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PREFACE

In the Summer of 2006, an International Conference, satellite of the International Congress of Mathematicians (ICM2006, Madrid), took place in Lisbon and was organized by the Group of Mathematical Physics of the University of Lisbon (GFMUL).

We decided to entitle "Stochastic Analysis in Mathematical Physics" the elaboration of the ideas presented there by some of the participants.

The last ten years were witness to a remarkable penetration of the methods of Stochastic Analysis in all fields of Mathematical Physics. What was regarded by many, not so long ago, as a set of esoteric tools, turned into a fundamental component of our understanding of natural phenomena. The works collected here illustrate the versatility of those stochastic methods and we warmly thank the authors.

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Near extinction of solution caused by strong absorption on a fine-grained set

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This article considers the large-time behaviour of solutions of a nonlinear parabolic equation modelling heat transfer in a medium with a highly oscillating absorption coefficient (e.g., one that is generated by a "random chessboard"or periodic). The minimal size of a cube where absorption is substantial irrespective of its position is the small parameter of the problem.

If the absorption coefficient is separated from zero on a disperse fine-grained set, the behaviour of a solution is shown to imitate extinction in finite time — even when the lack of absorption on a massive set makes *bona fide* extinction impossible. Namely, as long as the instantaneous value of thermal energy exceeds a small threshold value, it admits a decreasing majorant that vanishes after a finite time. For energies below this threshold, this majorant becomes unapplicable; it can be replaced by one which decays fast, but remains strictly positive.

It is also shown that the Dirichlet problem for a quasi-linear heat equation with nonlinear absorption term can be homogenized.

Keywords: Absorption-diffusion equation, medium with microstructure, random chessboard, finite time extinction, homogenization

1. Introduction

This paper is dedicated to the large-time behaviour of solutions of a model boundary problem describing heat transfer (or diffusion) in a bounded domain $G \Subset \mathbb{R}^d$ with regular boundary: for $t > 0$ and $x \in G$

$$\partial_t \left(|u_\varepsilon|^{\gamma-2} u_\varepsilon \right) = \operatorname{div} \left(a \, |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \right) - S_\varepsilon^\sigma \, |u_\varepsilon|^{\sigma-2} u_\varepsilon, \qquad (1.1)$$

where the matrix $a\,(x,t,u)$ is symmetric, bounded and strictly positive definite:

$$0 < A_* \, |\xi|^2 \le a_\varepsilon\,(x,t,u)\,\xi \cdot \xi \le A^* \, |\xi|^2 . \qquad (1.2)$$

The initial and boundary conditions are

$$u_\varepsilon|_{t=0} = u_0 \in L^{\gamma+k}(G), \; k \in \mathbb{R}_+, \quad u_\varepsilon|_{\partial G} = 0. \qquad (1.3)$$

The exponents of nonlinearities are constant and satisfy the condition

$$1 < \sigma < \gamma \le p \le d. \qquad (1.4)$$

The absorption coefficient $S_\varepsilon(x) \ge 0$ does not depend on time t and is a highly oscillating function of the spatial variable (e.g., ε-periodic or random). The existence of a solution is assumed as a prerequisite (some pertinent existence and uniqueness theorems are briefly discussed in §2.1).

Equation (1.1) is obviously a modification of the classical heat equation. In the context of thermal transport, the nonlinearities model the dependence of properties of the heat carrier on its temperature. For instance, the choice of $\gamma = p = 2$ and $\sigma < 2$ in Eq. (1.4) corresponds to a material which combines constant thermal conductivity with heat absorption that increases as it cools down. Other admissible choices of exponents maintain a similar property of the underlying physical process.

The influence of nonlinear absorption can produce a qualitative difference in the behaviour of solution. Without external forcing, the linear heat equation describes the process of cooling which never ends completely. By contrast, Eq. (1.1) with exponents of Eq. (1.4) defines a solution which vanishes completely after a finite time if the absorption coefficient S_ε is separated from zero on all domain (see, e.g., Ref. 1 (Ch.2 §2.3), the survey Ref. 2 and Refs. 3,4).

However, the above qualitative difference in behaviour of solutions is sensitive to seemingly slight alterations of the problem. Extinction of the solution in finite time does not occur if the non-absorbing set $F_\varepsilon = \{S_\varepsilon = 0\}$ has positive measure.

For instance, a solution of the simplest quasi-linear heat equation ($\gamma = p = 2$, $a = Id$) cannot vanish on all domain in finite time if its initial value is positive on an arbitrarily small ball contained in F_ε. This follows, e.g., from the Feynman-Kac representation for the solution of the heat equation because a Brownian trajectory can protract its stay in the ball indefinitely (albeit with a very low probability of late exit).

Theorem 2.1 of Sec. 2, which is the main result of this paper, shows that the behaviour of a solution to Eq. (1.1) can, nevertheless, imitate extinction in finite time even when the set F_ε has positive measure. This can occur under the following condition.

Condition 1.1 (Dispersiveness). *There exists a function* $K(\varepsilon) \in \mathbb{N}$ *such that* $\lim_{\varepsilon\to 0} \varepsilon K(\varepsilon) = 0$ *and*

$$\forall z \in \mathbb{G}(\varepsilon, K) \quad |C_{\varepsilon,K(\varepsilon),z}|^{-1} |C_{\varepsilon,K(\varepsilon),z} \cap \{S_\varepsilon > \beta\}| \geq \tau > 0,\ \beta > 0, \quad (1.5)$$

where $\mathbb{G}(\varepsilon, K) = \{z \in \mathbb{Z}^d : |C_{\varepsilon,K,z} \cap G| > 0\}$ *and*

$$C_{\varepsilon,K,z} = \{x : x - \varepsilon K z \in]0, \varepsilon K]^d\},\ K \in \mathbb{N}, K \geq K(\varepsilon). \quad (1.6)$$

The sets $C_{\varepsilon,K,z}$ are later called *εK-blocks*. Condition 1.1 is satisfied with $K = 1$ if S_ε is ε-periodic and not identically zero. When restrictions of S_ε to individual cells $Y_{\varepsilon,z} \equiv C_{\varepsilon,1,z}$ are independent, Eq. (1.5) holds with very high probability for blocks of size $\varepsilon \ln(1/\varepsilon)$ or greater (see Appendix A.3).

The proof of Theorem 2.1 combines the energy method[1] with techniques used to establish deterministic large-volume asymptotic behaviour of the principal eigenvalue (PE) for elliptic operators with random non-negative potential.[5,6] Its approach is related to that of Refs. 3,4 which detects finite-time extinction of solutions of nonlinear parabolic equations through the study of PE's of pertinent Schrödinger operators.

It seems appropriate to show the tools used to prove Theorem 2.1 in a heuristic argument. For the linear problem $\partial_t v = \Delta v - S_\varepsilon^\sigma v$, $v|_{\partial G} = 0$, the solution's norm $\|v(t)\|_2$ decays exponentially, and its half-life is inversely proportional to the Dirichlet PE $\lambda_\varepsilon = \inf_\phi \|\phi\|_2^{-2}(\|\nabla\phi\|_2 + \|S_\varepsilon^\sigma \phi^2\|_1)$. Decay is fast if S_ε satisfies some form of Cond. 1.1 and ε is small. (For example, calculations[5,6] done for $G = [0,1]^d$ and absorption restricted to ε-balls surrounding points of a Poisson cloud with intensity $\mu\varepsilon^{-d}$ show that $\lambda_\varepsilon \asymp \varepsilon^{-2}(\ln \frac{1}{\varepsilon})^{-2/d}$ as $\varepsilon \to 0$.)

For a solution of the non-linear equation $\partial_t u = \Delta u - |u|^{\sigma-2} S^\sigma u$ under the same boundary and initial conditions, the instantaneous rate of decay is proportional to $\Lambda_\varepsilon(t) = \inf_\phi \|\phi\|_2^{-2}(\|\nabla\phi\|_2^2 + \||u|^{\sigma-2} S_\varepsilon^\sigma \phi^2\|_1)$, so the decay of norm $\|u\|_2$ should accelerate as it nears zero. If the non-absorbing set $\{S_\varepsilon = 0\}$ is empty, this results in finite time extinction.[1,3,4] Otherwise, the decay slows down when the solution becomes negligible outside $\{S_\varepsilon = 0\}$.

Theorem 2.1 provides some majorants for an appropriate functional $U(t)$ of the solution (see §2.2). It shows that the evolution of $U(t)$ includes a phase of "attempted extinction" if $\varepsilon K(\varepsilon)$ of Eq. (1.5) is small. Namely, there exists a time interval where the solution's norm admits a majorant that vanishes after a finite time.

When $U(t)$ drops below a small ε-dependent threshold value, the majorant suggestive of extinction in finite time has to be substituted by one that never vanishes. However, the decay of the solution's norm remains fast (exponential if $\gamma = p$ or as a negative power of time otherwise). The times necessary to halve its value are of order $\mathcal{O}(\varepsilon^\lambda)$, $\lambda > 0$, so the difference between true and simulated extinction may not be easy to detect numerically.

One more result of this article is Theorem 3.1 which shows that Eq. (1.1) admits homogenization in the simple case when $\gamma = p = 2$ and the absorption term is its only nonlinearity. The homogenized problem, which can be written down explicitly, is one with finite extinction time.

2. Near Extinction of Solution

2.1. *Weak solutions*

Notation. For vectors $x = (x^{(i)}) \in \mathbb{R}^d$, the scalar product is $x \cdot y = \sum_{i=1}^d x^{(i)} y^{(i)}$ and $|x| = (x \cdot x)^{1/2}$. The Lebesgue measure of $A \subset \mathbb{R}^d$ is $|A|$.

The gradient of a scalar function is $\nabla\phi = \left(\partial\phi/\partial x^{(i)}\right)$, and $\partial_t \phi$ is its time derivative.

Notation for monomials similar to nonlinear terms of Eq. (1.1) is

$$u^{\diamond\mathcal{P}} \overset{\text{abbr}}{=} |u|^{\mathcal{P}-1} u,\ u \in \mathbb{R}. \quad (2.1)$$

Obviously, $u^{\diamond\alpha} u^{\diamond k} = |u|^{\alpha+k}$ and $u^{\diamond\alpha} |u|^k = u^{\diamond(\alpha+k)}$. For a smooth function $\partial_t |u|^k = k u^{\diamond k-1} \partial_t u$ and $\nabla |u|^k = k u^{\diamond k-1} \nabla u$ if $\kappa - 1 \geq 0$ or $u \neq 0$.

When misunderstanding is unlikely, notation is abbreviated: $\phi(t)$ may refer to a function $\phi(x,t)$ considered as a function-valued mapping $t \mapsto \phi(x,t)$, $\int \phi$ can substitute $\int_A \phi(x)\,dx$ if the nature of the argument and the domain of integration are clear from the context. The L^p-norm of a function is always $\|\phi\|_p$. Notation of Sobolev spaces is standard.[7,8]

Definition of weak solution. A measurable function $u = u_\varepsilon(x,t)$ on $Q_{T+} = G \times]0, T_+[$ is a weak solution of Eq. (1.1) with initial and boundary conditions of Eq. (1.3) if for each test function $\zeta \in C^\infty([0,T_+]; C_0^\infty(G))$

$$\int_G u^{\diamond(\gamma-1)}(x,T_+)\zeta(x,T_+)dx - \int_G u_0^{\diamond(\gamma-1)}(x)\,\zeta(x,0)dx$$
$$= \int_{Q(T_+)} \left(u^{\diamond(\gamma-1)}\partial_t\zeta - |\nabla u|^{p-2} a_\varepsilon \nabla u \cdot \nabla\zeta - S_\varepsilon^\sigma u^{\diamond(\sigma-1)}\zeta\right) dxdt. \quad (2.2)$$

The weak solutions u_ε considered below are, up to notation of exponents, those of Ref. 1 (see [Ch.2, §2.1, Def. 2.1). Namely, the function $u^{\diamond(1+k/p)}$

belongs to the space $V\left(Q_{T_+}\right)$ for a given value of parameter $k \geq 0$, i.e.,

$$u^{\diamond(1+k/p)} \in L^p\left(0, T_+; W_0^{1,p}(G)\right),$$
$$u \in L^\infty\left(0, T_+; L^{\gamma+k}(G)\right), \quad u \in L^{\sigma+k}\left(Q_{T_+}\right). \tag{2.3}$$

The possibility to choose the parameter $k \geq 0$ depending on the exponents γ, p, and σ proves important in the arguments below.

Some existence and uniqueness theorems. For the case of $\gamma = 2$, the methods of demonstration of existence and uniqueness theorems are classic.[7–10] For the linear equation with $\gamma = p = \sigma = 2$ and a not depending on the solution, the construction of the solution by the Galerkin method and proof of its uniqueness can be found in Ch.7 of Ref. 8.

For $\gamma = 2$ and $p \geq 2$, the existence and uniqueness of solution for Eq. (1.1) follow from known theorems[9,10] on parabolic equations containing monotone operators; the existence of a unique solution for Eq. (1.1) under the assumptions of Sec. 1 is established in Ch. 2 of Ref. 9 (Theorem 1.1 for $p > 2$, Theorem 1.4 and Examples 1.7.1-2 for $p > 1$).

For $\gamma < 2$ and $p > 1$, Eq. (1.1) is a special case of the doubly nonlinear parabolic equation $\partial_t b(u) = \nabla \cdot A(u, \nabla u) + f$, where $b : \mathbb{R} \to \mathbb{R}$ is nondecreasing and continuous (see survey Ref. 2). For $b(u) = u^{\diamond(\gamma-a)}$, $1 < \gamma < 2$, the existence of solution follows from the results of Refs. 11,12.

If for $v \in \mathbb{R}$, $\xi \in \mathbb{R}^d$ both $|A(v,\xi)| \leq c\left(|\xi|^{p-1} + |v|^{1/\gamma-1} + 1\right)$ and $(A(v,\xi) - A(v,\eta)) \cdot (\xi - \eta) \geq c|\xi - \eta|^p$, then there exists a solution with finite energy $\int_0^T \|u(t)\|_p^p dt + \sup_{t\in[0,T]} \|u(t)\|_\gamma^\gamma$. It is unique[12,13] provided that $|A(b(u_1,\xi) - A(b(u_2,\xi)| \leq c|u_1 - u_2|^{1-1/p}\left(|\xi|^{p-1} + |u_1| + |u_2| + 1\right)$.

The energy inequality. The calculations of this article are based on the so-called energy inequality for weak solutions satisfying conditions Eq. (2.3) (see Lemma 2.1 in Ch.2 of Ref. 1 and Lemma 3.1 of Ref. 14):

$$\|u(S)\|_{\gamma+k}^{\gamma+k} - \|u(T)\|_{\gamma+k}^{\gamma+k} \tag{2.4}$$
$$\geq C\textstyle\int_S^T \left(A_* \|\nabla u^{\diamond(1+k/p)}(t)\|_p^p + \|S_\varepsilon u^{\diamond 1+k/\sigma}(t)\|_\sigma^\sigma dt\right).$$

The following proposition is a special case of Lemma 3.1 in Ref. 14, adapted to integrals of $|u|^{\gamma+k}$ with $k > 0$ (its proof is included in Appendix A.1 to facilitate reading).

Lemma 2.1. *Consider a weak solution $u \in V(T_+)$ that corresponds to the initial value $u_0 \in L^{\gamma+k}(G)$. If $\limsup_{t\to 0+} \|u(t)\|_\gamma < \infty$ and $u(t)$ converges to u_0 in measure as $t \searrow 0$, then $u^{\diamond(1+k/p)} \in V\left(Q_{T_+}\right)$, inequality* (2.4) *holds true for $0 \leq S < T < T_+$, and $\|u(S)\|_{L^{\gamma+k}(G)} \leq \|u_0\|_{L^{\gamma+k}(G)}$ for $S \in]0, T_+[$.*

A bound on Rayleigh quotient for absorbing cubes. The following estimate of the L^{p+k}-norm of a function over a cube partly covered by the absorbing set in terms of similarly restricted "instantaneous" diffusion and absorption terms of Eq. (2.4) is used below.

Given a time $t > 0$, an integer $L = L(t)$ is used to partition the space into "large" cubes $H_z = \left\{x : (L\varepsilon K)^{-1}x - z \in]0,1]^d\right\}$ of size $L\varepsilon K$. Each of these cubes is the union of $L(t)^d$ blocks (1.6). If condition (1.5) is satisfied and $S_\varepsilon(x) = \beta$ outside G, then $|H_z \cap \{S_\varepsilon \geq \beta\}| \geq \tau |H_z|$ for $L \geq 1$.

Lemma 2.2. *If condition* (1.5) *is satisfied and $|G \cap \{|\phi| > \Phi\}| \leq \frac{1}{2}\tau (L\varepsilon K)^d$ for some $\Phi > 0$, then*

$$\int_G \left(|\nabla(\phi^{\diamond 1+k/p})|^p + |S_\varepsilon \phi^{\diamond 1+k/\sigma}|^\sigma\right) \geq \frac{c_2 \|\phi\|_{p+k}^{p+k}}{\max\left\{(L\varepsilon K)^p, \beta^{-\sigma}\Phi^{p-\sigma}\right\}}, \tag{2.5}$$

where the constant c_2 is determined by p, k, τ, and d.

Remark 2.1. When S_ε is separated from zero on G, the size of cubes can be selected arbitrarily. For $\Phi < 1$ and $L\varepsilon K \leq \Phi^{1-\sigma/p}$ inequality (2.5) becomes $\|\phi^{1+k/p}\nabla\phi)\|_p^p + \|S_\varepsilon \phi^{\diamond 1+k/\sigma}\|^\sigma \geq C\Phi^{-(p-\sigma)}\|\phi\|_{p+k}^{p+k}$.

The proof of Lemma 2.2 makes use of Lemma A.2.

Proof of Lemma 2.2. By the assumption, the sets $Q_z = H_z$ and $Q_{0,z} = H_z \cap \{S_\varepsilon \geq \beta\} \cap \{|\phi| \leq \Phi\}$ satisfy the inequality $|Q_z|/|Q_{0,z}| \geq 2/\tau > 1$ for each single cube. Thus, Lemma A.2 yields the estimate

$$\begin{aligned} c_2 \int_{H_z} |\phi|^{p+k} &\leq (L\varepsilon K)^p \int_{H_z} |\nabla\phi^{\diamond 1+k/p}|^p + \int_{Q_{0,z}} |\phi|^{p+k} \\ &\leq (L\varepsilon K)^p \int_{H_z} |\nabla\phi^{\diamond 1+k/p}|^p + \int_{Q_{0,z}} \frac{S_\varepsilon^\sigma}{\beta^\sigma}\Phi^{p-\sigma}|\phi|^{k+\sigma}, \end{aligned}$$

which is equivalent to (2.5) with G replaced by any single cube H_z (being trivial if it has no common points with G). Summing these inequalities for $z \in \mathbb{Z}^d$ yields (2.5). □

2.2. *The main result*

Initial phase of decay. Embedding theorems for the Sobolev spaces $W_0^{1,p}(G)$ (see, e.g., § II.2 of Ref. 7) and well-known inequalities for L^p-norms imply that

$$\|\nabla v_p(t)\|_p^p \geq c\|v_p(t)\|_p^p = c\|u(t)\|_{p+k}^{p+k} \geq c\|u(t)\|_{\gamma+k}^p,$$

so Eq. (2.4) results in the well-known "differential inequality"

$$\psi(t) + \Psi \int_0^t \psi^\kappa(s)\, ds \le C \tag{2.6}$$

for $\psi(t) = \|u(t)\|_{\gamma+k}^{\gamma+k}$ with $C = \psi(0)$ and $\kappa = \frac{p+k}{\gamma+k} \ge 1$. This ensures convergence of $\psi(t)$ to zero as $t \to \infty$. (The possible forms of the majorant for $\psi(t)$ are reproduced in Lemma A.1).

Attempted extinction. Below, some functions on G are identified with their trivial extension to all space for the sake of convenience: $u_\varepsilon(x,t) = 0$, $S_\varepsilon(x) = \beta$, $\phi(x) = 0$, etc., for $x \notin G$.

Theorem 2.1. *Assume that Cond. 1.1 is satisfied. Let $u = u_\varepsilon(x,t)$ be a weak solution of boundary problem (1.1)-(1.3) with $u^{\diamond(1+k/p)} \in V(Q_{T_+})$, and consider the function*[a] $U(t) = \|u(t)\|_{\gamma+k}^{\gamma+k}$.

(a) If k and the exponents of Eq. (1.4) *satisfy the inequality*

$$\gamma + k > \frac{(p-\sigma)}{(\gamma-\sigma)} \frac{d}{p} (p-\gamma), \tag{2.7}$$

then for small values of εK there exists a time interval $\Delta = [t_0, t_1]$ on which the solution $u = u_\varepsilon$ of problem (1.1)-(1.3) decays at a rate characteristic of extinction in finite time: for $t \in \Delta$

$$U(t) \le U(t_0)\left(1 - (1-\kappa)\Psi(t-t_0)\right)^{1/(1-\kappa)}, \tag{2.8}$$

where $\kappa = \alpha_* \frac{p+k}{\gamma+k} + (1-\alpha_*)\frac{\sigma+k}{\gamma+k} < 1$, the number Ψ does not depend on ε, $\alpha_* = \left(1 + \frac{p}{d} \cdot \frac{\gamma+k}{p-\sigma}\right)^{-1}$, and

$$t_0 = \sup\{t > 0 : U(t) \le 1\}, \tag{2.9}$$

$$t_1 = \sup\left\{t > t_0 : U(t) > \tfrac{1}{3}\tau(\varepsilon K)^{d/\alpha_*}\right\}. \tag{2.10}$$

The length of interval Δ satisfies the inequality $T = t_1 - t_0 \le \Psi(1-\kappa)$.

(b) If $p > \gamma$, then for $t > t_1$

$$\frac{U(t_1+t)}{U(t_1)} \le \left(1 + (\tilde\kappa - 1)U^{(\tilde\kappa-1)}(t_1)\frac{t}{(\varepsilon K)^p}\right)^{-1/(\tilde\kappa-1)},$$

where the time scale characterizing decay is small for small εK:

$$\frac{U^{(\tilde\kappa-1)}(t_1)}{(\varepsilon K)^p} = 0((\varepsilon K)^{-\mathcal{P}}),\ \mathcal{P} = \frac{p}{1-\alpha_*}\left(\frac{\gamma-\sigma}{p-\sigma} - \alpha_*\right) > 0. \tag{2.11}$$

[a]It plays the part of a clock in the calculations to follow.

If $p = \gamma$, then $\tilde\kappa = 1$ and the above majorant changes to $\exp\{-ct/(\varepsilon K)^p\}$.

2.3. *Proof of Theorem 2.1*

(a) The existence of a time t_0 such that $U(t) \le 1$ for $t \ge t_0$ follows from Lemma 2.1.

We apply the Chebyshev inequality to evaluate the measure of the set where the solution is large: for each $t \ge t_0$,

$$|\{|u(x,t)| \ge \|u\|_{\gamma+k}^{1-\alpha}\}| \le \textstyle\int_G (|u(x,t)|/\|u\|_{\gamma+k}^{1-\alpha})^{\gamma+k} dx = U^\alpha(t) \le 1. \tag{2.12}$$

The value of the free parameter $\alpha \in]0,1[$ will be specified later on.

It follows from condition (1.5) and (2.12) that

$$|H_z|^{-1}|H_z \cap \{S_\varepsilon \ge \beta\} \cap \{|u(t)| \le \|u\|_{\gamma+k}^{1-\alpha}\}| \ge \tfrac{2}{3}\tau$$

if $t \ge t_0$ and

$$|H_z|^{1/d} = L(t)\varepsilon K \ge (3/\tau)^{1/d} U^{\alpha/d}(t). \tag{2.13}$$

As long as condition (2.13) is satisfied, Lemma 2.2 yields a minorant for the integral on the left-hand side of (2.4). Namely, this lemma is applicable to $\phi = u$ with $\Phi = B\beta^{\sigma/(p-\sigma)}\|u\|_{\gamma+k}^{1-\alpha}$, where B is one more parameter to be selected later.

Under the additional condition $(L\varepsilon K)^p \le B^{p-\sigma}\|u(t)\|_{\gamma+k}^{(1-\alpha)(p-\sigma)}$, i.e.,

$$L(t)\varepsilon K \le B^{1-\sigma/p} U(t)^{(1-\alpha)(1-\sigma/p)/(\gamma+k)}, \tag{2.14}$$

inequality (2.5) holds true with

$$\max\left\{(L(t)\varepsilon K)^p, \beta^{-\sigma}\Phi^{p-\sigma}\right\} = B^{p-\sigma}\|u\|_{\gamma+k}^{(p-\sigma)(1-\alpha)}. \tag{2.15}$$

Since $\|u\|_{p+k} \ge c\|u\|_{\gamma+k}$, it follows in this case that

$$\|\nabla u^{\diamond 1+k/p}\|_p^p + \|S_\varepsilon u^{\diamond 1+k/\sigma}\|_\sigma^\sigma \ge c_2\|u\|_{p+k}^{p+k} B^{-(p-\sigma)}\|u\|_{\gamma+k}^{-(1-\alpha)(p-\sigma)}$$
$$\ge \tilde c(B)U^\kappa(t),\ \kappa(\alpha) = \alpha\tfrac{p+k}{\gamma+k} + (1-\alpha)\tfrac{\sigma+k}{\gamma+k} = \tfrac{\sigma+k}{\gamma+k} + \alpha\tfrac{p-\sigma}{\gamma+k} \tag{2.16}$$

When $U \le 1$, the quantity $U^{\kappa(\alpha)}$ is a non-increasing function of α, so the right-hand side is largest for the smallest possible value of α.

For small values of $U(t)$, conditions (2.13) and (2.14) are compatible only if $\alpha/d \ge (1-\alpha)(1-\sigma/p)(\gamma+k)^{-1}$, so the best minorant corresponds to the value of α that satisfies this condition as equality:

$$\alpha_* = \left(1 + \frac{p}{d} \cdot \frac{\gamma+k}{p-\sigma}\right)^{-1},\ p\alpha_* = d\frac{p-\sigma}{\gamma+k}(1-\alpha_*). \tag{2.17}$$

In the differential inequality (2.6) for $\psi(t) = U(t)$ that follows from (2.16), the rate of decay corresponds to extinction of $\psi(t)$:

$$\kappa(\alpha) < 1 \Leftrightarrow \alpha < (\gamma - \sigma)/(p - \sigma). \tag{2.18}$$

The exponent $\kappa(\alpha_*)$ is typical of finite time of extinction for α_* of Eq. (2.17) if the exponent $\gamma + k$ in the initial condition satisfies inequality (2.7) — in this case $\alpha_* < (\gamma - \sigma)/(p - \sigma)$ and

$$0 < 1 - \kappa(\alpha_*) = \frac{\gamma - \sigma}{\gamma + k} - \left(\frac{p - \sigma}{\gamma + k}\right)^2 \left(\frac{p}{d} + \frac{p - \sigma}{\gamma + k}\right)^{-1}.$$

An "optimal-order" minorant for the right-hand side of (2.16) results from the choice of the natural-valued function $L(t)$ as

$$L(t) = \left[(\varepsilon K)^{-1} U(t)^{\alpha_*/d}\right] = \left[(\varepsilon K)^{-1} \|u(t)\|_{\gamma+k}^{(1-\alpha_*)(1-\sigma/p)}\right] \tag{2.19}$$

where $[\cdot]$ is the integer part of a number.

As long as Eq. (2.16) is applicable with $\alpha = \alpha_*$ of (2.17), inequality (2.4) results in the differential inequality (2.6) with $\psi(t) = U(t_0 + t)$, the exponent $\kappa(\alpha_*)$ of (2.8), $C = U(t_0) = 1$, and the coefficient $\Psi = \widehat{c}(B)$.

However, estimate (2.16) applies only as long as $L(t) \geq 1$ in Eqs. (2.13) and (2.19). The phase of accelerating decay ends at time $t_1(\alpha_*)$ defined in Eq. (2.10), when

$$U(t_1) = \|u(t_1)\|_{\gamma+k}^{\gamma+k} = (\varepsilon K)^{d/\alpha_*}. \tag{2.20}$$

After that majorant (2.8) is no longer applicable.

To estimate the duration of the phase of "attempted extinction," note that it starts with $U(t_0) = 1$. Consequently, it cannot last longer than the time when the majorant vanishes (see Lemma A.1):

$$T(\alpha_*) \leq \left(\widehat{c}\beta^\sigma (1 - \kappa)\right)^{-1}.$$

(b) After time t_1, estimate (2.15) changes to

$$\max\left\{(\varepsilon K)^p, \beta^{-\sigma}\Phi^{p-\sigma}\right\} = (\varepsilon K)^p,$$

and (2.16) is replaced by the inequality

$$\begin{aligned}\|\nabla u^{\diamond 1+k/p}\|_p^p + \|S_\varepsilon u^{\diamond 1+k/\sigma}\|_\sigma^\sigma &\geq c_2 (\varepsilon K)^{-p} \|u\|_{p+k}^{p+k} \\ &\geq \widetilde{c}(\varepsilon K)^{-p} U^{\widetilde{\kappa}},\ \widetilde{\kappa} = \tfrac{p+k}{\gamma+k} \geq 1.\end{aligned}$$

If $\gamma < p$, then $\widetilde{\kappa} > 1$, so the corresponding differential inequality (2.6) produces the slow majorant of Eq. (2.11). By Eq. (2.20) the coefficient that accompanies t on the right-hand side of Eq. (2.11) is proportional to $(\varepsilon K)^{-\mathcal{P}}$, and the use of formulae (2.17) and (2.18) shows that $\mathcal{P}$ is a positive number:

$$\mathcal{P} = p - (\widetilde{\kappa} - 1)d/\alpha_* = \frac{p}{1 - \alpha_*}\left(\frac{\gamma - \sigma}{p - \sigma} - \alpha_*\right) > 0,$$

so the typical times for halving the majorant are small even at this phase of slower decay.

The majorant of Lemma A.1 decays exponentially if $p = \gamma$, $\widetilde{\kappa} = 1$. □

Remark 2.2. When the absorption coefficient S_ε is separated from zero, there is no need to restrict the use of the embedding inequality of Lemma 2.2 to cubes of size εK or greater. Hence (see Remark 2.1) the phase of accelerating decay continues until the solution really dies out in finite time.

3. A theorem on homogenization

In the simple case considered here, boundary problem (1.1)-(1.3) admits homogenization, which may be useful, e.g., for finding more accurate bounds on duration of the initial phases of the solution's decay.

Below $\gamma = p = 2$ and $1 < \sigma < 2$, while $a = A(x)$ does not depend on the solution or time and satisfies condition (1.2), so Eq. (1.1) reduces to the quasi-linear equation

$$\partial_t u_\varepsilon = \operatorname{div}(A\nabla u_\varepsilon) - S_\varepsilon^\sigma u_\varepsilon^{\diamond\sigma-1} \tag{3.1}$$

with the only nonlinearity in the absorption term (notation is that of Eq. (2.1)). The initial and boundary conditions are those of Eq. (1.3) with $k = 0$. As before, the existence of the solutions from $L^2(0, T; W_0^1(G)) \cap L^\infty(0, T; L^2(G))$ is assumed as a prerequisite.

For a function $\phi(x)$, notation $\langle\phi\rangle_{\varepsilon,K}(x)$ refers to the piecewise-constant function that equals $|C_{\varepsilon,K,z}|^{-1} \int_{C_{\varepsilon,K,z}} \phi(\xi)d\xi$ on the block $C_{\varepsilon,K,z}$ of Eq. (1.6).

Theorem 3.1. *Assume that the absorption coefficient $S_\varepsilon(x)$ is bounded, and there exists a a constant $\widehat{S}^\sigma$ such that for some $q > d$*

$$\|\langle S_{\varepsilon,K}^\sigma\rangle - \widehat{S}^\sigma\|_{L^q(G)} \leq \nu. \tag{3.2}$$

If the function $W : G \times [0, T]$ satisfies the homogenized equation

$$\partial_t W = \operatorname{div}(A\nabla W) - \widehat{S}^\sigma |W|^{\sigma-2} W \tag{3.3}$$

and the initial and boundary conditions of Eq. (1.2), *then for small* εK *the difference of solutions to* Eq. (3.1) *and* Eq. (3.3) *admits the estimate*

$$\|\nabla(u_\varepsilon - W)\|_{L^2(G\times[0,T])} \le C\left(\nu\|W\|^{\sigma-1}_{2d/(d-2)} + \varepsilon K\|S^\sigma\|_\infty\|W\|^{\sigma-1}_{2d/(d-2)} + (\varepsilon K/\delta)\|W\|_2^{\sigma-1} + \delta^{\sigma-1}\|\nabla W\|_2^{\sigma-1}\right),$$

where $\delta \in]0,1[$ *is a free parameter.*

Condition (3.2) holds true for blocks of size $K(\varepsilon) = O(\ln(1/\varepsilon))$ when S_ε is generated by a "random chessboard" structure with independent cells (see Lemma A.3). Convergence of u_ε to W follows from the inequality of the theorem if it is possible to choose $K = K(\varepsilon)$, $\delta = \delta(\varepsilon)$, and $\nu = \nu(\varepsilon)$ so that $K \to \infty$, $\nu(\varepsilon) \to 0$, $\varepsilon K/\delta \to 0$, and $\delta \to 0$ as $\varepsilon \to 0$.

Proof of Theorem 3.1. Below notation is abbreviated to $u = u_\varepsilon$, $S = S_\varepsilon$, and $\langle\phi\rangle = \langle\phi\rangle_{\varepsilon,K}$. The calculations deal with the case of $d > 2$.

The difference $V = u - W$ vanishes at $t = 0$ and satisfies the equation (in notation of Eq. (2.1))

$$\partial_t V - \nabla\cdot A\nabla V + S^\sigma(u^{\Diamond\sigma-1} - W^{\Diamond\sigma-1}) = \sum\nolimits_{k=1}^3 H_k, \qquad (3.4)$$

where

$$H_1 = (S^\sigma - \langle S^\sigma\rangle)(W^{\Diamond\sigma-1} - \langle W^{\Diamond\sigma-1}\rangle),$$
$$H_2 = (S^\sigma - \langle S^\sigma\rangle)\langle W^{\Diamond\sigma-1}\rangle,\ \ H_3 = (\langle S^\sigma\rangle - \widehat{S}^\sigma)W^{\Diamond\sigma-1}.$$

Using a sequence of smooth test functions convergent to $u - W$ in the integral identity, we arrive at the inequality

$$\int_0^T dt \int_G \nabla(u-W)\cdot A\nabla(u-W)\,dx + \int_0^T dt\int_G S^\sigma\left(u^{\Diamond\sigma-1} - W^{\Diamond\sigma-1}\right)(u-W)dx \le \int_0^T I(t)dt, \quad (3.5)$$

where $I(t) = \sum_{k=1}^3 \int_G (u(x,t) - W(x,t))H_k(x,t)dx$. Both terms on the left-hand side are non-negative.

The integral containing H_3 of Eq. (3.4) is estimated using the Hölder inequality (recall that $\|\phi\|_{2d/(d-2)} \le c\|\nabla\phi\|_2$):

$$\left|\int_G (u-W)H_3dx\right| \le \|u-W\|_2\|\langle S^\sigma\rangle - \widehat{S}^\sigma\|_d\|W\|^{\sigma-1}_{2d/(d-2)}. \qquad (3.6)$$

Since for each block $\int_{C_{\varepsilon,K,z}}(S^\sigma(x,t) - \langle S^\sigma\rangle)\left(\langle u-W\rangle\langle W^{\Diamond\sigma-1}\rangle\right)dx = 0$, it follows from the well-known embedding theorems that

$$\left|\int_G H_2(u-W)dx\right| = \left|\int_G (S^\sigma - \langle S^\sigma\rangle)(u - W - \langle u-W\rangle)\langle W^{\Diamond\sigma-1}\rangle dx\right|$$
$$\le c\varepsilon K\|\nabla(u-W)\|_2\|S^\sigma - \langle S^\sigma\rangle\|_d\|W\|^{\sigma-1}_{2d/(d-2)}. \qquad (3.7)$$

Finally, we evaluate the integral containing H_1 of Eq. (3.4)

$$\int_G (u-W)H_1dx = \int_G (u-W)(S^\sigma - \langle S^\sigma\rangle)(W^{\Diamond\sigma-1} - \langle W^{\Diamond\sigma-1}\rangle)dx.$$

To this end, we exploit smoothness of W and the following elementary inequality with $q = \sigma - 1 < 1$:

$$\forall a,b\in\mathbb{R}\ \left|a^{\Diamond q} - b^{\Diamond q}\right| \le 2\,|a-b|^q,\ 0 < q \le 1. \qquad (3.8)$$

The function $W^{\Diamond\sigma-1}$ may not have appropriately summable gradient. It is convenient to approximate this factor in the integrand on the right hand side of (3.5) by its convolution with a smooth kernel $h \in C_0^\infty(\mathbb{R}^d)$:

$$W_\delta(x) = W_\delta(x,t) \equiv \int_{\mathbb{R}^d} W(x+\delta y)\,h(y)\,dy,$$
$$W_\delta^{\Diamond\sigma-1}(x) = W_\delta^{\Diamond\sigma-1}(x,t) \equiv \int_{\mathbb{R}^d} W^{\Diamond\sigma-1}(x+\delta y)\,h(y)\,dy,$$

where $\delta > 0$, $h(y) \ge 0$, $\int_{\mathbb{R}^d} h(y)\,dy = 1$, and $h(y) = 0$ if $|y| > \eta$.

Inequality (3.8) furnishes the estimate

$$\left|W^{\Diamond\sigma-1}(x+\delta y) - W^{\Diamond\sigma-1}(x)\right| \le 2\,|W(x+\delta y) - W(x)|^{\sigma-1},$$

so an application of the Hölder inequality to integral in y shows that t-a.e. on $[0,T]$ for $Q = 2/(\sigma-1)$

$$\|W_\delta^{\Diamond\sigma-1} - W^{\Diamond\sigma-1}\|_Q^Q \le c\int dx\left(\int |W(x+\delta y) - W(x)|^{\sigma-1}h(y)\,dy\right)^Q$$
$$\le c\int h(y)\,dy\,|W(x+\delta y) - W(x)|^2\,dx.$$

It is well known (see, e.g., [15, §4.6]) that

$$\int_{\mathbb{R}^d} |W(x+\delta y) - W(x)|^2\,dx \le c\delta^2\,|y|^2\,\|\nabla W\|_2^2,$$

so $\|W_\delta^{\Diamond\sigma-1} - W^{\Diamond\sigma-1}\|_Q^Q \le c\delta^2\|\nabla W\|_2^2\int |y|^2\,h(y)\,dy = C\delta^2\,\|\nabla W\|_2^2$ and

$$\|W_\delta^{\Diamond\sigma-1} - W^{\Diamond\sigma-1}\|_Q \le C\delta^{\sigma-1}\,\|\nabla W\|_2^{\sigma-1}. \qquad (3.9)$$

The function $W_\delta^{\Diamond\sigma}$ is smooth. Its gradient admits representation $\nabla W_\delta^{\Diamond\sigma-1}(x) = \frac{1}{\delta}\int W^{\Diamond\sigma-1}(x+\delta y)\,\nabla h(y)\,dy$, and consequently

$$\|\nabla W_\delta^{\Diamond\sigma-1}\|_Q = \left(\int dx\left|\delta^{-1}\int_{\mathbb{R}^d} W^{\Diamond\sigma-1}(x+\delta y)\,\nabla h(y)\,dy\right|^Q\right)^{(\sigma-1)/2}$$
$$\le c\delta^{-1}\left(\int dx\int dy\,|W(x+\delta y)|^2\,|\nabla h(y)|^Q\right)^{(\sigma-1)/2} \le c\delta^{-1}\,\|W\|_2^{\sigma-1}.$$

It follows that

$$\|W_\delta^{\Diamond\sigma-1} - \langle W_\delta^{\Diamond\sigma-1}\rangle\|_Q \le c(\varepsilon K/\delta)\,\|W\|_2^{\sigma-1},$$

and Eq. (3.9) and the triangle inequality lead to the estimate

$$\|W^{\diamond\sigma-1}-\langle W^{\diamond\sigma-1}\rangle\|_Q\le C\left((\varepsilon K/\delta)\|W\|_2^{\sigma-1}+\delta^{\sigma-1}\|\nabla W\|_2^{\sigma-1}\right).$$

The inequality $\|\phi\|_{2d/(d-2)}\le c\|\nabla\phi\|_2$ leads to the conclusion that

$$\begin{aligned}|\textstyle\int_G(u-W)H_1dx| &= |\textstyle\int_G(u-W)(S^\sigma-\langle S^\sigma\rangle)(W^{\diamond\sigma-1}-\langle W^{\diamond\sigma-1}\rangle)dx|\\ &\le c\|\nabla(u-W)\|_2\Big(\|\langle S^\sigma\rangle-\widehat{S}^\sigma\|_d\|W\|_{2d/(d-2)}^{\sigma-1}\\ &\quad+\varepsilon K\|S^\sigma\|_\infty\|W\|_{2d/(d-2)}^{\sigma-1}\\ &\quad+(\varepsilon K/\delta)\|W\|_2^{\sigma-1}+\delta^{\sigma-1}\|\nabla W\|_2^{\sigma-1}\Big).\end{aligned}\tag{3.10}$$

Equations (3.6), (3.7), and (3.10) yield the estimate of the theorem. □

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Appendix A.

A.1. *Proof of Lemma 2.1.*

(a) First, we consider the case $T_+>T>S>0$. We apply integral identity (2.2) to a sequence of admissible test functions for which all its integrals converge to their counterparts for the function

$$Z(x,t)=\mathcal{K}_\varepsilon*1_{]S,T]}(t)\,\mathcal{K}^+_{\varepsilon,\delta}*u_L^{\diamond k+1}(x,t),\tag{A.1}$$

where $u_L=\mathrm{sign}(u)|u|\wedge L$, the parameters $\varepsilon,\delta>0$ are small and $L>0$ large. As usual, $*$ denotes convolution in t. The mollifiers $\mathcal{K}_\varepsilon(t)=\frac{1}{\varepsilon}\mathcal{K}(\frac{1}{\varepsilon}t)$ and $\mathcal{K}^+_{\varepsilon,\delta}=\mathcal{K}_\varepsilon*\frac{1}{\delta}1_{[-\delta,0]}(t)$ are smooth, and for the sake of convenience it is assumed that $\mathcal{K}(t)=\mathcal{K}(-t)$.

For small $\varepsilon,\delta>0$, function (A.1) vanishes for $t=0$ and $t=T_+$, so identity (2.2) results in the equality

$$\mathcal{T}_{\varepsilon,\delta}\equiv\textstyle\int_{G\times[S,T]}u^{\diamond\gamma-1}\partial_tZ=\mathcal{A}_{\varepsilon,\delta}+\mathcal{B}_{\varepsilon,\delta},\tag{A.2}$$

where $\mathcal{A}_{\varepsilon,\delta}=\int_{G\times\mathbb{R}}\nabla Z\cdot a|u_\varepsilon|^{p-1}$ and $\mathcal{S}_{\varepsilon,\delta}=\int_{G\times\mathbb{R}}S^\sigma u^{\diamond\sigma-1}Z$.

The symmetry of $\mathcal{K}$ implies that $\int_{\mathbb{R}}g(s)\,\mathcal{K}_\varepsilon*h(s)\,ds=\int_{\mathbb{R}}\mathcal{K}_\varepsilon*g(t)\,h(t)\,dt$ and $\int_{\mathbb{R}}g(s)\,\mathcal{K}'_\varepsilon*h(s)\,ds=-\int_{\mathbb{R}}h(t)\,\mathcal{K}'_\varepsilon*g(t)\,dt$, so in (A.2)

$$\begin{aligned}\mathcal{T}_{\varepsilon,\delta}&=-\textstyle\int_{G\times[S,T]}\left(\mathcal{K}'_\varepsilon*u^{\diamond\gamma-1}\right)\left(\mathcal{K}^+_{\varepsilon,\delta}*u_L^{\diamond k+1}\right)dxdt,\\ \mathcal{A}_{\varepsilon,\delta}&=\textstyle\int_{G\times[S,T]}\mathcal{K}^+_{\varepsilon,\delta}*\left(|u|^k1_{\{|u|<L\}}\nabla u\right)\cdot\mathcal{K}_\varepsilon*\left(|\nabla u|^{p-2}a\nabla u\right)dxdt,\\ \mathcal{B}_{\varepsilon,\delta}&=\textstyle\int_{G\times[S,T]}\mathcal{K}^+_{\varepsilon,\delta}*\left(u_L^{\diamond k+1}\right)\mathcal{K}_\varepsilon*\left(S_\varepsilon^\sigma u^{\diamond\sigma-1}\right)dxdt.\end{aligned}$$

It is easily seen that

$$\lim_{\delta\searrow0}\lim_{\varepsilon\searrow0}\mathcal{A}_{\varepsilon,\delta}=\mathcal{A}\equiv\textstyle\int_{G\times[S,T]}|u_L|^k1_{\{|u|<L\}}\left(\nabla u\cdot|\nabla u|^{p-2}a\nabla u\right),\tag{A.3}$$

$$\lim_{\delta\searrow0}\lim_{\varepsilon\searrow0}\mathcal{B}_{\varepsilon,\delta}=\mathcal{B}\equiv\textstyle\int_{G\times[S,T]}|u_L|^{k+1}S_\varepsilon^\sigma|u|^{\sigma-1}.\tag{A.4}$$

Integration by parts reduces the term of Eq. (A.2) containing time derivative to

$$\mathcal{T}_{\varepsilon,\delta}=\mathcal{T}_1(t)|_{t=T}^{t=S}+\mathcal{T}_2(S,T),\tag{A.5}$$

where $\mathcal{T}_1(t)=\int_G\left(\mathcal{K}_\varepsilon*u^{\diamond\gamma-1}\right)\left(\mathcal{K}^+_{\varepsilon,\delta}*u_L^{\diamond k+1}\right)$ and $\mathcal{T}_2(S,T)=\int_{G\times]S,T[}\mathcal{K}_\varepsilon*u^{\gamma-1}\partial_t\mathcal{K}^+_{\varepsilon,\delta}*u_L^{\diamond k+1}$.

For an integrable function $\lim_{\varepsilon\searrow0}\mathcal{K}^+_{\varepsilon,\delta}*\phi(t)=\phi_\delta(t)\equiv\frac{1}{\delta}\int_t^{t+\delta}\phi(s)\,ds$ and $\lim_{\delta\searrow0}\phi_\delta(t)=\phi(t)$ a.e. in t. Consequently, the limit of the first term in Eq. (A.5) is

$$\lim_{\delta\searrow0}\lim_{\varepsilon\searrow0}\mathcal{T}_1(t)|_{t=T}^{t=S}=\textstyle\int_Gu^{\gamma-1}(x,t)\,u_L^{\diamond k+1}(x,t)\,dx\,|_{t=T}^{t=S}.\tag{A.6}$$

It is easily seen that $\partial_t\mathcal{K}^+_{\varepsilon,\delta}*\phi=\frac{1}{\delta}\left(\mathcal{K}_\varepsilon*\phi(t+\delta)-\mathcal{K}_\varepsilon*\phi(t)\right)$, so

$$\lim_{\varepsilon\searrow0}\mathcal{T}_2(S,T)=\tfrac{1}{\delta}\textstyle\int_{G\times[S,T]}u^{\gamma-1}\left(u_L^{\diamond k+1}(t+\delta)-u_L^{\diamond k+1}(t)\right)dxdt.\tag{A.7}$$

By analogy with the function $j(u)$ of Ref. 14, define for $k\ge0$ and $u\in\,]-\infty,\infty[$ the nonnegative convex function

$$J_k(u)=\textstyle\int_0^uv^{\diamond\gamma-1}d\left(v^{\diamond k+1}\right)=\frac{k+1}{k+\gamma}|u|^{\gamma+k},\tag{A.8}$$

which satisfies the inequality (cf. Eq.(3.14) of Ref. 13)

$$\begin{aligned}D_L(t,\delta)&=J_k(u_L(t+\delta))-J_k(u_L(t))\\ &\quad-\psi(u(t))\left(u_L^{\diamond k+1}(t+\delta)-u_L^{\diamond k+1}(t)\right)\ge0.\end{aligned}\tag{A.9}$$

Indeed, by the definition

$$D_L(t,\delta)=\textstyle\int_{u_L(t)}^{u_L(t+\delta)}\left(v^{\diamond\gamma-1}-u^{\diamond\gamma-1}(t)\right)(k+1)|v|^kdv.$$

If $u_L(t+h) > u_L(t)$, then necessarily $u(t+h) > u(t)$ and $u(t) < u_L(t+h)$, so the integrand is non-negative. The integral is zero if $u_L(t+h) = u_L(t)$. If $u_L(t+h) < u_L(t)$, then $u(t+h) < u(t)$ and $u_L(t+h) < u(t)$, so the integrand is non-positive, and the integral non-negative.

It follows from Eq. (A.9) that in A.7

$$\lim_{\varepsilon\searrow 0} \mathcal{T}_2 \geq \tfrac{1}{\delta}\textstyle\int_G\int_S^T \left(J_k\left(u_L\left(t+\delta\right)\right) - J_k\left(u\left(t\right)\right)dt\right)dx$$
$$= \textstyle\int_G dx\left(\tfrac{1}{\delta}\int_T^{T+\delta} J_k\left(u_L\left(t\right)\right)dt - \tfrac{1}{\delta}\int_S^{S+\delta} J_k\left(u_L\left(t\right)\right)dt\right).$$

By the Lebesgue-Vitali theorem $\lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\mathcal{T}_2 = J_k\left(u_L\left(T\right)\right) - J_k\left(u_L\left(S\right)\right)$ a.e. in S,T, so by (A.6)

$$\limsup_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\mathcal{T} \leq \textstyle\int_G\left(u^{\diamond\gamma-1}\left(x,t\right)u_L^{\diamond k+1}\left(x,t\right) - J_k\left(u_L\left(x,t\right)\right)\right)dx\,\Big|_{t=T}^{t=S}.$$

Combined with Eqs. (A.3) and (A.4), this latter estimate shows that for $T > S > 0$ and J_k of Eq. (A.8)

$$\textstyle\int_G\left(u^{\gamma-1}u_L^{\diamond k+1}\left(x,t\right) - J_k\left(u_L\left(x,t\right)\right)\right)dx\,\Big|_{t=T}^{t=S} \tag{A.10}$$
$$\geq \textstyle\int_{G\times[S,T]}\left(\left|u\right|^k 1_{\{|u|<L\}}\left(\nabla u\cdot\left|\nabla u\right|^{p-2}a\nabla u\right) + \left|u_L\right|^{k+1}S_\varepsilon^\sigma\left|u\right|^{\sigma-1}\right).$$

(b) On the left-hand side of Eq. (A.10) the integrands have the form $\mathcal{W}(u(x,s),L)$ and $\mathcal{W}(u(x,T)$, where for $v\in\mathbb{R}$

$$\tfrac{\gamma-1}{k+\gamma}\left|v_L\right|^{\gamma+k} \leq \mathcal{W}(v,L) = \left(\left|v\right|^{\gamma-1} - \tfrac{k+1}{k+\gamma}\left|v_L\right|^{\gamma-1}\right)\left|v_L\right|^{k+1} \leq \left|v\right|^{\gamma-1}\left|v_L\right|^{k+1}.$$

We compare now $u_0(x) = \mathcal{W}(u_0(x),L)$ with $U\left(x,S\right) = \mathcal{W}(u(x,S),L)$ for small $S>0$.

By the assumption, $U\left(x,S\right)\to u_0\left(x\right)$ in measure as $S\searrow 0$. Moreover, $\left|U\left(x,S\right) - u_0\left(x\right)\right| \leq W$, where for a fixed $L>0$ and an arbitrary $\mu>0$

$$W = c_1\left(L,\gamma,k\right)\left(\left|u\left(x,S\right)\right|^{\gamma-1} + \left|u_0\left(x\right)\right|^{\gamma-1} + 1\right)1_{\{|U(S)-U_0|>\mu\}}$$
$$+ c_2\left(L,\gamma,k\right)\left(\mu^{\gamma-1} + \mu\left|u\left(x,S\right)\right|^{\gamma-1} + \mu\left|u_0\left(x\right)\right|^{\gamma-1}\right)1_{\{|u(x,S)-u_0(x)|\leq\mu\}}.$$

By the Hölder inequality for each $\mu>0$ and $\widehat{U} = \limsup_{S\searrow 0}\|u(S)\|_\gamma$

$$\limsup_{S\searrow 0} W \leq c_1\limsup_{S\searrow 0}\left(\mathrm{mes}\left\{\left|U\left(S\right) - u_0\right| > \mu\right\}\right)^{1/\gamma}\left(\widehat{U}^{\gamma-1} + \|u_0\|_\gamma^{\gamma-1}\right)$$
$$+ c_2\left(\mu^{\gamma-1} + \mu\left(\widehat{U}^{\gamma-1} + \|u_0\|_\gamma^{\gamma-1}\right)\right),$$

which proves that for all L

$$\textstyle\int_G u_0\left(x\right)dx = \lim_{S\to 0+}\int_G U\left(x,S\right)dx \geq \int_G U\left(x,S\right)dx.$$

If $u_0\in L^{\gamma+k}\left(G\right)$, then $u_0\left(x,L\right)\to\left|u_0\left(x\right)\right|^{\gamma+k}$, and $\left|u_0\left(x\right)\right|^{\gamma+k}$ is a majorant for this family of functions. Hence

$$\lim_{L\to\infty}\textstyle\int_G u_0\left(x\right)dx = \left(1 - \tfrac{k+1}{k+\gamma}\right)\int_G\left|u_{*,L}\left(x\right)\right|^{\gamma+k}\right)dx.$$

By the above, for each $S\in\left]0,T_+\right[$

$$\textstyle\int_G\left|u_L\left(x,S\right)\right|^{\gamma+k}dx \leq \left(1 - \tfrac{k+1}{k+\gamma}\right)^{-1}\int_G U(x,S)dx \leq \int_G\left|u_{*,L}\left(x\right)\right|^{\gamma+k}dx.$$

By the theorem on monotonic convergence $\|u\left(S\right)\|_{\gamma+k} < \infty$, and the existence of an integrable majorant justifies the passages to the limit as $L\to\infty$ in all integrals of Eq. (A.10). This proves the lemma. □

We also cite here the well-known differential inequality that plays an important part in detection of finite time extinction by the energy method (see Ref. 1, §2.1).

Lemma A.1. *If a non-increasing right-continuous function* $\psi\left(t\right)\geq 0$ *satisfies inequality* (2.6) *with* $\kappa>0$ *on an interval* $[0,T[$, *then on this interval*

$$\psi\left(t\right) \leq F\left(t,\Psi,C,\kappa\right),$$

where

$$F\left(t,\Psi,C,\kappa\right) = \begin{cases} C\left(1 - \left(1-\kappa\right)\dfrac{\Psi t}{C^{1-\kappa}}\right)_+^{1/(1-\kappa)}, & \kappa<1, \\ C\exp\left\{-\Psi t\right\}, & \kappa=1, \\ C\left(1 + \left(\kappa-1\right)C^{\kappa-1}\Psi t\right)^{-1/(\kappa-1)}, & \kappa>1. \end{cases}$$

A.2. *An embedding inequality*

The following embedding inequality is an adaptation to $p\neq 2$ of one well known in many forms for $p=2$ (the proof below follows that of Ref. 6).

Lemma A.2. *Consider a bounded convex set* $Q\subset\mathbb{R}^d$, $d\geq 2$, *containing a subset* Q_0 *of positive Lebesgue measure such that* $A^d = |Q|/|Q_0| \geq 1$. *The following inequality holds true with* $c = (2^d-2)/(d-1)$ *for each real function* $u\in W^{1,p}\left(Q\right)$, $p>1$:

$$\left|u\right|_{L^p(Q)}^p \leq cA^{d-1}\left(\mathrm{diam}^p\left(Q\right)\left|\nabla u\right|_{L^p(Q)}^p + \textstyle\int_{Q_0}\left|u^p\right|\right).$$

Proof. It is easily seen that

$$u(x) = |Q_0|^{-1} \int_{Q_0} u(y)dy + |Q_0|^{-1} \int_{Q_0} \int_0^1 (x-y) \cdot \nabla u(\xi_t)\, dtdy$$

for a smooth function u, an arbitrary pair of points $a, y \in Q$, and $\xi_t = \xi_t(x,y) = tx + (1-t)y$. Thus by Hölder's inequality

$$\|u\|^p_{L^p(Q)} \le 2^{p-1} \left(\frac{|Q|}{|Q_0|} \int_{Q_0} \|u\|^p + \frac{D^p}{|Q_0|} \int_0^1 dt \int_Q dx \int_{Q_0} \|\nabla u(\xi_t)\|^p dy \right),$$

where $D = \operatorname{diam}(Q)$. For a fixed value of t, the argument ξ_t is in Q by convexity, so starting with integration in variable $\xi = \xi_t(x,y)$ (with fixed $y \in Q_0$) or $\eta = \xi_t$ (with fixed $x \in Q$) leads to the inequality

$$|Q_0|^{-1} \int_Q dx \int_{Q_0} \|\nabla u(\xi_t)\|^p dy \le U(t) \|\nabla u\|^p_{L^p(Q)},$$

where $U(t) = \min\{t^{-d}, (1-t)^{-d}|Q|/|Q_0|\}$. Integration in t yields the estimate of the lemma[b]. □

A.3. *Inequalities for random chessboard*

Below, we consider the simplest random model of the absorbing medium. The following lemma combines the well-known exponential inequality of S. N. Bernstein (see Ref. 16, §3.4) with the estimate $\#(\mathbb{G}) \le c_1|G|/(\varepsilon K)^d$ for the number of pertinent blocks.

Lemma A.3. (a) *If there exist numbers $\beta, \lambda > 0$ and a family of i.i.d. binary random variables $\chi_z : \Omega \to \{0,1\}$ such that $\mathbf{P}\{\chi_z = 1\} = q > 0$ and $|\{S_\varepsilon \ge \beta\} \cap Y_{\varepsilon,z}| \ge \lambda\chi_z$ for all cells having common points with G, then there exist constants c_1 dependent on the shape of G and $c_2 = c_2(q,\lambda)$ such that for $\tau = \frac{1}{3}q\lambda$*

$$\mathbf{P}\left\{ \min_{z \in \mathbb{G}(\varepsilon,K)} \frac{|C_{\varepsilon,K,z} \cap \{S_\varepsilon < \beta\}|}{|C_{\varepsilon,K,z}|} \le \tau \right\} \le \frac{c_1|G|}{(\varepsilon K)^d} \exp\{-c_2 K^d\}.$$

(b) *If $S_\varepsilon(x,\omega) \in [0, S_+]$ is a bounded measurable random field and the random variables $\widehat{S}_z = |Y_{\varepsilon,z}|^{-1} \int_{Y_{\varepsilon,z}} S_\varepsilon$ are i.i.d., then in condition* (3.2)

$$\mathbf{P}\left\{ \|\langle S_\varepsilon\rangle_{\varepsilon,K} - \mathbf{E}\widehat{S}_0\|_{L^\infty} \ge \nu \right\} \le \frac{c_1|G|}{(\varepsilon K)^d} \exp\{-\widehat{c}\nu^2 K^d\}.$$

[b] With $c(d) = \max_{A \ge 1} \frac{A^{-d+1}}{d-1}\left((A+1)^d - A^d - 1\right)$, which is the constant of the lemma.

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