatorname{div}(k_m(\varepsilon\delta)^2\nabla\sigma_\lambda^{\varepsilon,\delta})=h_\lambda(x) & \text{in } \Omega_m^{\varepsilon,\delta};\\ \sigma_\lambda^{\varepsilon,\delta}=\rho_\lambda^{\varepsilon,\delta} & \text{on } \gamma_{mf}^{\varepsilon,\delta}.\end{cases} \tag{3.3}$$

Next we transform ε-dependent domain Ω^ε into the fixed domain

$$\Pi = \left(-\frac{1}{2}, +\frac{1}{2}\right) \times (0, L)$$

by the change of variables $z_1 = \frac{x_1}{\varepsilon}$, $z_2 = x_2$, and denote by $\Pi_f^{\varepsilon,\delta}$, $\Pi_m^{\varepsilon,\delta}$ the images, under this transformation, of $\Omega_f^{\varepsilon,\delta}$, $\Omega_m^{\varepsilon,\delta}$, respectively. The image of $\gamma_{mf}^{\varepsilon,\delta}$ is denoted by $\Gamma_{mf}^{\varepsilon,\delta}$. In the new variables $z_1,\ z_2$ Eq. (3.1) reads

$$\begin{cases} \lambda\omega^{\varepsilon,\delta}(\varepsilon z_1, z_2) W_\lambda^{\varepsilon,\delta} - \operatorname{div}(\mathbf{K}^{\varepsilon,\delta}(z)\nabla W_\lambda^{\varepsilon,\delta}) = G_\lambda^{\varepsilon,\delta}(\varepsilon z_1, z_2) & \text{in } \Pi; \\ k_f \varepsilon^{-2} \dfrac{\partial W_\lambda^{\varepsilon,\delta}}{\partial z_1}\nu_1 = 0 \text{ and } k_f \dfrac{\partial W_\lambda^{\varepsilon,\delta}}{\partial z_2}\nu_2 = 0 & \text{on } \partial\Pi, \end{cases} \tag{3.4}$$

where $W_\lambda^{\varepsilon,\delta}(z) = u_\lambda^{\varepsilon,\delta}(\varepsilon z_1, z_2)$, and the matrix $\mathbf{K}^{\varepsilon,\delta}$ is given by

$$\mathbf{K}^{\varepsilon,\delta}(z) = \begin{pmatrix} \varepsilon^{-2} k^{\varepsilon,\delta}(\varepsilon z_1, z_2) & 0 \\ 0 & k^{\varepsilon,\delta}(\varepsilon z_1, z_2) \end{pmatrix}.$$

As usual, we want to rewrite (3.4) separately in the fracture and matrix parts. To this end we denote

$$R_\lambda^{\varepsilon,\delta}(z) = \rho_\lambda^{\varepsilon,\delta}(\varepsilon z_1, z_2), \quad S_\lambda^{\varepsilon,\delta}(z) = \sigma_\lambda^{\varepsilon,\delta}(\varepsilon z_1, z_2).$$

Equation (3.2) now reads

$$\begin{cases} \lambda\omega_f R_\lambda^{\varepsilon,\delta} - k_f \varepsilon^{-2} \dfrac{\partial^2 R_\lambda^{\varepsilon,\delta}}{\partial z_1^2} - k_f \dfrac{\partial^2 R_\lambda^{\varepsilon,\delta}}{\partial z_2^2} = (g_\lambda + h_\lambda)(\varepsilon z_1, z_2) & \text{in } \Pi_f^{\varepsilon,\delta}; \\ k_f \dfrac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_1}\nu_1 = k_m(\varepsilon\delta)^2 \dfrac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_1}\nu_1 & \text{on } \Gamma_{mf}^{\varepsilon,\delta}; \\ k_f \dfrac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2}\nu_2 = k_m(\varepsilon\delta)^2 \dfrac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_2}\nu_2 & \text{on } \Gamma_{mf}^{\varepsilon,\delta}; \\ k_f \varepsilon^{-2} \dfrac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_1}\nu_1 = 0 \quad \text{and} \quad k_f \dfrac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2}\nu_2 = 0 & \text{on } \partial\Pi. \end{cases} \tag{3.5}$$

Similarly Eq. (3.3) reads

$$\begin{cases} \lambda\omega_m S_\lambda^{\varepsilon,\delta} - k_m \delta^2 \dfrac{\partial^2 S_\lambda^{\varepsilon,\delta}}{\partial z_1^2} - k_m(\varepsilon\delta)^2 \dfrac{\partial^2 S_\lambda^{\varepsilon,\delta}}{\partial z_2^2} = h_\lambda(\varepsilon z_1, z_2) & \text{in } \Pi_m^{\varepsilon,\delta}; \\ S_\lambda^{\varepsilon,\delta} = R_\lambda^{\varepsilon,\delta} & \text{on } \Gamma_{mf}^{\varepsilon,\delta}. \end{cases} \tag{3.6}$$

In what follows χ^δ denotes a y_2-periodic solution of the problem:

$$\begin{cases} -\Delta_y \chi^\delta = 0 & \text{in } \mathcal{F}^\delta; \\ \dfrac{\partial(\chi^\delta - y_2)}{\partial y_1}\nu_1 = 0 \text{ and } \dfrac{\partial(\chi^\delta - y_2)}{\partial y_2}\nu_2 = 0 & \text{on } \Gamma_{mf}^\delta \\ \dfrac{\partial(\chi^\delta - y_2)}{\partial y_1} = 0 & \text{on } \partial\mathcal{Y} \cap \left\{y = \pm\dfrac{1}{2}\right\}. \end{cases} \tag{3.7}$$

The function U_λ^δ is a solution of

$$\begin{cases} \lambda\omega_m U_\lambda^\delta - k_m\delta^2\,\Delta_y U_\lambda^\delta = 0 & \text{in } \mathcal{M}^\delta; \\ U_\lambda^\delta = 1 & \text{on } \Gamma_{mf}^\delta. \end{cases} \tag{3.8}$$

Finally, $\mathfrak{L}(V;W)$ stands for the space of linear and continuous operators from V to W, where V and W are real Banach spaces.

Remark 3.1. The variable z_1 is somehow twofold because in the original problem (2.1) it varies on the interval $[-\varepsilon/2, \varepsilon/2]$ and serves as a fast variable while in the rescaled problem (3.4) it becomes a slow variable. In this connection we set $y_1 \equiv z_1$ and use both symbols z_1 and y_1 for the notation convenience. Notice also that $x_2 \equiv z_2 \equiv \xi$.

We proceed with the main result of the section. We want to show that the homogenized model can be described in terms of the following equation:

$$\delta\text{-}model: \begin{cases} |\mathcal{F}^\delta|\lambda\omega_f R_\lambda^\delta - k_f K^\delta \dfrac{d^2R_\lambda^\delta}{d\xi^2} = G_\lambda^\delta(\xi) + \mathbf{S}(R_\lambda^\delta) \quad \text{in } (0,L); \\ \dfrac{dR_\lambda^\delta}{d\xi}(0) = \dfrac{dR_\lambda^\delta}{d\xi}(L) = 0 \end{cases} \tag{3.9}$$

whose coefficients K^δ, I_λ^δ, and $G_\lambda^\delta(\xi)$ are given by

$$\begin{gathered} K^\delta = |\mathcal{F}^\delta| - \int_{\mathcal{F}^\delta} \frac{\partial\chi^\delta}{\partial y_2}\,dy; \quad G_\lambda^\delta(\xi) = |\mathcal{F}^\delta|(g_\lambda + h_\lambda)(0,\xi); \\ \mathbf{S}(R_\lambda^\delta) = -I_\lambda^\delta R_\lambda^\delta + \frac{I_\lambda^\delta}{\lambda\omega_m} h_\lambda(0,\xi); \quad I_\lambda^\delta = \lambda\omega_m \int_{\mathcal{M}^\delta} U_\lambda^\delta(y)\,dy. \end{gathered} \tag{3.10}$$

Later on we will show that this equation has a unique solution.

Remark 3.2. All the coefficients of the equations in (3.9) are vanishing as $\delta \to 0$ and, in fact, are of order $|\mathcal{F}^\delta|$. To make the asymptotic behavior of these coefficients more visible for small δ, one has to divide Eq. (3.9) by $|\mathcal{F}^\delta|$. As will be shown later on (see (4.14), (4.15)), after this normalization the coefficients of the resulting equation have nontrivial limits as $\delta \to 0$.

Theorem 3.1. *A solution* $W_\lambda^{\varepsilon,\delta}$ *of* (3.4) *strongly two-scale converges, as* $\varepsilon \to 0$, *to a function* $W_\lambda^\delta(z,y_2) = \langle R_\lambda^\delta(z_2), S_\lambda^\delta(z,y_2)\rangle$, *where* R_λ^δ *is a solution of* (3.9), *and*

$$S_\lambda^\delta(z,y_2) = R_\lambda^\delta(z_2)U_\lambda^\delta(z_1,y_2) + \zeta_\lambda^\delta(z_1,y_2)h_\lambda(0,z_2), \tag{3.11}$$

here $U_\lambda^\delta(z_1,y_2)$ *is a solution of* (3.8), *and* $\zeta_\lambda^\delta(z_1,y_2) = \frac{1}{\lambda\omega_m}(1 - U_\lambda^\delta(z_1,y_2))$. *Moreover, there is an extension operator* $\mathsf{P}^{\varepsilon,\delta} \in \mathfrak{L}(L^2(\Pi_f^{\varepsilon,\delta}); L^2(\Pi)) \cap \mathfrak{L}(H^1(\Pi_f^{\varepsilon,\delta}); H^1(\Pi))$ *such that*

$$\mathsf{P}^{\varepsilon,\delta} R_\lambda^{\varepsilon,\delta} \to R_\lambda^\delta \quad \textit{weakly in } H^1(\Pi), \tag{3.12}$$

where $R_\lambda^{\varepsilon,\delta}$ *is the solution of* (3.5).

Due to the regularity of W_λ^δ, the strong two-scale convergence stated in Theorem 3.1 implies the following result.

Corollary 3.1. *For any* $\delta > 0$ *the convergence takes place*

$$\lim_{\varepsilon\to 0} \frac{1}{|\Omega^\varepsilon|} \int_{\Omega^\varepsilon} \left|(u_\lambda^{\varepsilon,\delta}(x) - W_\lambda^\delta\left(x_2, \frac{x}{\varepsilon}\right)\right|^2 dx = 0.$$

Notice that due to (3.7), one can represent K^δ in a slightly different form. Namely, multiplying (3.7) by χ^δ, integrating by parts, and using the boundary conditions in (3.7), one has

$$K^\delta = \frac{1}{|\mathcal{F}^\delta|} \alpha_{\mathcal{F}^\delta} \left(\chi^\delta - y_2, \chi^\delta - y_2\right), \tag{3.13}$$

where $\alpha_{\mathcal{F}^\delta}$ is the bilinear form associated with the Laplace operator. The formula (3.13) implies that $K^\delta > 0$. It is also easy to see that $I_\lambda^\delta > 0$. Therefore, problem (3.9) is well-posed on $(0, L)$.

3.1. *Proof of Theorem* 3.1

First we obtain *a priori* estimates. To this end we multiply (3.5) by $R_\lambda^{\varepsilon,\delta}$ and (3.6) by $S_\lambda^{\varepsilon,\delta}$, and then integrate the resulting relation by parts. After simple computations this gives the bounds

$$\begin{aligned} &\lambda\omega_f \|R_\lambda^{\varepsilon,\delta}\|^2_{L^2(\Pi_f^{\varepsilon,\delta})} + k_f \varepsilon^{-2} \left\| \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_1} \right\|^2_{L^2(\Pi_f^{\varepsilon,\delta})} + k_f \left\| \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2} \right\|^2_{L^2(\Pi_f^{\varepsilon,\delta})} \\ &\quad + \lambda\omega_m \|S_\lambda^{\varepsilon,\delta}\|^2_{L^2(\Pi_m^{\varepsilon,\delta})} + k_m \delta^2 \left\| \frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_1} \right\|^2_{L^2(\Pi_m^{\varepsilon,\delta})} + k_m (\varepsilon\delta)^2 \left\| \frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_2} \right\|^2_{L^2(\Pi_m^{\varepsilon,\delta})} \le C, \end{aligned} \tag{3.14}$$

where C is a constant independent of ε, δ. Therefore,

$$\|R_\lambda^{\varepsilon,\delta}\|_{L^2(\Pi_f^{\varepsilon,\delta})} + \varepsilon^{-1} \left\| \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_1} \right\|_{L^2(\Pi_f^{\varepsilon,\delta})} + \left\| \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2} \right\|_{L^2(\Pi_f^{\varepsilon,\delta})} \le C, \tag{3.15}$$

$$\|S_\lambda^{\varepsilon,\delta}\|_{L^2(\Pi_m^{\varepsilon,\delta})} \le C; \quad \left\| \frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_1} \right\|_{L^2(\Pi_m^{\varepsilon,\delta})} + \varepsilon \left\| \frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_2} \right\|_{L^2(\Pi_m^{\varepsilon,\delta})} \le \frac{C}{\delta}. \tag{3.16}$$

Now considering the properties of the extension operator $\mathsf{P}^{\varepsilon,\delta}$ (see Lemma 2.9 from Chap. 1, Sec. 2 in Ref. 14) we obtain (3.12). Indeed, by this lemma there exists an extension operator P^δ in $\mathcal{Y}$ such that

$$\|\mathsf{P}^\delta \varphi\|_{L^2(\mathcal{Y})} \le C_\delta \, \|\varphi\|_{L^2(\mathcal{F}^\delta)}, \quad \|\nabla \mathsf{P}^\delta \varphi\|_{(L^2(\mathcal{Y}))^2} \le C_\delta \, \|\nabla \varphi\|_{[L^2(\mathcal{F}^\delta)]^2}.$$

We associate with the operator P^δ the extension operator $\mathsf{P}^{\varepsilon,\delta}$ in Π, defined by scaling $\mathcal{Y}$ in the z_2-direction. Hence,

$$\begin{gathered} \left\| \mathsf{P}^{\varepsilon,\delta} R_\lambda^{\varepsilon,\delta} \right\|_{L^2(\Pi)} \le C_\delta \|R_\lambda^{\varepsilon,\delta}\|_{L^2(\Pi_f^{\varepsilon,\delta})}, \\ \left\| \frac{\partial \mathsf{P}^{\varepsilon,\delta} R_\lambda^{\varepsilon,\delta}}{\partial z_i} \right\|_{L^2(\Pi)} \le C_\delta \left\| \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_i} \right\|_{L^2(\Pi_f^{\varepsilon,\delta})}, \quad (i = 1, 2). \end{gathered} \tag{3.17}$$

From (3.15), we get

$$\left\| \mathsf{P}^{\varepsilon,\delta} R_\lambda^{\varepsilon,\delta} \right\|_{L^2(\Pi)} + \varepsilon^{-1} \left\| \frac{\partial \mathsf{P}^{\varepsilon,\delta} R_\lambda^{\varepsilon,\delta}}{\partial z_1} \right\|_{L^2(\Pi)} + \left\| \frac{\partial \mathsf{P}^{\varepsilon,\delta} R_\lambda^{\varepsilon,\delta}}{\partial z_2} \right\|_{L^2(\Pi)} \le C_\delta, \tag{3.18}$$

and thus (up to a subsequence) $\mathsf{P}^{\varepsilon,\delta} R_\lambda^{\varepsilon,\delta} \rightharpoonup R_\lambda^\delta$ weakly in $H^1(\Pi)$ as $\varepsilon \to 0$, and $\frac{\partial R_\lambda^\delta}{\partial z_1} = 0$. This yields $R_\lambda^\delta(z) = R_\lambda^\delta(z_2)$.

It remains to show that R_λ^δ satisfies (3.9). For this aim we use the two-scale convergence approach (see, e.g., Ref. 1). For the reader's convenience we recall the definition of two-scale convergence.

Definition 3.1. A sequence of functions $v^\varepsilon \in L^2(\Omega)$ two-scale converges to $v(x,y) \in L^2(\Omega \times Y)$, if $\|v^\varepsilon\|_{L^2(\Omega)} \le C$ and for any function $\varphi(x,y) \in \mathcal{D}(\Omega; C_\#^\infty(Y))$, it holds

$$\lim_{\varepsilon \to 0} \int_\Omega v^\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega \times Y} v(x,y) \varphi(x,y) \, dxdy.$$

This convergence is denoted by $v^\varepsilon(x) \stackrel{2s}{\rightharpoonup} v(x,y)$.

First we obtain a two-scale compactness result for the solution of (3.4).

Lemma 3.1. *Let* $W_\lambda^{\varepsilon,\delta} = \langle R_\lambda^{\varepsilon,\delta}, S_\lambda^{\varepsilon,\delta} \rangle$ *be a solution of problem* (3.4). *Then there exist* $R_\lambda^\delta \in H^1(0,L)$, $v_f^\delta \in L^2(0,L; H^1(-1/2,1/2) \times H_\#^1(0,1))$ *and* $S_\lambda^\delta \in L^2(0,L; H^1(-1/2,1/2) \times H_\#^1(0,1))$ *such that up to a subsequence*

$$\mathbf{1}_f^{\varepsilon,\delta} \, R_\lambda^{\varepsilon,\delta} \stackrel{2s}{\rightharpoonup} \mathbf{1}_f^\delta(y) \, R_\lambda^\delta(z_2), \tag{3.19}$$

$$\mathbf{1}_f^{\varepsilon,\delta} \left[\varepsilon^{-1} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_1} \right] \stackrel{2s}{\rightharpoonup} \mathbf{1}_f^\delta(y) \, \frac{\partial v_f^\delta}{\partial y_1}(z_2, y_1, y_2), \tag{3.20}$$

$$\mathbf{1}_f^{\varepsilon,\delta} \, \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2} \stackrel{2s}{\rightharpoonup} \mathbf{1}_f^\delta(y) \left[\frac{\partial R_\lambda^\delta}{\partial z_2}(z_2) + \frac{\partial v_f^\delta}{\partial y_2}(z_2, y_1, y_2) \right], \tag{3.21}$$

$$\mathbf{1}_m^{\varepsilon,\delta}\, S_\lambda^{\varepsilon,\delta} \xrightarrow{2s} \mathbf{1}_m^\delta(y)\, S_\lambda^\delta(z_2, y_1, y_2), \tag{3.22}$$

$$\mathbf{1}_m^{\varepsilon,\delta}\, \frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_1} \xrightarrow{2s} \mathbf{1}_m^\delta(y)\, \frac{\partial S_\lambda^\delta}{\partial y_1}(z_2, y_1, y_2), \tag{3.23}$$

$$\mathbf{1}_m^{\varepsilon,\delta} \left[\varepsilon \frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_2}\right] \xrightarrow{2s} \mathbf{1}_m^\delta(y)\, \frac{\partial S_\lambda^\delta}{\partial y_2}(z_2, y_1, y_2), \tag{3.24}$$

where $\mathbf{1}_f^{\varepsilon,\delta} = \mathbf{1}_f^{\varepsilon,\delta}(z)$ *and* $\mathbf{1}_m^{\varepsilon,\delta} = \mathbf{1}_m^{\varepsilon,\delta}(z)$ *denote the characteristic functions of the sets* $\Pi_f^{\varepsilon,\delta}$ *and* $\Pi_m^{\varepsilon,\delta}$, *respectively*; $\mathbf{1}_f^\delta = \mathbf{1}_f^\delta(y)$ *and* $\mathbf{1}_m^\delta = \mathbf{1}_m^\delta(y)$ *denote the characteristic functions of the sets* $\mathcal{F}^\delta$ *and* $\mathcal{M}^\delta$.

Proof of Lemma 3.1. The proof of the lemma is based on the *a priori* estimates (3.15)–(3.16) and two-scale convergence arguments similar to those in Ref. 1. However, we should take into account the fact that the partial derivatives of the functions $R_\lambda^{\varepsilon,\delta}, S_\lambda^{\varepsilon,\delta}$ appear in Eqs. (3.5)–(3.6) with different scale factors.

We first prove (3.19)–(3.21). It is clear that (3.19) follows immediately from Definition 3.1. For the partial derivatives of $R_\lambda^{\varepsilon,\delta}$ we have:

$$\mathbf{1}_f^{\varepsilon,\delta} \left[\varepsilon^{-1}\, \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_1}\right] \xrightarrow{2s} \mathbf{1}_f^\delta(y)\, \frac{\partial v_1^\delta}{\partial y_1}(z_2, y), \tag{3.25}$$

$$\mathbf{1}_f^{\varepsilon,\delta}\, \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2} \xrightarrow{2s} \mathbf{1}_f^\delta(y)\, \left[\frac{\partial R_\lambda^\delta}{\partial z_2}(z_2) + \frac{\partial v_2^\delta}{\partial y_2}(z_2, y)\right]. \tag{3.26}$$

To prove (3.20), (3.21), we have to show that there is $v_f^\delta \in L^2(0, L; H^1(-1/2, 1/2) \times H_\#^1(0,1))$ such that

$$\frac{\partial v_1^\delta}{\partial y_1} = \frac{\partial v_f^\delta}{\partial y_1}, \qquad \frac{\partial v_2^\delta}{\partial y_2} = \frac{\partial v_f^\delta}{\partial y_2}. \tag{3.27}$$

To this end we consider the integral

$$I^\varepsilon = \int_{\Pi_f^{\varepsilon,\delta}} \varepsilon^{-1} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_1}(z) \frac{\partial \Phi}{\partial z_2} \left(z_2, z_1, \frac{z_2}{\varepsilon}\right)\, dz$$

with an admissible test function Φ of the form $\Phi\left(z_2, z_1, \frac{z_2}{\varepsilon}\right) = \varepsilon \phi_0(z_2)\phi_1(z_1, \frac{z_2}{\varepsilon})$. After simple rearrangements we get

$$\begin{aligned} I^\varepsilon &= \int_{\Pi_f^{\varepsilon,\delta}} \varepsilon^{-1} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_1}(z) \left[\varepsilon \frac{\partial \phi_0}{\partial z_2}(z_2)\phi_1\left(z_1, \frac{z_2}{\varepsilon}\right)\right] dz \\ &\quad + \int_{\Pi_f^{\varepsilon,\delta}} \varepsilon^{-1} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_1}(z) \left[\phi_0(z_2) \frac{\partial \phi_1}{\partial y_2}\left(z_1, \frac{z_2}{\varepsilon}\right)\right] dz. \end{aligned} \tag{3.28}$$

The first term on the right-hand side here vanishes as $\varepsilon \to 0$. Therefore, by (3.25)

$$I^\varepsilon \to \int_{\mathcal{F}^\delta} \frac{\partial v_1^\delta}{\partial y_1}(z_2, y)\phi_0(z_2)\frac{\partial \phi_1}{\partial y_2}(y)\, dz_2 dy \quad \text{as } \varepsilon \to 0. \tag{3.29}$$

On the other hand, assuming that the support of $\phi_1(y)$ is a compact set in $\mathcal{F}^\delta$ and integrating I^ε by parts, we have

$$\begin{aligned} I^\varepsilon &= -\int_{\Pi_f^{\varepsilon,\delta}} R_\lambda^{\varepsilon,\delta}(z)\frac{\partial^2}{\partial z_1 \partial z_2}\left[\phi_0(z_2)\phi_1\left(z_1, \frac{z_2}{\varepsilon}\right)\right] dz \\ &= \int_{\Pi_f^{\varepsilon,\delta}} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2}(z)\phi_0(z_2)\frac{\partial \phi_1}{\partial z_1}\left(z_1, \frac{z_2}{\varepsilon}\right) dz. \end{aligned} \tag{3.30}$$

It follows from (3.26) that

$$I^\varepsilon \to \int_{\mathcal{F}^\delta} \left[\frac{\partial R_\lambda^\delta}{\partial z_2} + \frac{\partial v_2^\delta}{\partial y_2}(z_2, y)\right] \phi_0(z_2)\frac{\partial \phi_1}{\partial y_1}(y)\, dz_2 dy \quad \text{as } \varepsilon \to 0. \tag{3.31}$$

Since ϕ_0 is an arbitrary smooth function of variable z_2, then

$$\int_{\mathcal{F}^\delta} \frac{\partial v_1^\delta}{\partial y_1}(z_2, y)\frac{\partial \phi_1}{\partial y_2}(y)\, dy = \int_{\mathcal{F}^\delta} \frac{\partial v_2^\delta}{\partial y_2}(z_2, y)\frac{\partial \phi_1}{\partial y_1}(y)\, dy. \tag{3.32}$$

The existence of the function v_f^δ satisfying (3.27) is now a consequence of the following statement (generalization of the classical de Rham theorem (see, e.g., Ref. 13).

Lemma 3.2. *Let Ω be a Lipschitz domain, and let $\mathbf{g} = (g_1, g_2, \ldots, g_n) \in (L^2(\Omega))^n$ satisfy the relation*

$$\langle \mathbf{g}, \mathbf{u} \rangle_{(L^2(\Omega))^n} = \sum_{i=1}^n \int_\Omega g_i(x)u_i(x)\, dx = 0$$

for any $\mathbf{u} \in H_0^1(\mathrm{div}, \Omega) = \{\mathbf{u} \in H_0^1(\Omega) : \mathrm{div}\, \mathbf{u} = 0\}$. Then there exists $p \in H^1(\Omega)$ such that $\mathbf{g} = \nabla p$.

This proves (3.20)–(3.21). The assertions (3.22)–(3.24) can be proved in a similar way. It should be emphasized here that there is not "fast" variable in the z_1-direction.

Lemma 3.1 is proved. □

Choosing in the weak formulation of problem (3.4) a test function $\Phi(z_1, z_2, \frac{z_2}{\varepsilon})$ of the form $\Phi = \langle \phi_f, \phi_m \rangle$ with $\phi_f, \phi_m \in C^1(0, L; C^1(-1/2, 1/2) \times C_\#^1(0, 1))$ such that $\phi_f = \phi_m$ on $\Gamma_{mf}^{\varepsilon,\delta}$, we arrive at the following integral identity:

$$\begin{aligned}
&\omega_f \lambda \int_{\Pi_f^{\varepsilon,\delta}} R_\lambda^{\varepsilon,\delta}(z)\phi_f(z)\,dz + \varepsilon^{-2} k_f \int_{\Pi_f^{\varepsilon,\delta}} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_1}\frac{\partial \phi_f}{\partial z_1}\,dz \\
&\quad + k_f \int_{\Pi_f^{\varepsilon,\delta}} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2}\frac{\partial \phi_f}{\partial z_2}\,dz + \omega_m \lambda \int_{\Pi_m^{\varepsilon,\delta}} S_\lambda^{\varepsilon,\delta}(z)\phi_m(z)\,dz \\
&\quad + \delta^2 k_m \int_{\Pi_m^{\varepsilon,\delta}} \frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_1}\frac{\partial \phi_m}{\partial z_1}\,dz + (\varepsilon\delta)^2 k_m \int_{\Pi_m^{\varepsilon,\delta}} \frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_2}\frac{\partial \phi_m}{\partial z_2}\,dz \\
&= \int_{\Pi_f^{\varepsilon,\delta}} (g_\lambda + h_\lambda)(\varepsilon z_1, z_2)\phi_f(z)\,dz + \int_{\Pi_m^{\varepsilon,\delta}} h_\lambda(\varepsilon z_1, z_2)\phi_m(z)\,dz. \qquad (3.33)
\end{aligned}$$

In order to pass to the limit in (3.33) we introduce a smooth function $\vartheta = \vartheta(s)$ such that $0 \le \vartheta(s) \le 1$ and

$$\vartheta(s) = \begin{cases} 1 & \text{for } s \le 0; \\ 0 & \text{for } s \ge 1. \end{cases} \qquad (3.34)$$

For $\gamma \in (0,1)$ we set

$$\begin{aligned}
\vartheta_1^\varepsilon(z_1) &= \vartheta\left(\frac{z_1 + \frac{1-\delta}{2}}{\varepsilon^{1-\gamma}}\right)\vartheta\left(-\frac{z_1 + \frac{1-\delta}{2}}{\varepsilon^{1-\gamma}}\right), \\
\vartheta_2^\varepsilon(z_2) &= \vartheta\left(\frac{z_2 - \frac{\varepsilon\delta}{2}}{\varepsilon^{2-\gamma}}\right)\vartheta\left(\frac{-z_2 + \left(\varepsilon - \frac{\varepsilon\delta}{2}\right)}{\varepsilon^{2-\gamma}}\right)
\end{aligned} \qquad (3.35)$$

for $z_2 \in [0,\varepsilon]$. We extend ϑ_2^ε ε-periodically to the whole $\mathbb{R}$ and define the test functions ϕ_f, ϕ_m by

$$\phi_f = \varphi_f(z_2) + \varepsilon\zeta\left(z_2, z_1, \frac{z_2}{\varepsilon}\right), \qquad (3.36)$$

$$\phi_m = \varphi_f(z_2)U_\lambda^\delta\left(z_1, \frac{z_2}{\varepsilon}\right) + \varepsilon\vartheta_1^\varepsilon(z_1)\vartheta_2^\varepsilon(z_2)\zeta\left(z_2, z_1, \frac{z_2}{\varepsilon}\right), \qquad (3.37)$$

where $\varphi_f \in C^1(0,L)$, $\zeta \in C^1(0,L; C^1(-1/2,1/2) \times C^1_\#(0,1))$ and $U_\lambda^\delta = U_\lambda^\delta(y)$ is defined in (3.8). Denote:

$$J_f^{\varepsilon,\delta} \equiv \int_{\Pi_f^{\varepsilon,\delta}} \left\{\omega_f \lambda R_\lambda^{\varepsilon,\delta}(z)\phi_f(z) + \varepsilon^{-2}k_f \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_1}\frac{\partial \phi_f}{\partial z_1} + k_f \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2}\frac{\partial \phi_f}{\partial z_2}\right\} dz; \qquad (3.38)$$

$$J_m^{\varepsilon,\delta} \equiv \int_{\Pi_m^{\varepsilon,\delta}} \left\{\omega_m \lambda S_\lambda^{\varepsilon,\delta}(z)\phi_m(z) + \delta^2 k_m \frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_1}\frac{\partial \phi_m}{\partial z_1} + (\varepsilon\delta)^2 k_m \frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_2}\frac{\partial \phi_m}{\partial z_2}\right\} dz. \qquad (3.39)$$

The asymptotic behavior of the integrals $J_f^{\varepsilon,\delta}, J_m^{\varepsilon,\delta}$ is studied in the following lemma.

Lemma 3.3. *Let $J_f^{\varepsilon,\delta}$ be given by* (3.38). *Then*

$$\lim_{\varepsilon\to 0} J_f^{\varepsilon,\delta} = \omega_f \lambda |\mathcal{F}^\delta| \int_0^L R_\lambda^\delta(z_2)\varphi_f(z_2)\, dz_2 + k_f \int_0^L \int_{\mathcal{F}^\delta} \frac{\partial v_f^\delta}{\partial y_1}(y) \frac{\partial \zeta}{\partial y_1}(z_2, y)\, dy\, dz_2$$
$$+ k_f |\mathcal{F}^\delta| \int_0^L \frac{\partial R_\lambda^\delta}{\partial z_2} \frac{\partial \varphi_f}{\partial z_2}(z_2)\, dz_2 + k_f \int_0^L \int_{\mathcal{F}^\delta} \frac{\partial v_f^\delta}{\partial y_2}(y) \frac{\partial \varphi_f}{\partial z_2}(z_2)\, dy\, dz_2$$
$$+ k_f \int_0^L \int_{\mathcal{F}^\delta} \left[\frac{\partial R_\lambda^\delta}{\partial z_2}(z_2) + \frac{\partial v_f^\delta}{\partial y_2}(z_2, y)\right] \frac{\partial \zeta}{\partial y_2}(z_2, y)\, dy dz_2. \tag{3.40}$$

Proof of Lemma 3.3. After simple rearrangements $J_f^{\varepsilon,\delta}$ can be represented as follows

$$J_f^{\varepsilon,\delta} = \omega_f \lambda \int_{\Pi_f^{\varepsilon,\delta}} R_\lambda^{\varepsilon,\delta}(z) \left\{\varphi_f(z_2) + \varepsilon\zeta\left(z_2, z_1, \frac{z_2}{\varepsilon}\right)\right\} dz$$
$$+ k_f \int_{\Pi_f^{\varepsilon,\delta}} \varepsilon^{-2} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_1} \frac{\partial}{\partial z_1} \left\{\varphi_f(z_2) + \varepsilon\zeta\left(z_2, z_1, \frac{z_2}{\varepsilon}\right)\right\} dz$$
$$+ k_f \int_{\Pi_f^{\varepsilon,\delta}} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2} \frac{\partial}{\partial z_2} \left\{\varphi_f(z_2) + \varepsilon\zeta\left(z_2, z_1, \frac{z_2}{\varepsilon}\right)\right\} dz \equiv I_1^{\varepsilon,\delta} + I_2^{\varepsilon,\delta} + I_3^{\varepsilon,\delta}. \tag{3.41}$$

By Lemma 3.1 we have:

$$I_1^{\varepsilon,\delta} \to \omega_f \lambda |\mathcal{F}^\delta| \int_0^L R_\lambda^\delta(z_2)\varphi_f(z_2)\, dz_2 \quad \text{as } \varepsilon \to 0 \tag{3.42}$$

and

$$I_2^{\varepsilon,\delta} = k_f \int_{\Pi_f^{\varepsilon,\delta}} \varepsilon^{-1} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_1} \frac{\partial \zeta}{\partial z_1}\left(z_2, z_1, \frac{z_2}{\varepsilon}\right) dz \to k_f \int_0^L \int_{\mathcal{F}^\delta} \frac{\partial v_f^\delta}{\partial y_1}(y) \frac{\partial \zeta}{\partial y_1}(z_2, y)\, dy\, dz_2. \tag{3.43}$$

Consider the third term on the right-hand side of (3.41). Clearly,

$$I_3^{\varepsilon,\delta} = k_f \int_{\Pi_f^{\varepsilon,\delta}} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2} \frac{\partial \varphi_f}{\partial z_2}(z_2)\, dz + k_f \int_{\Pi_f^{\varepsilon,\delta}} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2} \frac{\partial}{\partial z_2} \left\{\varepsilon\zeta\left(z_2, z_1, \frac{z_2}{\varepsilon}\right)\right\} dz. \tag{3.44}$$

By Lemma 3.1 we get

$$k_f \int_{\Pi_f^{\varepsilon,\delta}} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2} \frac{\partial \varphi_f}{\partial z_2}(z_2)\, dz$$
$$\to k_f |\mathcal{F}^\delta| \int_0^L \frac{\partial R_\lambda^\delta}{\partial z_2} \frac{\partial \varphi_f}{\partial z_2}(z_2)\, dz_2 + k_f \int_0^L \int_{\mathcal{F}^\delta} \frac{\partial v_f^\delta}{\partial y_2}(y) \frac{\partial \varphi_f}{\partial z_2}(z_2)\, dy\, dz_2. \tag{3.45}$$

For the second integral on the right-hand side of (3.44) we have

$$
\begin{aligned}
&k_f \int_{\Pi_f^{\varepsilon,\delta}} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2} \frac{\partial}{\partial z_2} \left\{ \varepsilon\zeta \left(z_2, z_1, \frac{z_2}{\varepsilon} \right) \right\} dz \\
&\quad = k_f \int_{\Pi_f^{\varepsilon,\delta}} \varepsilon \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2} \frac{\partial \zeta}{\partial z_2} (z_2, z_1, y_2) \Big|_{y_2 = \frac{z_2}{\varepsilon}} dz + k_f \int_{\Pi_f^{\varepsilon,\delta}} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2} \frac{\partial \zeta}{\partial y_2} \left(z_2, z_1, \frac{z_2}{\varepsilon} \right) dz.
\end{aligned}
\tag{3.46}
$$

Clearly, the first term on the right-hand side goes to zero as $\varepsilon \to 0$, and by Lemma 3.1 we obtain

$$
\begin{aligned}
&k_f \int_{\Pi_f^{\varepsilon,\delta}} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2} \frac{\partial}{\partial z_2} \left\{ \varepsilon\zeta \left(z_2, z_1, \frac{z_2}{\varepsilon} \right) \right\} dz \\
&\qquad \to k_f \int_0^L \int_{\mathcal{F}^\delta} \left[\frac{\partial R_\lambda^\delta}{\partial z_2}(z_2) + \frac{\partial v_f^\delta}{\partial y_2}(z_2, y) \right] \frac{\partial \zeta}{\partial y_2}(z_2, y)\, dy dz_2.
\end{aligned}
\tag{3.47}
$$

Finally, (3.44)–(3.47) yield

$$
I_3^{\varepsilon,\delta} \to k_f \int_0^L \int_{\mathcal{F}^\delta} \left[\frac{\partial R_\lambda^\delta}{\partial z_2}(z_2) + \frac{\partial v_f^\delta}{\partial y_2}(z_2, y) \right] \left(\frac{\partial \zeta}{\partial y_2}(z_2, y) + \frac{\partial \varphi_f}{\partial z_2}(z_2) \right) dy dz_2 \quad \text{as } \varepsilon \to 0.
\tag{3.48}
$$

Now the desired statement follows from (3.42), (3.43) and (3.48). This completes the proof of Lemma 3.3. □

Lemma 3.4. *Let $J_m^{\varepsilon,\delta}$ be given by* (3.38). *Then*

$$
\lim_{\varepsilon \to 0} J_m^{\varepsilon,\delta} = I_\lambda^\delta \int_0^L \varphi_f(z_2) R_\lambda^\delta(z_2)\, dz_2,
\tag{3.49}
$$

where I_λ^δ is defined in (3.10).

Proof of Lemma 3.4. By the definition of ϕ_m we have $J_m^{\varepsilon,\delta} = J_\Theta^{\varepsilon,\delta} + J_0^{\varepsilon,\delta}$, where

$$
J_\Theta^{\varepsilon,\delta} = \int_{\Pi_m^{\varepsilon,\delta}} \left\{ \omega_m \lambda S_\lambda^{\varepsilon,\delta} \Theta^{\varepsilon,\delta} + \delta^2 k_m \frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_1} \frac{\partial \Theta^{\varepsilon,\delta}}{\partial z_1} + (\varepsilon\delta)^2 k_m \frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_2} \frac{\partial \Theta^{\varepsilon,\delta}}{\partial z_2} \right\} dz
\tag{3.50}
$$

with

$$
\Theta^{\varepsilon,\delta} \left(z_2, z_1, \frac{z_2}{\varepsilon} \right) = \varepsilon \vartheta_1^\varepsilon(z_1) \vartheta_2^\varepsilon(z_2) \zeta \left(z_2, z_1, \frac{z_2}{\varepsilon} \right)
\tag{3.51}
$$

and $\vartheta_1^\varepsilon, \vartheta_2^\varepsilon$ defined in (3.35),

$$J_0^{\varepsilon,\delta} = \int_{\Pi_m^{\varepsilon,\delta}} \left\{ \omega_m \lambda U_\lambda^{\varepsilon,\delta}(z)\varphi_f(z_2)S_\lambda^{\varepsilon,\delta}(z) + \delta^2 k_m \frac{\partial U_\lambda^{\varepsilon,\delta}}{\partial z_1}(z)\varphi_f(z_2)\frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_1} \right.$$

$$\left. + (\varepsilon\delta)^2 k_m \frac{\partial U_\lambda^{\varepsilon,\delta}}{\partial z_2}(z)\varphi_f(z_2)\frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_2} + (\varepsilon\delta)^2 k_m U_\lambda^{\varepsilon,\delta}(z)\frac{\partial \varphi}{\partial z_2}(z_2)\frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_2} \right\} dz. \tag{3.52}$$

Taking into account (3.51), the choice of the cutoff functions ϑ_1^ε and ϑ_2^ε, the fact that $0 < \gamma < 1$ and the *a priori* estimates one can show that the integral $J_\Theta^{\varepsilon,\delta}$ vanishes as $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} J_\Theta^{\varepsilon,\delta} = 0. \tag{3.53}$$

Consider now the functional $J_0^{\varepsilon,\delta}$. By Lemma 3.1, we have

$$\omega_m \lambda \int_{\Pi_m^{\varepsilon,\delta}} U_\lambda^{\varepsilon,\delta}(z)\varphi_f(z_2)S_\lambda^{\varepsilon,\delta}(z)\,dz \to \omega_m \lambda \int_0^L \int_{\mathcal{M}^\delta} U_\lambda^\delta(y)\varphi_f(z_2)S_\lambda^\delta(z_2,y)\,dydz_2 \quad \text{as } \varepsilon \to 0, \tag{3.54}$$

and

$$\delta^2 k_m \int_{\Pi_m^{\varepsilon,\delta}} \frac{\partial U_\lambda^{\varepsilon,\delta}}{\partial z_1}(z)\varphi_f(z_2)\frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_1}\,dz \to \delta^2 k_m \int_0^L \int_{\mathcal{M}^\delta} \frac{\partial U_\lambda^\delta}{\partial y_1}(y)\varphi_f(z_2)\frac{\partial S_\lambda^\delta}{\partial y_1}\,dydz_2 \quad \text{as}\, \varepsilon \to 0. \tag{3.55}$$

For the other two terms on the right-hand side of (3.52), by Lemma 3.1 we get

$$\int_{\Pi_m^{\varepsilon,\delta}} (\varepsilon\delta)^2 k_m \frac{\partial U_\lambda^{\varepsilon,\delta}}{\partial z_2}(z)\varphi_f(z_2)\frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_2}\,dz = \int_{\Pi_m^{\varepsilon,\delta}} \varepsilon\delta^2 k_m \frac{\partial U_\lambda^{\varepsilon,\delta}}{\partial y_2}(z)\varphi_f(z_2)\frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_2}\,dz$$

$$\to \delta^2 k_m \int_0^L \int_{\mathcal{M}^\delta} \frac{\partial U_\lambda^\delta}{\partial y_2}(y)\varphi_f(z_2)\frac{\partial S_\lambda^\delta}{\partial y_2}\,dydz_2 \quad \text{as } \varepsilon \to 0 \tag{3.56}$$

and

$$\int_{\Pi_m^{\varepsilon,\delta}} (\varepsilon\delta)^2 k_m U_\lambda^{\varepsilon,\delta}(z)\varphi_f(z_2)\frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_2}\,dz \to 0 \quad \text{as}\, \varepsilon \to 0. \tag{3.57}$$

The formulas (3.53)–(3.57) imply that

$$\lim_{\varepsilon\to 0} J_m^{\varepsilon,\delta} = J_m^\delta \equiv \omega_m\lambda \int_0^L \int_{\mathcal{M}^\delta} U_\lambda^\delta(y)\varphi_f(z_2) S_\lambda^\delta(z_2,y)\,dydz_2$$
$$+ \delta^2 k_m \int_0^L \int_{\mathcal{M}^\delta} \frac{\partial U_\lambda^\delta}{\partial y_1}(y)\varphi_f(z_2)\frac{\partial S_\lambda^\delta}{\partial y_1}\,dydz_2$$
$$+ \delta^2 k_m \int_0^L \int_{\mathcal{M}^\delta} \frac{\partial U_\lambda^\delta}{\partial y_2}(y)\varphi_f(z_2)\frac{\partial S_\lambda^\delta}{\partial y_2}\,dydz_2. \tag{3.58}$$

Intergrating J_m^δ by parts we obtain

$$J_m^\delta = \int_0^L \int_{\mathcal{M}^\delta} \left\{\omega_m\lambda U_\lambda^\delta - k_m\delta^2\Delta_y U_\lambda^\delta\right\}\varphi_f(z_2)S_\lambda^\delta(z_2,y)\,dydz_2$$
$$+ k_m\delta^2 \int_0^L \int_{\Gamma_{mf}^\delta} \left(\nabla_y U_\lambda^\delta\cdot\boldsymbol{\nu}\right) S_\lambda^\delta(z_2,y)\,ds_y\varphi_f(z_2)\,dz_2$$
$$= I_\lambda^\delta \int_0^L \varphi_f(z_2)R_\lambda^\delta(z_2)\,dz_2, \tag{3.59}$$

where I_λ^δ is defined in (3.10). Lemma 3.4 is proved. □

We now pass to the limit on the right-hand side of (3.33). It is clear that

$$\lim_{\varepsilon\to 0}\int_{\Pi_f^{\varepsilon,\delta}} (g_\lambda + h_\lambda)(\varepsilon z_1, z_2)\left(\varphi_f(z_2) + \varepsilon\zeta\left(z_2, z_1, \frac{z_2}{\varepsilon}\right)\right)\,dz$$
$$= |\mathcal{F}^\delta| \int_0^L (g_\lambda + h_\lambda)(0,z_2)\varphi_f(z_2)\,dz_2, \tag{3.60}$$

$$\lim_{\varepsilon\to 0}\int_{\Pi_m^{\varepsilon,\delta}} h_\lambda(\varepsilon z_1, z_2)\phi_m\left(z_2, z_1, \frac{z_2}{\varepsilon}\right)\,dz = \frac{I_\lambda^\delta}{\lambda\omega_m}\int_0^L h_\lambda(0,z_2)\varphi_f(z_2)\,dz_2. \tag{3.61}$$

Finally, from (3.40), (3.49), (3.60), (3.61) we deduce the following limit relation:

$$\omega_f\lambda|\mathcal{F}^\delta| \int_0^L R_\lambda^\delta\varphi_f\,dz_2 + k_f\int_0^L\int_{\mathcal{F}^\delta} \frac{\partial v_f^\delta}{\partial y_1}\frac{\partial\zeta}{\partial y_1}\,dy\,dz_2$$
$$+ k_f|\mathcal{F}^\delta| \int_0^L \frac{\partial R_\lambda^\delta}{\partial z_2}\frac{\partial\varphi_f}{\partial z_2}\,dz_2 + k_f\int_0^L\int_{\mathcal{F}^\delta}\frac{\partial v_f^\delta}{\partial y_2}\frac{\partial\varphi_f}{\partial z_2}\,dy\,dz_2$$
$$+ k_f\int_0^L\int_{\mathcal{F}^\delta}\left[\frac{\partial R_\lambda^\delta}{\partial z_2} + \frac{\partial v_f^\delta}{\partial y_2}\right]\frac{\partial\zeta}{\partial y_2}\,dydz_2 + I_\lambda^\delta\int_0^L\varphi_f R_\lambda^\delta\,dz_2$$
$$= |\mathcal{F}^\delta|\int_0^L (g_\lambda + h_\lambda)(0,z_2)\varphi_f(z_2)\,dz_2 + \frac{I_\lambda^\delta}{\lambda\omega_m}\int_0^L h_\lambda(0,z_2)\varphi_f(z_2)\,dz_2. \tag{3.62}$$

Now we proceed in a standard way. Letting $\varphi_f = 0$, we obtain that

$$v_f^\delta(z_2, y) = -\frac{\partial R_\lambda^\delta}{\partial z_2}(z_2)\chi^\delta(y), \tag{3.63}$$

where χ^δ is the solution of (3.7). Then we set $\zeta = 0$ and obtain the weak formulation of the macroscopic equation (3.9) or δ-model.

The strong two-scale convergence of $R_\lambda^{\varepsilon,\delta}$ is a consequence of the weak compactness, for each fixed $\delta > 0$, of $\{\mathsf{P}^{\varepsilon,\delta} R_\lambda^{\varepsilon,\delta}\}$ in $H^1(\Pi)$.

To complete the proof of Theorem 3.1 it remains to describe the two-scale limit of $S_\lambda^{\varepsilon,\delta}$. To this end we substitute in the integral identity (3.33) an arbitrary test function $\phi_m = \phi_m(z_2, z_1, \frac{z_2}{\varepsilon})$ with a compact support in $\Pi_m^{\varepsilon,\delta}$ and $\phi_f = 0$. Then passing to the two-scale limit in (3.33) and making the same rearrangements as in the proof of Lemma 3.4, we obtain the relation

$$\begin{aligned}
&\omega_m\lambda \int_0^L \int_{\mathcal{M}^\delta} \phi_m(z_2, y) S_\lambda^\delta(z_2, y)\, dy dz_2 + \delta^2 k_m \int_0^L \int_{\mathcal{M}^\delta} \frac{\partial \phi_m}{\partial y_1}(z_2, y)\frac{\partial S_\lambda^\delta}{\partial z_1}\, dy dz_2 \\
&\quad + \delta^2 k_m \int_0^L \int_{\mathcal{M}^\delta} \frac{\partial \phi_m}{\partial y_2}(z_2, y)\frac{\partial S_\lambda^\delta}{\partial y_2}\, dy dz_2 \\
&\quad = \int_0^L \int_{\mathcal{M}^\delta} h(0, z_2)\phi_m(z_2, y)\, dy dz_2.
\end{aligned} \tag{3.64}$$

The fact that $S_\lambda^\delta = R_\lambda^\delta$ on $\partial\mathcal{M}^\delta$ can be justified in the standard way. Together with (3.64) this yields (3.11).

To show the strong two-scale convergence of $S_\lambda^{\varepsilon,\delta}$, we use the solution $W_\lambda^{\varepsilon,\delta}(z) = \langle R_\lambda^{\varepsilon,\delta}(z),\ S_\lambda^{\varepsilon,\delta}(z)\rangle$ as a test function in (3.33). This yields

$$\begin{aligned}
&\omega_f\lambda \int_{\Pi_f^{\varepsilon,\delta}} (R_\lambda^{\varepsilon,\delta}(z))^2\, dz + \varepsilon^{-2} k_f \int_{\Pi_f^{\varepsilon,\delta}} \left|\frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_1}\right|^2 dz \\
&\quad + k_f \int_{\Pi_f^{\varepsilon,\delta}} \left|\frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2}\right|^2 dz + \omega_m\lambda \int_{\Pi_m^{\varepsilon,\delta}} (S_\lambda^{\varepsilon,\delta}(z))^2\, dz \\
&\quad + \delta^2 k_m \int_{\Pi_m^{\varepsilon,\delta}} \left|\frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_1}\right|^2 dz + (\varepsilon\delta)^2 k_m \int_{\Pi_m^{\varepsilon,\delta}} \left|\frac{\partial S_\lambda^{\varepsilon,\delta}}{\partial z_2}\right|^2 dz \\
&\quad = \int_{\Pi_f^{\varepsilon,\delta}} (g_\lambda + h_\lambda)(\varepsilon z_1, z_2) R_\lambda^{\varepsilon,\delta}(z)\, dz + \int_{\Pi_m^{\varepsilon,\delta}} h_\lambda(\varepsilon z_1, z_2) S_\lambda^{\varepsilon,\delta}(z)\, dz.
\end{aligned} \tag{3.65}$$

For brevity, denote the left-hand side by $X^{\varepsilon,\delta}$. Each term of $X^{\varepsilon,\delta}$ is lower semi-continuous with respect to the two-scale convergence (see Ref. 1). Therefore, by

Lemma 3.1,

$$
\begin{aligned}
\varliminf_{\varepsilon\to 0} X^{\varepsilon,\delta} \geq \omega_f\lambda \int_0^L \int_{\mathcal{F}^\delta} (R_\lambda^\delta)^2\, dydz_2 \\
&+ k_f \int_0^L \int_{\mathcal{F}^\delta} \left|\frac{\partial R_\lambda^\delta}{\partial z_2}\right|^2 \left\{ \left|\frac{\partial \chi^\delta}{\partial y_1}\right|^2 + \left|\left(1 - \frac{\partial \chi^\delta}{\partial y_2}\right)\right|^2 \right\} dydz_2 \\
&+ \omega_m\lambda \int_0^L \int_{\mathcal{M}^\delta} (S_\lambda^\delta)^2\, dydz_2 + \delta^2 k_m \int_0^L \int_{\mathcal{M}^\delta} \left|\frac{\partial S_\lambda^\delta}{\partial y_1}\right|^2 dydz_2 \\
&+ \delta^2 k_m \int_0^L \int_{\mathcal{M}^\delta} \left|\frac{\partial S_\lambda^\delta}{\partial y_2}\right|^2 dydz_2.
\end{aligned}
\tag{3.66}
$$

By (3.13) we have

$$
\frac{1}{|\mathcal{F}^\delta|} \int_{\mathcal{F}^\delta} \left\{ \left|\frac{\partial \chi^\delta}{\partial y_1}\right|^2 + \left|\left(1 - \frac{\partial \chi^\delta}{\partial y_2}\right)\right|^2 \right\} dy = K^\delta.
$$

On the other hand, passing to the limit on the right-hand side of (3.65), one gets

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \left(\int_{\Pi_f^{\varepsilon,\delta}} (g_\lambda + h_\lambda)(\varepsilon z_1, z_2) R_\lambda^{\varepsilon,\delta}(z)\, dz + \int_{\Pi_m^{\varepsilon,\delta}} h_\lambda(\varepsilon z_1, z_2) S_\lambda^{\varepsilon,\delta}(z)\, dz \right) \\
&\quad = \int_0^L \int_{\mathcal{F}^\delta} (g_\lambda + h_\lambda)(0, z_2) R_\lambda^\delta\, dydz_2 + \int_0^L \int_{\mathcal{M}^\delta} h_\lambda(0, z_2) S_\lambda^\delta\, dydz_2.
\end{aligned}
$$

According to (3.9), (3.62) and (3.64), the right-hand side here is equal to the right-hand side of (3.66). Thus, (3.66) happens to be an equality. This implies, in particular, that the limit of each term on the left-hand side of (3.65) exists and equals to the corresponding term on the right-hand side of (3.66). This completes the proof of the strong two-scale convergence and Theorem 3.1.

4. Step 2. Passage to the Limit as $\delta \to 0$ in (3.9)

Here we pass to the limit, as $\delta \to 0$, in (3.9) and obtain the stationary limit (homogenized) problem as $\varepsilon, \delta \to 0$. This homogenized problem takes the form

$$
\textit{Stationary macromodel}: \begin{cases} \lambda\omega_f \rho_\lambda^* - \dfrac{k_f}{2} \dfrac{\partial^2 \rho_\lambda^*}{\partial \xi^2} = G_\lambda(\xi) + \mathbf{S}(\rho_\lambda^*) \quad \text{in } (0, L); \\ \dfrac{\partial \rho_\lambda^*}{\partial \xi}(0) = \dfrac{\partial \rho_\lambda^*}{\partial \xi}(L) = 0, \end{cases}
\tag{4.1}
$$

with

$$
G_\lambda(\xi) = \frac{1}{\lambda}(g + h)(0, \xi); \quad \mathbf{S}(\rho_\lambda^*) = -2\sqrt{\lambda\omega_m k_m}\,\rho_\lambda^* + \frac{2}{\lambda\sqrt{\lambda}} \sqrt{\frac{k_m}{\omega_m}}\, h(0, \xi).
\tag{4.2}
$$

The precise statement of the convergence result is as follows.

Proposition 4.1. *Let* $u_\lambda^{\varepsilon,\delta}$ *be a solution of problem* (3.1). *Then*

(I) *the function* $u_\lambda^{\varepsilon,\delta}$ *converges to* $\lambda^{-2}h(x)$, *namely*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{|\Omega^\varepsilon|}\left\|\omega^{\varepsilon,\delta}u^{\varepsilon,\delta}-\lambda^{-2}h\right\|^2_{L^2(\Omega^\varepsilon)}=0; \tag{4.3}$$

(II) *the function* $\rho_\lambda^{\varepsilon,\delta}$ *converges to a function* ρ_λ^* *in the following sense*:

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{|\Omega_f^{\varepsilon,\delta}|}\left\|\rho_\lambda^{\varepsilon,\delta}-\rho_\lambda^*\right\|^2_{L^2(\Omega_f^{\varepsilon,\delta})}=0. \tag{4.4}$$

(III) *For any function* $\phi=\phi(x)$ *continuous in the vicinity of the segment* $\{x\in\mathbb{R}^2:x_1=0;\ 0\le x_2\le L\}$,

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{L}{|\Omega_f^{\varepsilon,\delta}|}\int_{\Omega^\varepsilon}k^{\varepsilon,\delta}(x)\nabla u_\lambda^{\varepsilon,\delta}\phi(x)\,dx=\frac{k_f}{2}\int_0^L\vec{\mathbf{R}}_\lambda^*(\xi)\phi(0,\xi)\,d\xi, \tag{4.5}$$

where

$$\vec{\mathbf{R}}_\lambda^*(\xi)=\left(0,\frac{d\rho_\lambda^*}{d\xi}(\xi)\right).$$

4.1. *Proof of Proposition* 4.1

The proof of Proposition 4.1 will be given in Secs. 4.1.1–4.1.3. In Sec. 4.1.1 we establish the uniform estimates for the function $u_\lambda^{\varepsilon,\delta}$ and the convergence result (4.3). Then in Sec. 4.1.2 we obtain the homogenized equation (4.1) and prove the convergence result (4.4). Finally, in Sec. 4.1.3 we prove the convergence of the fluxes.

4.1.1. *Proof of assertion* (I)

Consider problem (3.1). The solution $u_\lambda^{\varepsilon,\delta}=\langle\rho_\lambda^{\varepsilon,\delta},\sigma_\lambda^{\varepsilon,\delta}\rangle$ of this problem minimizes the functional:

$$\begin{aligned}\mathbf{J}^\varepsilon\left[u^{\varepsilon,\delta}\right]&=\mu^{\varepsilon,\delta}\int_{\Omega^\varepsilon}\left\{k^{\varepsilon,\delta}(x)\left|\nabla u_\lambda^{\varepsilon,\delta}\right|^2+\lambda\omega^{\varepsilon,\delta}(x)\left|u_\lambda^{\varepsilon,\delta}\right|^2-2G_\lambda^{\varepsilon,\delta}(x)u_\lambda^{\varepsilon,\delta}\right\}dx\\&=\mu^{\varepsilon,\delta}\int_{\Omega_f^{\varepsilon,\delta}}\left\{k_f\left|\nabla\rho_\lambda^{\varepsilon,\delta}\right|^2+\lambda\omega_f\left|\rho_\lambda^{\varepsilon,\delta}\right|^2-2(g_\lambda+h_\lambda)(x)\rho_\lambda^{\varepsilon,\delta}\right\}dx\\&\quad+\mu^{\varepsilon,\delta}\int_{\Omega_m^{\varepsilon,\delta}}\left\{k_m(\varepsilon\delta)^2\left|\nabla\sigma_\lambda^{\varepsilon,\delta}\right|^2+\lambda\omega_m\left|\sigma_\lambda^{\varepsilon,\delta}\right|^2-2h_\lambda(x)\sigma_\lambda^{\varepsilon,\delta}\right\}dx,\end{aligned} \tag{4.6}$$

where $\mu^{\varepsilon,\delta}=1/|\Omega_f^{\varepsilon,\delta}|$. To simplify this functional we set

$$\rho_\lambda^{\varepsilon,\delta}=\frac{1}{\omega_f}\left(\mathsf{r}_\lambda^{\varepsilon,\delta}+\lambda^{-2}h\right),\quad\sigma_\lambda^{\varepsilon,\delta}=\frac{1}{\omega_m}\left(\mathsf{s}_\lambda^{\varepsilon,\delta}+\lambda^{-2}h\right),\quad\mathsf{u}_\lambda^{\varepsilon,\delta}=\langle\mathsf{r}_\lambda^{\varepsilon,\delta},\mathsf{s}_\lambda^{\varepsilon,\delta}\rangle, \tag{4.7}$$

and after straightforward rearrangements, rewrite the right-hand side of (4.6) in terms of the functions $\mathsf{r}_\lambda^{\varepsilon,\delta}, \mathsf{s}_\lambda^{\varepsilon,\delta}$ as follows:

$$\begin{aligned}\mathbf{J}^\varepsilon\left[u^{\varepsilon,\delta}\right] &= \frac{\mu^{\varepsilon,\delta}}{(\omega_f)^2}\int_{\Omega_f^{\varepsilon,\delta}}\left\{k_f\left|\nabla\left(\mathsf{r}_\lambda^{\varepsilon,\delta}+\lambda^{-2}h\right)\right|^2+\lambda\omega_f\left|\mathsf{r}_\lambda^{\varepsilon,\delta}\right|^2-2g_\lambda\omega_f\mathsf{r}_\lambda^{\varepsilon,\delta}\right\}dx\\ &\quad+\frac{\mu^{\varepsilon,\delta}}{(\omega_m)^2}\int_{\Omega_m^{\varepsilon,\delta}}\left\{k_m(\varepsilon\delta)^2\left|\nabla\left(\mathsf{s}_\lambda^{\varepsilon,\delta}+\lambda^{-2}h\right)\right|^2+\lambda\omega_m\left|\mathsf{s}_\lambda^{\varepsilon,\delta}\right|^2\right\}dx+Q^{\varepsilon,\delta}\\ &\equiv\overline{\mathbf{J}^\varepsilon}\left[\mathsf{u}^{\varepsilon,\delta}\right]+Q^{\varepsilon,\delta},\end{aligned}\tag{4.8}$$

where the functional $Q^{\varepsilon,\delta}$ does not depend on $\mathsf{r}_\lambda^{\varepsilon,\delta}$ and $\mathsf{s}_\lambda^{\varepsilon,\delta}$. Since $\mathsf{u}_\lambda^{\varepsilon,\delta}$ minimizes the functional $\overline{\mathbf{J}^\varepsilon}$ and

$$\overline{\mathbf{J}^\varepsilon}[0]\le c\mu^{\varepsilon,\delta}\left(\varepsilon\delta+|\Omega_f^{\varepsilon,\delta}|\right)\le C,$$

then

$$\overline{\mathbf{J}^\varepsilon}\left[\mathsf{u}^{\varepsilon,\delta}\right]\le C. \tag{4.9}$$

Together with the definition (4.7) this gives

$$\left\|\nabla\rho_\lambda^{\varepsilon,\delta}\right\|^2_{L^2(\Omega_f^{\varepsilon,\delta})}+\left\|\omega_f\rho_\lambda^{\varepsilon,\delta}-\lambda^{-2}h\right\|^2_{L^2(\Omega_f^{\varepsilon,\delta})}+\left\|\omega_m\sigma_\lambda^{\varepsilon,\delta}-\lambda^{-2}h\right\|^2_{L^2(\Omega_m^{\varepsilon,\delta})}\le C\left|\Omega_f^{\varepsilon,\delta}\right| \tag{4.10}$$

and

$$\left\|\nabla\sigma_\lambda^{\varepsilon,\delta}\right\|^2_{L^2(\Omega_m^{\varepsilon,\delta})}\le\frac{C}{\varepsilon\delta}, \tag{4.11}$$

where the constant C does not depend on ε,δ. This yields

$$\mu^{\varepsilon,\delta}\left\|\rho_\lambda^{\varepsilon,\delta}\right\|^2_{H^1(\Omega_f^{\varepsilon,\delta})}\le C. \tag{4.12}$$

The assertion (I) is proved.

4.1.2. *Proof of assertion* (II)

Inspired by Remark 3.2 we will show that the renormalized coefficients of Eq. (3.9) converge, as $\delta\to 0$, to the corresponding coefficients in (4.1) and prove the following statement.

Lemma 4.1. *Let* R_λ^δ *be the solution of problem* (3.9). *Then*

$$R_\lambda^\delta\rightharpoonup\rho_\lambda^*\quad\textit{weakly in } H^1(0,L), \tag{4.13}$$

where ρ_λ^* *solves problem* (4.1).

Proof of Lemma 4.1. It is clear that the dependence on δ comes from the coefficients I_λ^δ and K^δ. First we study the asymptotic behavior of I_λ^δ as $\delta \to 0$. In the same way as in the proof of Lemma 7.2 in Ref. 19, we get

$$\lim_{\delta\to 0} \frac{1}{|\mathcal{F}^\delta|} I_\lambda^\delta = 2\sqrt{\lambda\omega_m k_m}. \tag{4.14}$$

It is also known from Chap. 5 of Ref. 14 that

$$\lim_{\delta\to 0} \frac{1}{|\mathcal{F}^\delta|} K^\delta = \frac{1}{2}. \tag{4.15}$$

Using (4.14) and (4.15) one can derive Eq. (4.1) from (3.9) by passing to the limit as $\delta \to 0$. The desired convergence (4.13) is now a consequence of the continuous dependence of solutions of (4.1) on the data. Lemma 4.1 is proved. □

We proceed with the convergence (4.4). It relies on (3.12) and (4.13). We have

$$\begin{aligned} &\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{1}{|\Omega_f^{\varepsilon,\delta}|} \left\|\rho_\lambda^{\varepsilon,\delta} - \rho_\lambda^*\right\|^2_{L^2(\Omega_f^{\varepsilon,\delta})} \\ &\quad \le \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{1}{2L\delta}\left(\left\|\mathsf{P}^{\varepsilon,\delta} R_\lambda^{\varepsilon,\delta} - R_\lambda^\delta\right\|^2_{L^2(\Pi)} + \left\|R_\lambda^\delta - \rho_\lambda^*\right\|^2_{L^2(\Pi_f^{\varepsilon,\delta})}\right), \end{aligned} \tag{4.16}$$

where $\mathsf{P}^{\varepsilon,\delta}$ is the extension operator defined in Theorem 3.1. By (3.12)

$$\frac{1}{2L\delta}\lim_{\varepsilon\to 0}\left\|\mathsf{P}^{\varepsilon,\delta} R_\lambda^{\varepsilon,\delta} - R_\lambda^\delta\right\|^2_{L^2(\Pi)} = 0. \tag{4.17}$$

Consider the second term on the right-hand side of (4.16). Since $R_\lambda^\delta = R_\lambda^\delta(z_2)$, $\rho_\lambda^* = \rho_\lambda^*(z_2)$, then taking into account the compactness of embedding $C[0,L]$ into $H^1(0,L)$ one can deduce from (4.13) that

$$\lim_{\delta\to 0}\left\|R_\lambda^\delta - \rho_\lambda^*\right\|_{C[0,L]} = 0.$$

Therefore,

$$\frac{1}{2L\delta}\left\|R_\lambda^\delta - \rho_\lambda^*\right\|^2_{L^2(\Pi_f^{\varepsilon,\delta})} \le C\left\|R_\lambda^\delta - \rho_\lambda^*\right\|^2_{C[0,L]} \underset{\delta\to 0}{\longrightarrow} 0. \tag{4.18}$$

Thus by (4.13)

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{2L\delta}\left\|R_\lambda^\delta - \rho_\lambda^*\right\|^2_{L^2(\Pi_f^{\varepsilon,\delta})} \le C\lim_{\delta\to 0}\left\|R_\lambda^\delta - \rho_\lambda^*\right\|^2_{C(0,L)} = 0. \tag{4.19}$$

The convergence (4.4) follows from (4.16), (4.17) and (4.19). The assertion (II) is proved.

4.1.3. *Proof of assertion* (III)

For an arbitrary function $\phi = \phi(x)$ continuous in the vicinity of the segment $\{x \in \mathbb{R}^2 : \ x_1 = 0; \ 0 \le x_2 \le L\}$, consider the integral

$$I_{\nabla}^{\varepsilon,\delta} \equiv \frac{L}{|\Omega_f^{\varepsilon,\delta}|} \int_{\Omega^\varepsilon} k^{\varepsilon,\delta}(x) \nabla u_\lambda^{\varepsilon,\delta} \phi dx$$

$$= \frac{L}{|\Omega_f^{\varepsilon,\delta}|} \int_{\Omega_f^{\varepsilon,\delta}} k_f \nabla \rho_\lambda^{\varepsilon,\delta} \phi dx + \frac{L}{|\Omega_f^{\varepsilon,\delta}|} \int_{\Omega_m^{\varepsilon,\delta}} k_m (\varepsilon\delta)^2 \nabla \sigma_\lambda^{\varepsilon,\delta} \phi dx. \tag{4.20}$$

For the second term on the right-hand side due to (4.11) we have:

$$\left| \frac{L}{|\Omega_f^{\varepsilon,\delta}|} \int_{\Omega_m^{\varepsilon,\delta}} k_m (\varepsilon\delta)^2 \nabla \sigma_\lambda^{\varepsilon,\delta} \phi(x)\, dx \right| \le C\,(\varepsilon\delta) \left\| \nabla \sigma_\lambda^{\varepsilon,\delta} \right\|_{L^2(\Omega_m^{\varepsilon,\delta})} \left| \Omega_m^{\varepsilon,\delta} \right|^{1/2} \le C\,\varepsilon\sqrt{\delta}.$$

Hence

$$\lim_{\varepsilon\to 0} \frac{L}{|\Omega_f^{\varepsilon,\delta}|} \int_{\Omega_m^{\varepsilon,\delta}} k_m (\varepsilon\delta)^2 \nabla \sigma_\lambda^{\varepsilon,\delta} \phi(x)\, dx = \mathbf{0}. \tag{4.21}$$

For the first term on the right-hand side of (4.20) we have

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{L}{|\Omega_f^{\varepsilon,\delta}|} \int_{\Omega_f^{\varepsilon,\delta}} k_f \frac{\partial \rho_\lambda^{\varepsilon,\delta}}{\partial x_1} \phi(x)\, dx = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{k_f}{2\delta} \int_{\Pi_f^{\varepsilon,\delta}} \varepsilon^{-1} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_1} \phi(\varepsilon z_1, z_2)\, dz \tag{4.22}$$

and

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{L}{|\Omega_f^{\varepsilon,\delta}|} \int_{\Omega_f^{\varepsilon,\delta}} k_f \frac{\partial \rho_\lambda^{\varepsilon,\delta}}{\partial x_2} \phi(x)\, dx = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{k_f}{2\delta} \int_{\Pi_f^{\varepsilon,\delta}} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2} \phi(\varepsilon z_1, z_2)\, dz. \tag{4.23}$$

Due to Lemma 3.1 and (3.63) we have

$$\lim_{\varepsilon\to 0} \frac{k_f}{2\delta} \int_{\Pi_f^{\varepsilon,\delta}} \varepsilon^{-1} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_1} \phi(\varepsilon z_1, z_2)\, dz = -\frac{k_f}{2\delta} \int_0^L \int_{\mathcal{F}^\delta} \frac{\partial R_\lambda^{\delta}}{\partial z_2}(z_2) \frac{\partial \chi^\delta}{\partial y_1}(y) \phi(0, z_2)\, dy dz_2 \tag{4.24}$$

and

$$\lim_{\varepsilon\to 0} \frac{k_f}{2\delta} \int_{\Pi_f^{\varepsilon,\delta}} \frac{\partial R_\lambda^{\varepsilon,\delta}}{\partial z_2} \phi(\varepsilon z_1, z_2)\, dz$$

$$= \frac{k_f}{2\delta} \int_0^L \int_{\mathcal{F}^\delta} \left[\frac{\partial R_\lambda^{\delta}}{\partial z_2}(z_2) - \frac{\partial R_\lambda^{\delta}}{\partial z_2}(z_2) \frac{\partial \chi^\delta}{\partial y_2}(y) \right] \phi(0, z_2)\, dy dz_2. \tag{4.25}$$

The integral in (4.24) vanishes because

$$\int_{\mathcal{F}^\delta} \frac{\partial \chi^\delta}{\partial y_1}(y)\, dy = 0$$

and we get

$$\lim_{\varepsilon\to 0}\frac{L}{|\Omega_f^{\varepsilon,\delta}|}\int_{\Omega_f^{\varepsilon,\delta}} k_f\frac{\partial\rho_\lambda^{\varepsilon,\delta}}{\partial x_1}\phi(x)\,dx = 0. \tag{4.26}$$

Consider the integral on the right-hand side of (4.25). By the definition of K^δ we have

$$\begin{aligned}
&\frac{k_f}{2\delta}\int_0^L\int_{\mathcal{F}^\delta}\left[\frac{\partial R_\lambda^\delta}{\partial z_2}(z_2)-\frac{\partial R_\lambda^\delta}{\partial z_2}(z_2)\frac{\partial\chi^\delta}{\partial y_2}(y)\right]\phi(0,z_2)\,dydz_2\\
&\quad = k_f\,\frac{K^\delta}{2\delta}\int_0^L\frac{\partial R_\lambda^\delta}{\partial z_2}(z_2)\phi(0,z_2)\,dz_2.
\end{aligned}$$

It remains to pass to the limit in δ and use (4.13) and (4.15) to obtain the relation

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{k_f}{2\delta}\int_0^L\int_{\mathcal{F}^\delta}\left[\frac{\partial R_\lambda^\delta}{\partial z_2}(z_2)-\frac{\partial R_\lambda^\delta}{\partial z_2}(z_2)\frac{\partial\chi^\delta}{\partial y_2}(y)\right]\phi(0,z_2)\,dydz_2\\
&\quad = \frac{k_f}{2}\int_0^L\frac{\partial\rho_\lambda^*}{\partial z_2}(z_2)\phi(0,z_2)\,dz_2
\end{aligned}$$

and, finally,

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{L}{|\Omega_f^{\varepsilon,\delta}|}\int_{\Omega_f^{\varepsilon,\delta}} k_f\frac{\partial\rho_\lambda^{\varepsilon,\delta}}{\partial x_2}\phi(x)\,dx = \frac{k_f}{2}\int_0^L\frac{\partial\rho_\lambda^*}{\partial\xi}(\xi)\phi(0,\xi)\,d\xi. \tag{4.27}$$

Now the desired flux convergence (4.5) follows from (4.21), (4.26) and (4.27). Proposition 4.1 is proved. □

5. Step 3. Proof of Theorem 2.1

The proof of Theorem 2.1 relies on the results obtained in the previous sections for the stationary case. It will be given in Secs. 5.1–5.3.

5.1. *Proof of assertion* (I)

In order to obtain the *a priori* estimates for Eq. (2.1) we rewrite it as follows:

$$\left[\omega^{\varepsilon,\delta}(x)u_t^{\varepsilon,\delta}-h(x)\right]-\operatorname{div}(k^{\varepsilon,\delta}\nabla\rho^{\varepsilon,\delta}) = g^{\varepsilon,\delta}(x)\quad\text{in }\Omega_T^\varepsilon, \tag{5.1}$$

where $g^{\varepsilon,\delta}(x)=g(x)$ in $\Omega_f^{\varepsilon,\delta}$ and $g^{\varepsilon,\delta}(x)=0$ in $\Omega_m^{\varepsilon,\delta}$, and $\Omega_T^\varepsilon=(0,T)\times\Omega^\varepsilon$. Then we multiply (5.1) by $\left[\omega_m u^{\varepsilon,\delta}-th(x)\right]$ and integrate the resulting relation over $(0,t)\times\Omega^\varepsilon$. Considering the regularity properties of the functions g,h and applying the Cauchy inequality and Gronwall's lemma we arrive at the estimate

$$\begin{aligned}
&\|\omega^{\varepsilon,\delta}u^{\varepsilon,\delta}(t)-th\|^2_{L^2(\Omega^\varepsilon)}+\int_0^t\int_{\Omega_f^{\varepsilon,\delta}}\left|\nabla\rho^{\varepsilon,\delta}\right|^2dxd\tau+(\varepsilon\delta)^2\int_0^t\int_{\Omega_m^{\varepsilon,\delta}}\left|\nabla\sigma^{\varepsilon,\delta}\right|^2dxd\tau\\
&\quad\le C_T\,|\Omega_f^{\varepsilon,\delta}|
\end{aligned}\tag{5.2}$$

with constant C_T independent of ε,δ.

In a similar way, multiplying (5.1) by $[\omega_m u_t^{\varepsilon,\delta} - h(x)]$ and integrating over $(0,t)\times\Omega^\varepsilon$, we get

$$\|\omega^{\varepsilon,\delta} u_t^{\varepsilon,\delta}(t) - h\|^2_{L^2(\Omega_T^\varepsilon)} + \left\|\nabla\rho^{\varepsilon,\delta}(t)\right\|^2_{L^2(\Omega_f^{\varepsilon,\delta})} + (\varepsilon\delta)^2 \left\|\nabla\sigma^{\varepsilon,\delta}(t)\right\|^2_{L^2(\Omega_m^{\varepsilon,\delta})} \le C_T\, |\Omega_f^{\varepsilon,\delta}|. \tag{5.3}$$

The estimates (5.2) and (5.3) imply the following uniform bounds

$$\frac{1}{|\Omega_f^{\varepsilon,\delta}|}\left\|\rho^{\varepsilon,\delta}(t)\right\|^2_{H^1(\Omega_f^{\varepsilon,\delta})} \le C_T, \quad \frac{1}{|\Omega^\varepsilon|}\left\|\omega^{\varepsilon,\delta} u^{\varepsilon,\delta}(t) - th\right\|^2_{L^2(\Omega^\varepsilon)} \le C_T\delta, \tag{5.4}$$

and (I) is proved.

5.2. *Proof of assertion* (II)

By the change of variables $z_1 = \frac{x_1}{\varepsilon}$, $z_2 = x_2$ from (5.4) we obtain that

$$\|R^{\varepsilon,\delta}(t)\|_{L^2(\Pi_f^{\varepsilon,\delta})} + \varepsilon^{-1}\left\|\frac{\partial R^{\varepsilon,\delta}}{\partial z_1}(t)\right\|_{L^2(\Pi_f^{\varepsilon,\delta})} + \left\|\frac{\partial R^{\varepsilon,\delta}}{\partial z_2}(t)\right\|_{L^2(\Pi_f^{\varepsilon,\delta})} \le C_T\sqrt{\delta}, \tag{5.5}$$

where $R^{\varepsilon,\delta}(t) = R^{\varepsilon,\delta}(t,z) = \rho^{\varepsilon,\delta}(t,\varepsilon z_1, z_2)$. It is not difficult to show that the extension operator $\mathsf{P}^{\varepsilon,\delta}$ can be constructed in such a way that the constant C_δ in (3.17), (3.18) is equal to C/δ. Under such a choice of $\mathsf{P}^{\varepsilon,\delta}$ we derive from (5.5) that

$$\left\|\mathsf{P}^{\varepsilon,\delta}R^{\varepsilon,\delta}(t)\right\|_{L^2(\Pi)} + \varepsilon^{-1}\left\|\frac{\partial \mathsf{P}^{\varepsilon,\delta}R^{\varepsilon,\delta}}{\partial z_1}(t)\right\|_{L^2(\Pi)} + \left\|\frac{\partial \mathsf{P}^{\varepsilon,\delta}R^{\varepsilon,\delta}}{\partial z_2}(t)\right\|_{L^2(\Pi)} \le C_T. \tag{5.6}$$

Also, the estimate (5.3) implies the bound

$$\left\|\frac{\partial}{\partial t}\left(\mathsf{P}^{\varepsilon,\delta}R^{\varepsilon,\delta}\right)\right\|_{L^2((0,T)\times\Pi)} = \left\|\left(\mathsf{P}^{\varepsilon,\delta}\frac{\partial}{\partial t}R^{\varepsilon,\delta}\right)\right\|_{L^2((0,T)\times\Pi)} \le C_T.$$

By the embedding theorem for each $\delta > 0$ there is a function $V^\delta = V^\delta(t,z)$ such that

$$\lim_{\varepsilon\to 0}\left\|\mathsf{P}^{\varepsilon,\delta}R^{\varepsilon,\delta} - V^\delta\right\|^2_{L^\infty((0,T);L^2(\Pi))} = 0 \tag{5.7}$$

and

$$\|V^\delta\|_{L^2(0,T;H^1(\Pi))} + \left\|\frac{\partial}{\partial t}V^\delta\right\|_{L^2(0,T;L^2(\Pi))} \le C_T.$$

The estimate (5.6) also yields that V^δ does not depend on z_1, i.e. $V^\delta(t,z) = V^\delta(t,z_2)$. Thus, there is a function $V = V(t,z_2)$ such that, along a subsequence,

$$\lim_{\delta\to 0}\left\|V^\delta - V\right\|^2_{L^\infty(0,T;L^2(0,L))} = 0. \tag{5.8}$$

Using the interpolation inequality we obtain

$$\begin{aligned}\left\|V^\delta(t) - V(t)\right\|^2_{L^\infty(0,L)} &\le C\left\|V^\delta(t) - V(t)\right\|^2_{H^{3/4}(0,L)}\\ &\le \left\|V^\delta(t) - V(t)\right\|^{1/2}_{L^2(0,L)}\left\|V^\delta(t) - V(t)\right\|^{3/2}_{H^1(0,L)}.\end{aligned}$$

Together with (5.6) and (5.8) this gives

$$\lim_{\delta\to 0}\left\|V^\delta - V\right\|_{L^\infty((0,T)\times(0,L))} = \lim_{\delta\to 0}\sup_{0\le t\le T}\left\|V^\delta(t) - V(t)\right\|_{L^\infty(0,L)} = 0.$$

By the same arguments as in (4.18) and (4.19), we obtain

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{2L\delta}\left\|V^\delta - V\right\|_{L^\infty(0,T;L^2(\Pi_f^{\varepsilon,\delta}))} = 0.$$

Now, returning to the variables x_1, x_2 and considering (5.7), we obtain

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{|\Omega_f^{\varepsilon,\delta}|}\left\|\rho^{\varepsilon,\delta} - V\right\|^2_{L^\infty(0,T;L^2(\Omega_f^{\varepsilon,\delta}))} = 0. \tag{5.9}$$

The convergence result (2.7) will immediately follow from (5.9) if we show that $V = V(t,\xi)$ is a solution of problem (2.4). To this end, for any $t \in (0,T)$ and any $\varphi \in C_0^\infty(0,L)$, we consider the integral

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{L}{|\Omega_f^{\varepsilon,\delta}|}\int_{\Omega_f^{\varepsilon,\delta}}\rho^{\varepsilon,\delta}(t,x)\varphi(x_2)\,dx = \int_0^L V(t,\xi)\varphi(\xi)\,d\xi. \tag{5.10}$$

Let $u_\lambda^{\varepsilon,\delta} = \langle\rho_\lambda^{\varepsilon,\delta}, \sigma_\lambda^{\varepsilon,\delta}\rangle$ be the solution of problem (2.1) with an arbitrary complex λ such that $\arg\lambda \ne \pi$. Then $\rho_\lambda^{\varepsilon,\delta}$ is an analytic function in the complex λ-plane $\mathbb{C}\setminus\{\arg\lambda = \pi\}$ and

$$\|\rho_\lambda^{\varepsilon,\delta}\|^2_{L^2(\Omega_f^{\varepsilon,\delta})} \le C_2\frac{|\Omega_f^{\varepsilon,\delta}|}{|\lambda|^4}, \quad |\arg\lambda - \pi| \ge \vartheta_0 > 0, \tag{5.11}$$

where C_2 is a constant independent of ε, δ. Moreover, $\rho^{\varepsilon,\delta}$ may be represented by the inverse Laplace transform which reads

$$\rho^{\varepsilon,\delta}(t,x) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\exp(\lambda t)\rho_\lambda^{\varepsilon,\delta}(x)\,d\lambda, \quad \theta > 0. \tag{5.12}$$

Now let ρ_λ^* be the solution of problem (4.1)–(4.2) with an arbitrary complex λ such that $\arg\lambda \ne \pi$. The solution ρ_λ^* of this problem is an analytic function of λ in the complex λ-plane $\mathbb{C}\setminus\{\arg\lambda = \pi\}$ and

$$\|\rho_\lambda^*\|^2_{L^2(0,L)} \le \frac{C_3}{|\lambda|^4} \tag{5.13}$$

for $|\arg\lambda - \pi| \geq \vartheta_0 > 0$. Moreover, the solution of problem (2.4)–(2.5) can be represented as follows

$$\rho^*(t,\xi) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \exp(\lambda t)\rho^*_\lambda(\xi)\,d\lambda, \quad \theta > 0. \tag{5.14}$$

From (5.11)–(5.14) it follows that

$$\frac{L}{|\Omega_f^{\varepsilon,\delta}|}\int_{\Omega_f^{\varepsilon,\delta}} \rho^{\varepsilon,\delta}(t,x)\varphi(x_2)\,dx = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\left\{\frac{L}{|\Omega_f^{\varepsilon,\delta}|}\int_{\Omega_f^{\varepsilon,\delta}} \rho_\lambda^{\varepsilon,\delta}(x)\varphi(x_2)\,dx\right\}d\lambda \tag{5.15}$$

and

$$\int_0^L \rho^*(t,\xi)\varphi(\xi)\,d\xi = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\left\{\int_0^L \rho^*_\lambda(\xi)\varphi(\xi)\,d\xi\right\}d\lambda. \tag{5.16}$$

From Proposition 4.1, for any $\lambda > 0$, we have

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{L}{|\Omega_f^{\varepsilon,\delta}|}\int_{\Omega_f^{\varepsilon,\delta}} \rho_\lambda^{\varepsilon,\delta}(x)\varphi(x_2)\,dx = \int_0^L \rho^*_\lambda(\xi)\varphi(\xi)\,d\xi. \tag{5.17}$$

Here $\rho_\lambda^{\varepsilon,\delta}$ is an analytic function in λ variable such that $\mu^{\varepsilon,\delta}\left\|\rho_\lambda^{\varepsilon,\delta}\right\|^2_{H^1(\Omega_f^{\varepsilon,\delta})} \leq C(\lambda)$. Therefore, (5.17) is valid for any complex λ and the convergence is uniform on compact sets in the domain $|\arg\lambda - \pi| \geq \vartheta_0 > 0$. Now from (5.15) to (5.17) we obtain that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{L}{|\Omega_f^{\varepsilon,\delta}|}\int_{\Omega_f^{\varepsilon,\delta}} \rho^{\varepsilon,\delta}(t,x)\varphi(x_2)dx = \int_0^L \rho^*(t,\xi)\varphi(\xi)\,d\xi, \tag{5.18}$$

for any $t \in (0,T)$. Comparing (5.10) and (5.18) we conclude that $V(t,\xi) = \rho^*(t,\xi)$.

Thus the assertion (II) of Theorem 2.1 is proved.

5.3. *Proof of assertion* (III)

Let us show now that, for any $t \in (0,T)$ and any function $\phi = \phi(x)$ continuous in the vicinity of the segment $\{x \in \mathbb{R}^2 :\ x_1 = 0;\ 0 \leq x_2 \leq L\}$,

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{L}{|\Omega_f^{\varepsilon,\delta}|}\int_{\Omega^\varepsilon} k^{\varepsilon,\delta}(x)\nabla u^{\varepsilon,\delta}\phi(x)\,dx = \frac{k_f}{2}\int_0^L \vec{\mathbf{R}}^*(t,\xi)\phi(0,\xi)\,d\xi, \tag{5.19}$$

where

$$\vec{\mathbf{R}}^*(t,\xi) = \left(0, \frac{\partial\rho^*}{\partial\xi}(t,\xi)\right).$$

To this end we fix $\theta > 0$ and consider the integral

$$I_\nabla^{\varepsilon,\delta} = \frac{L}{|\Omega_f^{\varepsilon,\delta}|}\int_{\Omega^\varepsilon} k^{\varepsilon,\delta}(x)\nabla u_\lambda^{\varepsilon,\delta}\phi(x)\,dx, \tag{5.20}$$

where $\lambda \in \Upsilon_\theta = \{s \in \mathbb{C} : \operatorname{Re} s > \theta/2\}$. This function is analytic in Υ_θ, moreover, using (5.11) one can show that

$$\left| I_\nabla^{\varepsilon,\delta} \right| \le C\,\lambda^{-3/2}, \tag{5.21}$$

where C is a constant independent of ε, δ and λ. Since the function ρ_λ^* is analytic, the convergence (4.5) occurs for all $\lambda \in \Upsilon_\theta$. Then we make use of the inverse Laplace transform and, finally, get:

$$\begin{aligned}
&\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{L}{|\Omega_f^{\varepsilon,\delta}|} \int_{\Omega^\varepsilon} k^{\varepsilon,\delta}(x)\nabla u^{\varepsilon,\delta}(t,x)\phi(x)\,dx \\
&\quad = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_{\theta-i\infty}^{\theta+i\infty} e^{\lambda t} I_\nabla^{\varepsilon,\delta}\,d\lambda \\
&\quad = \frac{1}{2\pi} \int_{\theta-i\infty}^{\theta+i\infty} e^{\lambda t} \left(\frac{k_f}{2} \int_0^L \vec{\mathbf{R}}_\lambda^*(\xi)\phi(0,\xi)\,d\xi \right) d\lambda \\
&\quad = \frac{k_f}{2} \int_0^L \vec{\mathbf{R}}^*(t,\xi)\phi(0,\xi)\,d\xi.
\end{aligned} \tag{5.22}$$

Thus the assertion (III) of Theorem 2.1 is proved. This completes the proof of Theorem 2.1.

6. Nonstationary Effective δ-Model

Here we formulate and justify the homogenization result for problem (2.1) in the case when the thickness of the fractures is of the same order as the structure period, i.e. δ is a fixed positive constant.

Consider the following auxiliary problem

$$\begin{cases}
\omega_m \zeta_t^\delta - k_m \delta^2 \Delta_y \zeta^\delta = 0 & \text{in } (0,T)\times\mathcal{M}^\delta, \\
\zeta^\delta(t,y) = 0 & \text{on } (0,T)\times\partial\mathcal{M}^\delta, \\
\zeta^\delta(0,y) = 1 & \text{in } \mathcal{M}^\delta,
\end{cases} \tag{6.1}$$

and denote

$$Y^\delta(t) = \omega_m \int_{\mathcal{M}^\delta} \zeta^\delta(t,y)\,dy. \tag{6.2}$$

The limit nonstationary δ-model reads

$$\begin{cases}
\omega_f |\mathcal{F}^\delta| R_t^\delta - k_f K^\delta \dfrac{\partial^2 R^\delta}{\partial \xi^2} = |\mathcal{F}^\delta|(g+h)(\xi) + \mathbf{S}(R^\delta) & \text{in } (0,T)\times(0,L), \\
\dfrac{\partial R^\delta}{\partial \xi}(t,0) = \dfrac{\partial R^\delta}{\partial \xi}(t,L) = 0 & \text{on } (0,T), \\
R^\delta(0,\xi) = 0 & \text{in } (0,L),
\end{cases} \tag{6.3}$$

where

$$\mathbf{S}(R^\delta) = -\frac{\partial Y^\delta}{\partial t} \star \frac{\partial R^\delta}{\partial t}(t) + \frac{1}{\omega_m} Y^\delta(t) h(0,\xi)$$

and $\star$ stands for the convolution operator.

Theorem 6.1. *The solution* $u^{\varepsilon,\delta} = \langle \rho^{\varepsilon,\delta}, \sigma^{\varepsilon,\delta} \rangle$ *of* (2.1) *converges, as* $\varepsilon \to 0$, *to a function* $W^\delta(t,\xi,y) = \langle R^\delta(t,\xi), S^\delta(t,\xi,y) \rangle$ *in the following sense*

$$\lim_{\varepsilon\to 0} \frac{1}{|\Omega^\varepsilon|} \int_0^T \int_{\Omega^\varepsilon} \left| (u^{\varepsilon,\delta}(t,x) - W^\delta\left(t, x_2, \frac{x}{\varepsilon}\right) \right|^2 dxdt = 0, \tag{6.4}$$

where R^δ *is a solution of* (6.3) *and*

$$S^\delta(t,\xi,y) = R^\delta(t,\xi) + \zeta^\delta(\cdot,y) \star \left(h(0,\xi) - \frac{\partial R^\delta}{\partial t}(\cdot,\xi) \right)(t). \tag{6.5}$$

The proof of this theorem relies on the statement of Theorem 3.1 and can be derived from this statement by means of the inverse Laplace transform in exactly the same way as in the proof of Theorem 2.1.

7. A Homogenization Result of Flow in a 3D Porous Medium with a Thin Layer

The convergence results of Theorem 2.1 remains valid (after natural modifications) for 3D thin domains. In this section we study a model problem of a single phase flow in a porous medium with a thin plate.

Denote by Ω^ε a rectangle parallelepiped in $\mathbb{R}^3$ defined by $\Omega^\varepsilon = (-\varepsilon/2, \varepsilon/2) \times \mathcal{P}$ with $\mathcal{P} = (0, L_2) \times (0, L_3)$. Letting $\mathcal{Y} = (0,1)^3$ we introduce the reference fracture part $\mathcal{F}^\delta = \{y \in \mathcal{Y}, \text{dist}\,(y, \partial\mathcal{Y}) < \frac{\delta}{2}\}$ and the reference matrix block $\mathcal{M}^\delta = \mathcal{Y} \setminus \overline{\mathcal{F}^\delta}$. Assuming that L_1 and L_2 are integer multipliers of ε, i.e. $L_2 = N_2\varepsilon$, $L_3 = N_3\varepsilon$, we define

$$\Omega_m^{\varepsilon,\delta} = \bigcup_{\ell_2=0}^{N_2-1} \bigcup_{\ell_3=0}^{N_3-1} \varepsilon\Big(\mathcal{M}^\delta + (0, \ell_2, \ell_3)\Big), \quad \Omega_f^{\varepsilon,\delta} = \Omega^\varepsilon \setminus \overline{\Omega_m^{\varepsilon,\delta}}.$$

The flow in the matrix-fracture medium Ω^ε is described by the following equation:

$$\textit{3D Micromodel}: \begin{cases} \omega^{\varepsilon,\delta}(x) u_t^{\varepsilon,\delta} - \text{div}\,(k^{\varepsilon,\delta}(x) \nabla u^{\varepsilon,\delta}) = G^{\varepsilon,\delta}(x) & \text{in } (0,T) \times \Omega^\varepsilon; \\ \nabla u^{\varepsilon,\delta} \cdot \boldsymbol{\nu} = 0 & \text{on } (0,T) \times \partial\Omega^\varepsilon; \\ u^{\varepsilon,\delta}(0,x) = 0 & \text{in } \Omega^\varepsilon, \end{cases} \tag{7.1}$$

where

$$\omega^{\varepsilon,\delta}(x) = \begin{cases} \omega_f & \text{in } \Omega_f^{\varepsilon,\delta}; \\ \omega_m & \text{in } \Omega_m^{\varepsilon,\delta}; \end{cases} \quad k^{\varepsilon,\delta}(x) = \begin{cases} k_f & \text{in } \Omega_f^{\varepsilon,\delta}; \\ k_m(\varepsilon\delta)^2 & \text{in } \Omega_m^{\varepsilon,\delta}; \end{cases}$$

$$G^{\varepsilon,\delta}(x) = \begin{cases} (g+h)(x) & \text{in } \Omega_f^{\varepsilon,\delta}; \\ h(x) & \text{in } \Omega_m^{\varepsilon,\delta}. \end{cases}$$

Here $\omega_f, \omega_m, k_f, k_m$ are positive constants and $g, h \in C^1(\mathbb{R}^3)$. As in the previous sections we introduce the notation:

$$u^{\varepsilon,\delta} = \begin{cases} \rho^{\varepsilon,\delta} & \text{in } \Omega_f^{\varepsilon,\delta}; \\ \sigma^{\varepsilon,\delta} & \text{in } \Omega_m^{\varepsilon,\delta} \end{cases}$$

and rewrite problem (7.1) separately in the fracture and matrix parts (see problems (2.2) and (2.3)).

The goal of this section is to extend the results on the asymptotic behavior of $u^{\varepsilon,\delta}$ obtained in the previous sections, to the 3D model under consideration. Following the lines of Theorems 2.1 and 6.1, we show that for any fixed δ problem (7.1) admits homogenization (as $\varepsilon \to 0$) and that the homogenized solution converges, as $\delta \to 0$, to a solution of the effective problem:

$$\textit{3D Macromodel}: \begin{cases} \omega_f \rho_t^* - \dfrac{2}{3} k_f \Delta \rho^* = G(\varkappa) + \mathbf{S}(\rho^*) & \text{in } (0,T) \times \mathcal{P}; \\ \nabla \rho^* \cdot \boldsymbol{\nu} = 0 & \text{on } (0,T) \times \partial \mathcal{P}; \\ \rho^*(0,\varkappa) = 0 & \text{in } \mathcal{P} \end{cases} \tag{7.2}$$

with $G(\varkappa) = (g+h)(0,\varkappa)$ and the additional source term $\mathbf{S}(\rho^*)$ defined in (2.5), here $\varkappa$ stands for (x_2, x_3). More precisely, the following result holds.

Theorem 7.1. *Let* $u^{\varepsilon,\delta} = \langle \rho^{\varepsilon,\delta}, \sigma^{\varepsilon,\delta} \rangle$ *be the solution of* (7.1). *Then, for any* $t \in (0,T)$,

(I) *the function* $u^{\varepsilon,\delta}$ *converges to* $t\,h(x)$, *namely*:

$$\lim_{\delta\to 0} \lim_{\varepsilon \to 0} \frac{1}{|\Omega^\varepsilon|} \left\| \omega^{\varepsilon,\delta} u^{\varepsilon,\delta} - \mathrm{th} \right\|^2_{L^2(\Omega^\varepsilon)} = 0; \tag{7.3}$$

(II) *the function* $\rho^{\varepsilon,\delta}$ *satisfies the limit relation*

$$\lim_{\delta\to 0} \lim_{\varepsilon \to 0} \frac{1}{|\Omega_f^{\varepsilon,\delta}|} \left\| \rho^{\varepsilon,\delta} - \rho^* \right\|^2_{L^2(\Omega_f^{\varepsilon,\delta})} = 0, \tag{7.4}$$

where $\rho^* = \rho(t,\varkappa)$ *is a solution of* (7.2), (2.5).

(III) *For any* $t \in (0,T)$, *and any function* $\phi = \phi(x)$ *continuous in the vicinity of the rectangle* $\{x \in \mathbb{R}^3 : \ x_1 = 0;\ 0 \le x_2 \le L_2;\ 0 \le x_3 \le L_3\}$, *it holds*

$$\lim_{\delta\to 0} \lim_{\varepsilon \to 0} \frac{|\mathcal{P}|}{|\Omega_f^{\varepsilon,\delta}|} \int_{\Omega^\varepsilon} k^{\varepsilon,\delta}(x) \nabla u^{\varepsilon,\delta} \phi(x)\, dx = \frac{2k_f}{3} \int_{\mathcal{P}} \vec{\mathbf{R}}^*(t,\varkappa) \phi(0,\varkappa)\, d\varkappa \tag{7.5}$$

with

$$\vec{\mathbf{R}}^*(t,\varkappa) = \left(0, \frac{\partial \rho^*}{\partial \varkappa_2}(t,\varkappa), \frac{\partial \rho^*}{\partial \varkappa_3}(t,\varkappa) \right).$$

For a fixed $\delta > 0$ the result similar to that of Theorem 6.1 holds true. In order to formulate this result, define ζ^δ and Y^δ as in (6.1) and (6.2). In the 3D case the limit nonstationary δ-model reads

$$\begin{cases} \omega_f|\mathcal{F}^\delta|R_t^\delta - k_f K^\delta \Delta_\varkappa R^\delta(t,\varkappa) = |\mathcal{F}^\delta|(g+h)(0,\varkappa) + \mathbf{S}(R^\delta) & \text{in } (0,T)\times\mathcal{P}; \\ \nabla_\varkappa R^\delta \cdot \boldsymbol{\nu} = 0 & \text{on } (0,T)\times\partial\mathcal{P}; \\ R^\delta(0,\varkappa) = 0 & \text{in } \mathcal{P}, \end{cases} \tag{7.6}$$

where

$$\mathbf{S}(R^\delta) = -\frac{\partial Y^\delta}{\partial t} \star \frac{\partial R^\delta}{\partial t}(t) + \frac{1}{\omega_m} Y^\delta(t) h(0,\varkappa)$$

and

$$K^\delta = \frac{1}{|\mathcal{F}^\delta|}\alpha_{\mathcal{F}^\delta}\left(\chi_2^\delta - y_2, \chi_2^\delta - y_2\right) = \frac{1}{|\mathcal{F}^\delta|}\alpha_{\mathcal{F}^\delta}\left(\chi_3^\delta - y_3, \chi_3^\delta - y_3\right);$$

here the vector-function χ^δ is one-periodic in the variables y_2 and y_3, and satisfies the equation

$$\begin{cases} -\Delta_y \chi_{2,3}^\delta = 0 & \text{in } \mathcal{F}^\delta; \\ \nabla_y(\chi_{2,3}^\delta - y_{2,3})\cdot\boldsymbol{\nu} = 0 & \text{on } \Gamma_{mf}^\delta; \\ \dfrac{\partial}{\partial y_1}(\chi_{2,3}^\delta - y_{2,3}) = 0 & \text{on } \partial\mathcal{Y} \cap \left\{ y_1 = \pm\dfrac{1}{2} \right\} \end{cases}$$

Theorem 7.2. *The solution* $u^{\varepsilon,\delta} = \langle \rho^{\varepsilon,\delta}, \sigma^{\varepsilon,\delta}\rangle$ *of* (7.1) *converges, as* $\varepsilon \to 0$, *to a function* $W^\delta(t,\varkappa,y) = \langle R^\delta(t,\varkappa), S^\delta(t,\varkappa,y)\rangle$ *in the following sense*

$$\lim_{\varepsilon\to 0} \frac{1}{|\Omega^\varepsilon|}\int_0^T \int_{\Omega^\varepsilon} \left| u^{\varepsilon,\delta}(t,x) - W^\delta\left(t, x_2, x_3, \frac{x}{\varepsilon}\right)\right|^2 dxdt = 0, \tag{7.7}$$

where R^δ *is a solution of* (7.6) *and* S^δ *is defined by*

$$S^\delta(t,\varkappa,y) = R^\delta(t,\varkappa) + \zeta^\delta(\cdot,y) \star \left(h(0,\varkappa) - \frac{\partial R^\delta}{\partial t}(\cdot,\varkappa)\right)(t).$$

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