

Scaling Limit of Symmetric Random Walk in High-Contrast Periodic Environment

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Received: 18 May 2017 / Accepted: 19 September 2017 / Published online: 23 September 2017
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Abstract The paper deals with the asymptotic properties of a symmetric random walk in a high contrast periodic medium in \mathbb{Z}^d , $d \geq 1$. From the existing homogenization results it follows that under diffusive scaling the limit behaviour of this random walk need not be Markovian. The goal of this work is to show that if in addition to the coordinate of the random walk in \mathbb{Z}^d we introduce an extra variable that characterizes the position of the random walk inside the period then the limit dynamics of this two-component process is Markov. We describe the limit process and observe that the components of the limit process are coupled. We also prove the convergence in the path space for the said random walk.

Keywords High-contrast periodic media · Symmetric random walk · Correctors · Multi-component Markov process · Invariance principle

Mathematics Subject Classification 60J05 · 60J25 · 60F05 · 60F17 · 39A70

1 Introduction

We study in this work the scaling limit of a symmetric random walk in \mathbb{Z}^d , $d \geq 1$, under the assumptions that the medium is periodic, elliptic and high-contrast. More precisely, we assume that the transition probabilities of the random walk depend on a small parameter $\varepsilon > 0$ and that they are of order one for some links on the period and of order ε^2 for other links. It is assumed, moreover, that the graph of links associated with transition probabilities of order one forms an unbounded connected set in \mathbb{Z}^d . Denoting this random walk $\widehat{X}(n)$ we study the large time behaviour of the process under the diffusive scaling that is the limit behaviour of

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the process $\widehat{X}_\varepsilon(t) = \varepsilon \widehat{X}(\lfloor t/\varepsilon^2 \rfloor)$, as $\varepsilon \rightarrow 0$. Our aim is to rearrange the process in such a way that the limit dynamics remains Markovian. We show that to this end it suffices to add to the random walk one more component which characterises the position of the random walk inside the period. We also describe the limit behaviour of this extended process and prove the convergence in the path space.

The study of random walks in high-contrast media with the properties described above is motivated by the so-called double porosity models which are of great importance in mechanics. Various phenomena in media with a high-contrast microstructure have been widely studied by the specialists in applied sciences and then since the '90s high-contrast homogenization problems have been attracting the attention of mathematicians. Homogenization problems for partial differential equations describing high-contrast periodic media have been widely investigated in the existing mathematical literature. In the pioneer work [2] a parabolic equation with high-contrast periodic coefficients has been considered. It was shown that the effective equation contains a non-local in time term which represents the memory effect. In the literature on porous media these models are usually called double porosity models. Later on in [1], with the help of two-scale convergence techniques, it was proved that the solutions of the original parabolic equations two-scale converge to a function which depends both on slow and fast variables, and, as a function of fast and slow variables, satisfies a system of local PDEs.

In the case of spectral problems the homogenized spectral problem turns out to be non-linear with respect to the spectral parameter. The convergence of spectra and the structure of the limit operator pencils have been considered in [3, 9] and other works.

A number of works have been devoted to nonlinear double porosity models, see [4, 6] and references therein. In particular, for the evolution nonlinear models the memory effect was also observed.

In the discrete setting homogenization problems for high-contrast equations and Lagrangians were studied in [5]. It was shown in particular that for evolution high-contrast difference equations the two-scale limit of solutions is a function of continuous "slow" variables and discrete "fast" variables.

The appearance of a non-local term in the homogenized equation means that the limit in law of the scaled random walks need not be a Markov process. Our goal is to study the large time behaviour of the random walk $\widehat{X}(n)$. The presence of transition probabilities of order ε^2 leads to essential slowing down the random walk in some parts of the domain that can be treated as traps. Moreover, the exit time from a trap is of order ε^{-2} .

It turns out that in order to keep the Markovity of the limit process one can equip the coordinate process $X(n)$ with an additional variable, $k(\widehat{X}(n))$, that specifies the position of the random walk in the period. Although in the original process $(\widehat{X}(n), k(\widehat{X}(n)))$ the last component is a function of $\widehat{X}(n)$, in the limit process the last component is independent of the other component. The limit process is a two-component continuous time Markov process $\mathcal{X}(t) = (\widehat{\mathcal{X}}(t), k(t))$, its first component $\widehat{\mathcal{X}}(t)$ lives in the space \mathbb{R}^d , while the second component is a jump Markov process $k(t)$ with a finite number of states $k(t) \in \{0, 1, \dots, M\}$. The process $k(t)$ does not depend on $\widehat{\mathcal{X}}(t)$; the intensities $\lambda(k)$ and transition probabilities μ_{kj} , $k \neq j$, $k, j = 0, 1, \dots, M$, of its jumps are expressed in terms of the transition probabilities of the original symmetric random walk. When $k(t) = 0$, the first component $\widehat{\mathcal{X}}(t)$ evolves along the trajectories of a Brownian motion in \mathbb{R}^d , but when $k(t) \neq 0$, then the first component remains still until the second component of the process takes again the value equal to 0. Thus the trajectories of $\widehat{\mathcal{X}}(t)$ coincide with the trajectories of a Brownian motion in \mathbb{R}^d on those time intervals where $k(t) = 0$. As long as $k(t) \neq 0$, then $\widehat{\mathcal{X}}(t)$ does not

move, and only the second component of the process evolves, that is, figuratively speaking, the process lives during this period in the “astral” space $A = \{x_1, \dots, x_M\}$.

We also study the generalization of this model to the case of several fast components. More precisely, we assume that the set of links to which transition probabilities of order one are assigned consists of a finite number of non-intersecting unbounded connected components. In this case we also equip the random walk with an additional variable, however it indicates not only whether the random walk is in the “astral” space or not, but also specifies the “fast” subset to which the random walk belongs. Also we associate to each fast component the corresponding effective covariance matrix. The limit two-component process is Markov, its second coordinate is a Markov jump process with a finite number of states. When the second coordinate indicates the “astral” state, the first one does not move. Otherwise, the first coordinate is a diffusion in \mathbb{R}^d , however its covariance matrix depends on the value of the second coordinate.

Our approach relies on approximation results from [7]. A crucial step here is constructing several periodic correctors which are introduced as solutions of auxiliary difference elliptic equations on the period. The coefficients of the corresponding difference operator on the discrete torus are defined as the transition probabilities of $\widehat{X}(n)$ with $\varepsilon = 0$. Earlier the corrector techniques in the discrete framework have been developed in [8] for proving the homogenization results for uniformly elliptic difference schemes.

We prove the convergence, as $\varepsilon \rightarrow 0$, of semigroups generated by $(\widehat{X}_\varepsilon(t), k(\widehat{X}_\varepsilon(t)))$ and determine the generator of the limit semigroup. This yields the convergence of finite dimensional distributions of $(\widehat{X}_\varepsilon(t), k(\widehat{X}_\varepsilon(t)))$. We then improve this result and show that $(\widehat{X}_\varepsilon(t), k(\widehat{X}_\varepsilon(t)))$ converges in law in the topology of $D[0, \infty)$.

It is interesting to observe that, unlike diffusion models, the high-contrast discrete models are feasible in any dimension including $d = 1$, at the price of admitting not only nearest neighbour interactions. The graph of non-vanishing transition probabilities should be large enough to ensure the existence of unbounded connected component. This is illustrated by an example in Sect. 3.

2 Problem Setup

We consider a symmetric random walk $\widehat{X}(n)$ on \mathbb{Z}^d , $d \geq 1$, with transition probabilities $p(x, y) = \Pr(x \rightarrow y)$, $(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d$:

$$p(x, y) = p(y, x), \quad (x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d; \quad \sum_{y \in \mathbb{Z}^d} p(x, y) = 1 \quad \forall x \in \mathbb{Z}^d. \quad (1)$$

We assume that the random walk satisfies the following properties:

- *Periodicity* The functions $p(x, x + \xi)$ are periodic in x with a period Y for all $\xi \in \mathbb{Z}^d$. In what follows we identify the period Y with the corresponding d -dimensional discrete torus \mathbb{T}^d .
- *Finite range of interactions* There exists $c > 0$ such that

$$p(x, x + \xi) = 0, \quad \text{if } |\xi| > c. \quad (2)$$
- *Irreducibility* The random walk is irreducible in \mathbb{Z}^d .

We denote the transition matrix of the random walk by $P = \{p(x, y), x, y \in \mathbb{Z}^d\}$.

In this paper we consider a family of transition probabilities $P^{(\varepsilon)} = \{p^{(\varepsilon)}(x, y)\}$ that satisfy the properties formulated above and depend on a small parameter $\varepsilon > 0$. These transition

probabilities describe the so-called high-contrast periodic structure of the environment. We suppose that the transition matrix $P^{(\varepsilon)}$ is a small perturbation of a fixed transition matrix P^0 and can be represented as

$$P^{(\varepsilon)} = P^0 + \varepsilon^2 V. \tag{3}$$

In the sequel the upper index (ε) is dropped.

In order to characterize the matrices P^0 and V we divide the periodicity cell into two sets

$$Y = A \cup B; \quad A, B \neq \emptyset, \quad A \cap B = \emptyset, \tag{4}$$

and assume that B is a connected set such that its periodic extension denoted B^\sharp is unbounded and connected. Here the connectedness is understood in terms of the transition matrix P^0 that is two points $x', x'' \in \mathbb{Z}^d$ are connected if there exists a path x^1, \dots, x^L in \mathbb{Z}^d such that $x^1 = x', x^L = x''$ and $p_0(x^j, x^{j+1}) > 0$ for all $j = 1, \dots, L - 1$. We also denote by A^\sharp the periodic extension of A . Then $\mathbb{Z}^d = A^\sharp \cup B^\sharp$.

We impose the following conditions on P^0 and V :

- P^0 satisfies conditions (1)
- $p_0(x, x) = 1$, if $x \in A^\sharp$;
- $p_0(x, y) = 0$, if $x, y \in A^\sharp, x \neq y$;
- $p_0(x, y) = 0$, if $x \in B^\sharp, y \in A^\sharp$;
- $v(x, y) = 0$, if $x, y \in B^\sharp, x \neq y$;
- the elements of matrix V satisfy the relation

$$\sum_{y \in \mathbb{Z}^d} v(x, y) = 0 \quad \forall x \in \mathbb{Z}^d, \tag{5}$$

Notice that, as a consequence of the above conditions, B^\sharp is a maximal connected component and, consequently, P^0 is irreducible on B^\sharp . From the periodicity of V it also follows that

$$v_{max} := \max_{x, y \in \mathbb{Z}^d} |v(x, y)| < \infty.$$

Under these conditions, for the transition probabilities defined in (3), if $p(x, y) \neq 0$, then

- $p(x, y) \asymp 1$, when $x, y \in B^\sharp$ (rapid movement);
- $p(x, x) = 1 + O(\varepsilon^2)$, when $x \in A^\sharp$;
- $p(x, y) \asymp \varepsilon^2$, when $x, y \in A^\sharp, x \neq y$ (slow movement);
- $p(x, y) \asymp \varepsilon^2$, when $x \in B^\sharp, y \in A^\sharp$ (rare exchange between A^\sharp and B^\sharp).

Notice that for $x, y \in B^\sharp$ we have $p_0(x, y) = \Pr(x \rightarrow y)|_{\varepsilon=0} = \lim_{\varepsilon \rightarrow 0} \Pr(x \rightarrow y | \text{no entry to } A^\sharp)$. The above choice of the transition probabilities reflects a significant slowdown of the random walk inside of high-contrast periodic environments. Our goal is to study the large time behavior of this random walk, ε being used as the corresponding scaling factor.

Let $l_0^\infty(\mathbb{Z}^d)$ be the Banach space of bounded functions on \mathbb{Z}^d vanishing at infinity with the norm $\|f\| = \sup_{x \in \mathbb{Z}^d} |f(x)|$.

Denote $\varepsilon\mathbb{Z}^d = \{z : \frac{z}{\varepsilon} \in \mathbb{Z}^d\}$, then $\varepsilon\mathbb{Z}^d = \varepsilon A^\sharp \cup \varepsilon B^\sharp$. In what follows the symbols x and y are used for the variables on \mathbb{Z}^d (fast variables), while the symbols z and w for the variables on $\varepsilon\mathbb{Z}^d$ (slow variables).

We introduce now the rescaled process. Denote by T_ε the transition operator

$$T_\varepsilon f(z) = \sum_{w \in \varepsilon\mathbb{Z}^d} p_\varepsilon(z, w) f(w), \quad f \in l_0^\infty(\varepsilon\mathbb{Z}^d), \tag{6}$$

where $p_\varepsilon(z, w) = p\left(\frac{z}{\varepsilon}, \frac{w}{\varepsilon}\right)$, and $p(x, y)$, $x = \frac{z}{\varepsilon}$, $y = \frac{w}{\varepsilon}$, is defined above in (1)–(3). Then the operator

$$L_\varepsilon = \frac{1}{\varepsilon^2}(T_\varepsilon - I) \tag{7}$$

is the difference generator of the rescaled random walk $\widehat{X}_\varepsilon(t) = \varepsilon \widehat{X}\left(\left[\frac{t}{\varepsilon^2}\right]\right)$ on $\varepsilon\mathbb{Z}^d$ with transition operator T_ε .

The goal of the paper is to describe the limit behavior of the rescaled random walk $\widehat{X}_\varepsilon(t)$, as $\varepsilon \rightarrow 0$, and to construct the limit process.

3 Semigroup Convergence

In this section we equip the random walk $\widehat{X}_\varepsilon(t)$ with an additional component, and, for the extended process, prove the convergence of the corresponding semigroups. Assume that the set A defined in (4) contains $M \in \mathbb{N}$ sites of \mathbb{T}^d : $A = \{x_1, \dots, x_M\}$. For each $k = 1, \dots, M$ we denote by $\{x_k\}^\sharp$ the periodic extension of the point $x_k \in A$, then

$$\varepsilon\mathbb{Z}^d = \varepsilon B^\sharp \cup \varepsilon A^\sharp = \varepsilon B^\sharp \cup \varepsilon\{x_1\}^\sharp \cup \dots \cup \varepsilon\{x_M\}^\sharp. \tag{8}$$

We assign to each $z \in \varepsilon\mathbb{Z}^d$ the index $k(z) \in \{0, 1, \dots, M\}$ depending on the component in decomposition (8) to which z belongs:

$$k(z) = \begin{cases} 0, & \text{if } z \in \varepsilon B^\sharp; \\ j, & \text{if } z \in \varepsilon\{x_j\}^\sharp, \quad j = 1, \dots, M. \end{cases} \tag{9}$$

With this construction in hands we introduce the metric space

$$E_\varepsilon = \left\{ (z, k(z)), z \in \varepsilon\mathbb{Z}^d, k(z) \in \{0, 1, \dots, M\} \right\}, \quad E_\varepsilon \subset \varepsilon\mathbb{Z}^d \times \{0, 1, \dots, M\} \tag{10}$$

with a metric that coincides with the metric in $\varepsilon\mathbb{Z}^d$ for the first component of $(z, k(z)) \in E_\varepsilon$. We denote by $\mathcal{B}(E_\varepsilon)$ the space of bounded functions on E_ε and introduce the transition operator T_ε of the random walk $X_\varepsilon(t) = (\widehat{X}_\varepsilon(t), k(\widehat{X}_\varepsilon(t)))$ on E_ε using the transition operator (6) of the random walk on $\varepsilon\mathbb{Z}^d$:

$$(T_\varepsilon f)(z, k(z)) = \sum_{w \in \varepsilon\mathbb{Z}^d} p_\varepsilon(z, w) f(w, k(w)), \quad f \in \mathcal{B}(E_\varepsilon). \tag{11}$$

Then T_ε is the contraction on $\mathcal{B}(E_\varepsilon)$:

$$\|T_\varepsilon f\|_{\mathcal{B}(E_\varepsilon)} = \sup_{(z, k(z))} |T_\varepsilon f(z, k(z))| \leq \sup_{(z, k(z))} |f(z, k(z))|, \quad f \in \mathcal{B}(E_\varepsilon).$$

Remark 1 Since the point $(z, k(z)) \in E_\varepsilon$ is uniquely defined by its first coordinate $z \in \varepsilon\mathbb{Z}^d$, then we can use $z \in \varepsilon\mathbb{Z}^d$ as a coordinate in E_ε (considering E_ε as a graph of the mapping $k : \varepsilon\mathbb{Z}^d \rightarrow \{0, 1, \dots, M\}$). In particular, for the transition probabilities of the random walk on E_ε we keep the same notations $p_\varepsilon(z, w)$ as in (6).

We proceed to constructing the limit semigroup. We denote $E = \mathbb{R}^d \times \{0, 1, \dots, M\}$, and $C_0(E)$ stands for the Banach space of continuous functions vanishing at infinity. A function $F = F(z, k) \in C_0(E)$ can be represented as a vector function

$$F(z, k) = \{f_k(z) \in C_0(\mathbb{R}^d), k = 0, 1, \dots, M\}.$$

The norm in $C_0(E)$ is given by

$$\|F\|_{C_0(E)} = \max_{k=0,1,\dots,M} \|f_k\|_{C_0(\mathbb{R}^d)}.$$

Consider the operator

$$LF(z, k) = (\Theta, \nabla \nabla f_0(z)) \mathbf{1}_{\{k=0\}} + L_A F(z, k), \tag{12}$$

where $\mathbf{1}_{\{k=0\}}$ is the indicator function, Θ is a positive definite matrix defined below in (62), $(\Theta, \nabla \nabla f_0) = \text{Tr}(\Theta \nabla \nabla f_0)$, and L_A is a generator of a Markov jump process

$$L_A F(z, k) = \lambda(k) \sum_{\substack{j=0 \\ j \neq k}}^M \mu_{kj} (f_j(z) - f_k(z)) \tag{13}$$

with

$$\begin{aligned} \alpha_{0j} &= \frac{1}{|B|} \sum_{x \in B} \sum_{y \in \{x_j\}^\sharp} v(x, y), & \alpha_{j0} &= \sum_{x \in B^\sharp} v(x_j, x), & j &= 1, \dots, M, \\ \alpha_{kj} &= \sum_{y \in \{x_j\}^\sharp} v(x_k, y), & j, k &= 1, \dots, M, & j &\neq k, \end{aligned} \tag{14}$$

$$\lambda(k) = \sum_{\substack{j=0 \\ j \neq k}}^M \alpha_{kj}, \quad \mu_{kj} = \frac{\alpha_{kj}}{\lambda(k)}, \quad j \neq k; \quad \mu_{jj} = 0. \tag{15}$$

Observe that

$$0 < \lambda_0 \leq \min_k \lambda(k) \leq \max_k \lambda(k) \leq \lambda_1 < \infty, \quad \mu_{kj} \geq 0, \quad \sum_{\substack{j=0 \\ j \neq k}}^M \mu_{kj} = 1 \quad \forall k.$$

Remark 2 The parameters α_{jk} , $j, k = 0, 1, \dots, M$, define intensities of the limit Markov jump process on the period Y .

In order to clarify the relation between the random walk and the limit process we consider the following example.

Example Take $d = 1$, $A^\sharp = \{1, 2\}(\text{mod } 3)$ and $B^\sharp = \{0\}(\text{mod } 3)$, so that the period $Y = \{0, 1, 2\}$. Define

$$\begin{aligned} p_0(x, x) &= 1 \text{ if } x \in A^\sharp, \\ p_0(x, x \pm 3) &= p_0(x, x) = \frac{1}{3} \text{ if } x \in B^\sharp, \\ v(x, x \pm 2) &= v(x, x \pm 4) = 1 \text{ and } v(x, x) = -4 \text{ if } x \in B^\sharp, \\ v(x, x - 4) &= v(x, x + 2) = 1 \text{ and } v(x, x) = -2 \text{ if } x = 1(\text{mod } 3), \\ v(x, x + 4) &= v(x, x - 2) = 1 \text{ and } v(x, x) = -2 \text{ if } x = 2(\text{mod } 3). \end{aligned}$$

Then $|B| = 1$; $\alpha_{0j} = \alpha_{j0} = 2$, $j = 1, 2$; $\alpha_{12} = \alpha_{21} = 0$, and the generator of the limit process takes the form (12) with $\Theta = 3$, $\lambda(0) = 4$, $\lambda(1) = \lambda(2) = 2$, $\mu_{01} = \mu_{02} = \frac{1}{2}$, $\mu_{10} = \mu_{20} = 1$, and $\mu_{12} = 0$.

The operator L is defined on the core

$$D = \{(f_0, f_1, \dots, f_M), f_0 \in C_0^\infty(\mathbb{R}^d), f_j \in C_0(\mathbb{R}^d), j = 1, \dots, M\} \subset C_0(E) \tag{16}$$

which is a dense set in $C_0(E)$. One can check that the operator L on $C_0(E)$ satisfies the positive maximum principle, i.e. if $F \in C_0(E)$ and $\max_E F(z, k) = F(z_0, k_0) = f_{k_0}(z_0)$, then $LF(z_0, k_0) \leq 0$. Indeed, from (12) to (13) we obtain

$$LF(z_0, 0) = (\Theta, \nabla \nabla f_0(z_0)) + L_A F(z_0, 0) \leq 0 \text{ in the case } (z_0, k_0) = (z_0, 0),$$

and

$$LF(z_0, k) = L_A F(z_0, k) \leq 0 \text{ in the case } (z_0, k_0) = (z_0, k), k \neq 0.$$

Since L_A is a bounded operator in $C_0(E)$, the operator $\lambda - L$ is invertible for sufficiently large λ . Then by the Hille-Yosida theorem the closure of L is a generator of a strongly continuous, positive, contraction semigroup $T(t)$ on $C_0(E)$, that is a Feller semigroup.

For every $F \in C_0(E)$ we define on E_ε the function $\pi_\varepsilon F$ as follows:

$$(\pi_\varepsilon F)(z, k(z)) = \begin{cases} f_0(z), & \text{if } z \in \varepsilon B^\sharp, k(z) = 0; \\ f_1(z), & \text{if } z \in \varepsilon \{x_1\}^\sharp, k(z) = 1; \\ \dots & \\ f_M(z), & \text{if } z \in \varepsilon \{x_M\}^\sharp, k(z) = M. \end{cases} \tag{17}$$

Let $l_0^\infty(E_\varepsilon)$ be a Banach space of functions on E_ε vanishing as $|z| \rightarrow \infty$ with the norm

$$\|f\|_{l_0^\infty(E_\varepsilon)} = \sup_{(z, k(z)) \in E_\varepsilon} |f(z, k(z))| = \sup_{z \in \varepsilon \mathbb{Z}^d} |f(z, k(z))|. \tag{18}$$

Then π_ε defines a bounded linear mapping $\pi_\varepsilon : C_0(E) \rightarrow l_0^\infty(E_\varepsilon)$:

$$\|\pi_\varepsilon F\|_{l_0^\infty(E_\varepsilon)} = \sup_{(z, k(z)) \in E_\varepsilon} |(\pi_\varepsilon F)(z, k(z))| \leq \|F\|_{C_0(E)}, \quad \sup_\varepsilon \|\pi_\varepsilon\| \leq 1. \tag{19}$$

Theorem 1 *Let $T(t)$ be a strongly continuous, positive, contraction semigroup on $C_0(E)$ with generator L defined by (12)–(15), and T_ε be the linear operator on $l_0^\infty(E_\varepsilon)$ defined by (11).*

Then for every $F \in C_0(E)$

$$T_\varepsilon \left[\frac{t}{\varepsilon^2} \right] \pi_\varepsilon F \rightarrow T(t)F \text{ for all } t \geq 0 \tag{20}$$

as $\varepsilon \rightarrow 0$.

Proof In view of (18) to prove (20) it suffices to show that

$$\begin{aligned} & \|T_\varepsilon \left[\frac{t}{\varepsilon^2} \right] \pi_\varepsilon F - \pi_\varepsilon T(t)F\|_{l_0^\infty(E_\varepsilon)} \\ &= \sup_{z \in \varepsilon \mathbb{Z}^d} \left| T_\varepsilon \left[\frac{t}{\varepsilon^2} \right] \pi_\varepsilon F(z, k(z)) - \pi_\varepsilon T(t)F(z, k(z)) \right| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{aligned} \tag{21}$$

The proof of (21) relies on the following approximation theorem [7, Theorem 6.5, Ch.1]. \square

Theorem [7]. For $n = 1, 2, \dots$, let T_n be a linear contraction on the Banach space \mathcal{L}_n , let ε_n be a positive number, and put $A_n = \varepsilon_n^{-1}(T_n - I)$. Assume that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Let $\{T(t)\}$ be a strongly continuous contraction semigroup on the Banach space \mathcal{L} with generator A , and let D be a core for A . Assume that $\pi_n : \mathcal{L} \rightarrow \mathcal{L}_n$ are bounded linear transformations with $\sup_n \|\pi_n\| < \infty$. Then the following are equivalent:

- a) For each $f \in \mathcal{L}$, $T_n^{\lfloor \frac{t}{\varepsilon n} \rfloor} \pi_n f \rightarrow T(t)f$ for all $t \geq 0$ as $\varepsilon \rightarrow 0$.
- b) For each $f \in D$, there exists $f_n \in \mathcal{L}_n$ for each $n \geq 1$ such that $f_n \rightarrow f$ and $A_n f_n \rightarrow Af$.

According to this theorem the semigroups convergence stated in item a) is equivalent to the statement in item b) which is the subject of the next lemma.

Lemma 2 *Let the operator L be defined by (12)–(15) and the core $D \subset C_0(E)$ of L be defined by (16); assume that a bounded linear transformation $\pi_\varepsilon : C_0(E) \rightarrow l_0^\infty(E_\varepsilon)$ is defined by (17), and $L_\varepsilon = \frac{1}{\varepsilon^2}(T_\varepsilon - I)$. Then for every $F \in D$, there exists $F_\varepsilon \in l_0^\infty(E_\varepsilon)$ such that*

$$\|F_\varepsilon - \pi_\varepsilon F\|_{l_0^\infty(E_\varepsilon)} \rightarrow 0 \tag{22}$$

and

$$\|L_\varepsilon F_\varepsilon - \pi_\varepsilon L F\|_{l_0^\infty(E_\varepsilon)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \tag{23}$$

Proof For any $F = (f_0, f_1, \dots, f_M) \in D$ we consider the following $F_\varepsilon \in l_0^\infty(E_\varepsilon)$

$$F_\varepsilon(z, k(z)) = \begin{cases} f_0(z) + \varepsilon \left(\nabla f_0(z), h\left(\frac{z}{\varepsilon}\right) \right) + \varepsilon^2 \left(\nabla \nabla f_0(z), g\left(\frac{z}{\varepsilon}\right) \right) \\ \quad + \varepsilon^2 \sum_{j=1}^M q_j\left(\frac{z}{\varepsilon}\right) (f_0(z) - f_j(z)), & \text{if } z \in \varepsilon B^\sharp, k(z) = 0, \\ f_1(z), & \text{if } z \in \varepsilon \{x_1\}^\sharp, k(z) = 1, \\ \dots & \\ f_M(z), & \text{if } z \in \varepsilon \{x_M\}^\sharp, k(z) = M. \end{cases} \tag{24}$$

Here $h(y), g(y), q_j(y), j = 1, \dots, M$, are periodic bounded functions defined below. From (17) and (24) it immediately follows that

$$\sup_{z \in \varepsilon \mathbb{Z}^d} |F_\varepsilon(z, k(z)) - \pi_\varepsilon F(z, k(z))| = \|F_\varepsilon - \pi_\varepsilon F\|_{l_0^\infty(E_\varepsilon)} \rightarrow 0$$

as $\varepsilon \rightarrow 0$. Thus convergence (22) is valid. □

In compliance with decomposition (3) for the transition matrix P we introduce the operators:

$$T_\varepsilon = T_\varepsilon^0 + \varepsilon^2 V_\varepsilon, \tag{25}$$

where

$$T_\varepsilon^0 f(z, k(z)) = \sum_{w \in \varepsilon \mathbb{Z}^d} p_0\left(\frac{z}{\varepsilon}, \frac{w}{\varepsilon}\right) f(w, k(w)),$$

$$V_\varepsilon f(z, k(z)) = \sum_{w \in \varepsilon \mathbb{Z}^d} v\left(\frac{z}{\varepsilon}, \frac{w}{\varepsilon}\right) f(w, k(w)).$$

Let us note that due to the structure of the matrix P^0 , the operator T_ε^0 has a block structure: $T_\varepsilon^0 f(z, k(z)) = f(z, k(z))$ for $z \in \varepsilon A^\sharp$, and $T_\varepsilon^0|_{z \in \varepsilon B^\sharp}$ is defined by the transition probabilities of the random walk on the perforated lattice $\varepsilon B^\sharp = \varepsilon \mathbb{Z}^d \setminus \varepsilon A^\sharp$.

According to (25) the operator $L_\varepsilon = \frac{1}{\varepsilon^2}(T_\varepsilon - I)$ can be written as

$$L_\varepsilon = \frac{1}{\varepsilon^2}(T_\varepsilon^0 + \varepsilon^2 V_\varepsilon - I) = L_\varepsilon^0 + V_\varepsilon,$$

where

$$L_\varepsilon^0 = \frac{1}{\varepsilon^2}(T_\varepsilon^0 - I) \tag{26}$$

is the generator of the random walk on the perforated lattice $\varepsilon B^\sharp = \varepsilon \mathbb{Z}^d \setminus \varepsilon A^\sharp$.

To prove that

$$\|L_\varepsilon F_\varepsilon - \pi_\varepsilon LF\|_{l^\infty(E_\varepsilon)} = \sup_{z \in \varepsilon \mathbb{Z}^d} |L_\varepsilon F_\varepsilon(z, k(z)) - \pi_\varepsilon LF(z, k(z))| \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0, \quad (27)$$

we consider separately the case when $z \in \varepsilon B^\sharp$, and $z \in \varepsilon A^\sharp$. Since the second component in E_ε is a function of the first one, in the remaining part of the proof for brevity write $F_\varepsilon(z)$ instead of $F_\varepsilon(z, k(z))$.

Let $z \in \varepsilon B^\sharp$, then the first component of F_ε can be written as a sum

$$F_\varepsilon(z) = F_\varepsilon^P(z) + F_\varepsilon^Q(z), \quad z \in \varepsilon B^\sharp, \quad (28)$$

where

$$F_\varepsilon^P(z) = f_0(z) + \varepsilon \left(\nabla f_0(z), h\left(\frac{z}{\varepsilon}\right) \right) + \varepsilon^2 \left(\nabla \nabla f_0(z), g\left(\frac{z}{\varepsilon}\right) \right), \quad (29)$$

$$F_\varepsilon^Q(z) = \varepsilon^2 \sum_{j=1}^M q_j\left(\frac{z}{\varepsilon}\right) (f_0(z) - f_j(z)). \quad (30)$$

Then

$$L_\varepsilon F_\varepsilon = (L_\varepsilon^0 + V_\varepsilon)F_\varepsilon = L_\varepsilon^0(F_\varepsilon^P + F_\varepsilon^Q) + V_\varepsilon F_\varepsilon = L_\varepsilon^0 F_\varepsilon^P + L_\varepsilon^0 F_\varepsilon^Q + V_\varepsilon F_\varepsilon. \quad (31)$$

Proposition 3 *There exist bounded periodic functions $h(y) = \{h_i(y)\}_{i=1}^d$ and $g(y) = \{g_{im}(y)\}_{i,m=1}^d$ (correctors) and a positive definite matrix $\Theta > 0$, such that*

$$L_\varepsilon^0 F_\varepsilon^P \rightarrow (\Theta, \nabla \nabla f_0), \quad \text{i.e.} \quad \sup_{z \in \varepsilon B^\sharp} |L_\varepsilon^0 F_\varepsilon^P(z) - (\Theta, \nabla \nabla f_0(z))| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \quad (32)$$

where F_ε^P is defined in (29).

The proof of this proposition is based on the corrector techniques, it is given in Appendix.

The last Proposition allows us to pass to the limit in the first term on the right-hand side of (31). We now turn to rearranging the other two terms. Using (26), (30) and (5) we have for $z \in \varepsilon B^\sharp$

$$\begin{aligned} (L_\varepsilon^0 F_\varepsilon^Q + V_\varepsilon F_\varepsilon)(z) &= \sum_{j=1}^M \left((T_\varepsilon^0 - I)q_j\left(\frac{z}{\varepsilon}\right) \right) (f_0(z) - f_j(z)) \\ &\quad + \sum_{j=1}^M \sum_{w \in \varepsilon\{x_j\}^\sharp} v_\varepsilon(z, w) F_\varepsilon(w) - F_\varepsilon(z) \sum_{j=1}^M \sum_{w \in \varepsilon\{x_j\}^\sharp} v_\varepsilon(z, w), \end{aligned}$$

where $v_\varepsilon(z, w) = v\left(\frac{z}{\varepsilon}, \frac{w}{\varepsilon}\right)$. Then it follows from (24), (28)–(30) together with (16) that the last two terms can be rewritten as

$$\begin{aligned} &\sum_{j=1}^M \sum_{w \in \varepsilon\{x_j\}^\sharp} v_\varepsilon(z, w) F_\varepsilon(w) - F_\varepsilon(z) \sum_{j=1}^M \sum_{w \in \varepsilon\{x_j\}^\sharp} v_\varepsilon(z, w) \\ &= \sum_{j=1}^M \sum_{w \in \varepsilon\{x_j\}^\sharp} v_\varepsilon(z, w) (f_j(w) - f_0(z)) + \mathcal{O}(\varepsilon). \end{aligned}$$

Considering the continuity of functions f_j and the fact that $|w - z| \leq c\varepsilon$, we have

$$\begin{aligned} &(L_\varepsilon^0 F_\varepsilon^Q + V_\varepsilon F_\varepsilon)(z) \\ &= \sum_{j=1}^M \left((T_\varepsilon^0 - I)q_j\left(\frac{z}{\varepsilon}\right) \right) (f_0(z) - f_j(z)) + \sum_{j=1}^M \sum_{w \in \varepsilon\{x_j\}^\sharp} v_\varepsilon(z, w)(f_j(z) - f_0(z)) + o(1), \end{aligned} \tag{33}$$

where $o(1)$ tends to 0 as $\varepsilon \rightarrow 0$.

Proposition 4 *There exist bounded periodic functions $q_j(x)$ on B^\sharp and non-negative constants $\alpha_{0j} \geq 0$, $j = 1, \dots, M$, such that*

$$\sup_{z \in \varepsilon B^\sharp} \left| (L_\varepsilon^0 F_\varepsilon^Q + V_\varepsilon F_\varepsilon)(z) - \sum_{j=1}^M \alpha_{0j} (f_j(z) - f_0(z)) \right| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \tag{34}$$

The proof of the proposition is given in Appendix.

Combining (32), (33) and (34) yields

$$\sup_{z \in \varepsilon B^\sharp} |L_\varepsilon F_\varepsilon(z) - \pi_\varepsilon L F(z)| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \tag{35}$$

The next step is to prove that

$$\sup_{z \in \varepsilon A^\sharp} |L_\varepsilon F_\varepsilon(z) - \pi_\varepsilon L F(z)| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \tag{36}$$

Let $z \in \varepsilon\{x_k\}^\sharp \subset \varepsilon A^\sharp$. From (24) and continuity of functions f_k it follows that

$$\begin{aligned} (L_\varepsilon F_\varepsilon)(z) &= (L_\varepsilon^0 + V_\varepsilon)F_\varepsilon(z) = V_\varepsilon F_\varepsilon(z) \\ &= \sum_{\substack{j=1 \\ j \neq k}}^M \sum_{w \in \varepsilon\{x_j\}^\sharp} v_\varepsilon(z, w)(f_j(z) - f_k(z)) + \sum_{w \in \varepsilon B^\sharp} v_\varepsilon(z, w)(f_0(z) - f_k(z)) + o(1) \\ &= \sum_{\substack{j=1 \\ j \neq k}}^M \sum_{y \in \{x_j\}^\sharp} v(x_k, y)(f_j(z) - f_k(z)) \\ &\quad + \sum_{x \in B^\sharp} v(x_k, x)(f_0(z) - f_k(z)) + o(1) \text{ as } \varepsilon \rightarrow 0, \end{aligned} \tag{37}$$

where we have used the fact that $f_k(z') = f_k(z) + o(1)$ when $|z - z'| \rightarrow 0$. Here $x \in Y$ are variables on the periodicity cell, and $v(x_k, x_j)$ are the elements of the matrix V given by (3). Thus if for every $j, k = 1, \dots, M$, $j \neq k$, we set:

$$\alpha_{kj} = \sum_{y \in \{x_j\}^\sharp} v(x_k, y), \quad \alpha_{k0} = \sum_{x \in B^\sharp} v(x_k, x), \quad \lambda(k) = \sum_{\substack{j=0 \\ j \neq k}}^M \alpha_{kj}, \quad \mu_{kj} = \frac{\alpha_{kj}}{\lambda(k)},$$

then relation (37) implies (36).

Finally, (27) is a consequence of (35) and (36), and Lemma 2 is proved. □

It remains to recall that (21) is a straightforward consequence of the above approximation theorem. This completes the proof of Theorem 1. □

Remark 3 In the next section we show that there exists a Markov process $\mathcal{X}(t)$ corresponding to the Feller semigroup $T(t)$. From Theorem 1 one can easily derive the convergence of finite dimensional distributions of the processes $X_\varepsilon(t)$ (random walks on E_ε defined by (11)) to those of $\mathcal{X}(t)$.

4 Invariance Principle, Convergence of the Processes

For the original process $X_\varepsilon(t) = (\widehat{X}_\varepsilon(t), k(\widehat{X}_\varepsilon(t)))$, which is the random walk on E_ε (see (11)), the second component $k(\widehat{X}_\varepsilon(t)) \in \{0, 1, \dots, M\}$ is the function of the first component $\widehat{X}_\varepsilon(t) \in \varepsilon\mathbb{Z}^d$ (see (6)). Thus Markov processes $\widehat{X}_\varepsilon(t)$ and $X_\varepsilon(t)$ are equivalent, i.e. the trajectories of $\{\widehat{X}_\varepsilon(t)\}$ are isomorphic to trajectories of $\{X_\varepsilon(t)\}$. However, the second component of $X_\varepsilon(t)$ plays the crucial role when passing to the limit $\varepsilon \rightarrow 0$. As has been shown in Sect. 3 the limit process $\mathcal{X}(t)$ preserves the Markov property only in the presence of the second component $k(t) \in \{0, 1, \dots, M\}$, and this is an interesting asymptotic property of the processes $X_\varepsilon(t)$. It should be noted that in the process $\mathcal{X}(t)$ the second component is not a function of the first one anymore. This can be observed, in particular, from the structure of the limit generator, see (12)–(13).

In the previous section we justified the convergence of the semigroups, and consequently, the finite dimensional distributions of $X_\varepsilon(t)$. The goal of this section is to prove the existence of the limit process $\mathcal{X}(t)$ in E with sample paths in $D_E[0, \infty)$ and to establish the invariance principle for the processes $X_\varepsilon(t)$. Namely, we show that $X_\varepsilon(t)$ converges in distributions to $\mathcal{X}(t)$ as $\varepsilon \rightarrow 0$ in the Skorokhod topology of $D_E[0, \infty)$.

Theorem 5 *For any initial distribution $\nu \in \mathcal{P}(E)$ there exists a Markov process $\mathcal{X}(t)$ corresponding to the semigroup $T(t) : C_0(E) \rightarrow C_0(E)$ with generator L defined by (12)–(15) and with sample paths in $D_E[0, \infty)$.*

If ν is the limit law of $X_\varepsilon(0)$, then

$$X_\varepsilon(t) \Rightarrow \mathcal{X}(t) \text{ in } D_E[0, \infty) \text{ as } \varepsilon \rightarrow 0. \tag{38}$$

Proof The main idea of the proof is to combine the convergence of the finite dimensional distributions of $X_\varepsilon(t)$ (that is a consequence of Theorem 1, see Remark 3) and the tightness of $X_\varepsilon(t)$ in $D_E[0, \infty)$.

We apply here Theorem 2.12 from [7], Chapter 4. For the reader convenience we formulate it here.

Theorem [7]. Let E, E_1, E_2, \dots be metric spaces with E locally compact and separable. For $n = 1, 2, \dots$ let $\eta_n : E_n \rightarrow E$ be measurable, let $\mu_n(x, \Gamma)$ be a transition function on $E_n \times \mathcal{B}(E_n)$, and suppose $\{Y_n(k), k = 0, 1, 2, \dots\}$ is a Markov chain in E_n corresponding to $\mu_n(x, \Gamma)$. Let $\varepsilon_n > 0$ satisfy $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Define $X_n(t) = \eta_n(Y_n(\lfloor t/\varepsilon_n \rfloor))$,

$$T_n f(x) = \int f(y) \mu_n(x, dy), \quad f \in B(E_n),$$

and $\pi_n : B(E) \rightarrow B(E_n)$ by $\pi_n f = f \circ \eta_n$. Suppose that $\{T(t)\}$ is a Feller semigroup on $C_0(E)$ and that for each $f \in C_0(E)$ and $t \geq 0$

$$\lim_{n \rightarrow \infty} T_n^{\lfloor t/\varepsilon_n \rfloor} f = T(t)f. \tag{39}$$

If $\{X_n(0)\}$ has limiting distribution $\nu \in \mathcal{P}(E)$, then there is a Markov process X corresponding to $\{T(t)\}$ with initial distribution ν and sample paths in $D_E[0, \infty)$, and $X_n \Rightarrow X$.

In our case, $E = \mathbb{R}^d \times \{0, 1, \dots, M\}$, $E_n = E_\varepsilon \subset E$, $\varepsilon = \frac{1}{n}$, and $\eta_n = \eta_\varepsilon : E_\varepsilon \rightarrow E$ is the measurable mapping for every ε , it is embedding of the set E_ε , isomorphic to the lattice $\varepsilon\mathbb{Z}^d$, to E . The Markov chain $Y_n(m)$, $m = 0, 1, \dots$ is the same as the random walk $X_\varepsilon(m) = (\widehat{X}_\varepsilon(m), k(\widehat{X}_\varepsilon(m)))$ on E_ε (see (11)) with the transition function $\mu_n((z, k(z)), (w, k(w))) = p_\varepsilon(z, w)$. The semigroup $T(t)$ on $C_0(E)$ generated by the operator L , see (12)–(15), is the Feller semigroup by the Hille-Yosida theorem as was mentioned in the beginning of Sect. 3. Setting $\varepsilon_n = \frac{1}{n^2}$ in (39) we see that the convergence in (20) ensures the convergence in (39).

Thus, all assumptions of Theorem 2.12 from [7] are fulfilled. Consequently, if we set $X_\varepsilon(t) = Y_n\left(\left[\frac{t}{\varepsilon^2}\right]\right) = \eta_n\left(Y_n\left(\left[\frac{t}{\varepsilon^2}\right]\right)\right)$, then these processes convergence in law in the space $D_E[0, \infty)$. Theorem 5 is completely proved. \square

5 Generalization: Several Fast Components

In the final part of the paper we consider some generalizations of the model studied above. We keep all the assumptions on $p(x, y)$, in particular we assume that these transition probabilities are periodic, have a finite range of interaction and define an irreducible random walk, and that (3) holds. We also keep all the assumptions on $p_0(x, y)$ except for that on the structure of the set B^\sharp . Here we assume that B^\sharp is the union of N , $N > 1$, non-intersecting unbounded periodic sets such that P^0 is invariant and irreducible on each of these sets.

We denote these sets $B_1^\sharp, \dots, B_N^\sharp$ and assume that each B_j^\sharp , $j = 1, \dots, N$, is connected with respect to P^0 and, moreover, is a maximal connected component. Our assumptions on the matrix P^0 now take the form

- P^0 satisfies conditions (1)
- $p_0(x, x) = 1$, if $x \in A^\sharp$;
- $p_0(x, y) = 0$, if $x, y \in A^\sharp$, $x \neq y$;
- $p_0(x, y) = 0$, if $x \in A^\sharp$, $y \in B^\sharp$;
- $p_0(x, y) = 0$, if $x \in B_i^\sharp$, $y \in B_j^\sharp$, $i, j = 1, \dots, N$, $i \neq j$.

As in Sect. 3 we introduce the extended process on $\varepsilon\mathbb{Z}^d$, $\varepsilon \in (0, 1)$. For each $k = 1, \dots, M$ we denote by $\{x_k\}^\sharp$ the periodic extension of the point $x_k \in A$, then

$$\varepsilon\mathbb{Z}^d = \varepsilon B^\sharp \cup \varepsilon A^\sharp = \varepsilon B_1^\sharp \cup \dots \cup \varepsilon B_N^\sharp \cup \varepsilon\{x_1\}^\sharp \cup \dots \cup \varepsilon\{x_M\}^\sharp. \tag{40}$$

We assign to each $z \in \varepsilon\mathbb{Z}^d$ the index $k(z) \in \{1, \dots, N + M\}$ depending on the component in decomposition (40) to which z belongs:

$$k(z) = \begin{cases} j, & \text{if } z \in \varepsilon B_j^\sharp, \quad j = 1, \dots, N; \\ N + j, & \text{if } z \in \varepsilon\{x_j\}^\sharp, \quad j = 1, \dots, M. \end{cases} \tag{41}$$

With this construction in hands we introduce the metric space

$$E_\varepsilon = \left\{ (z, k(z)), \quad z \in \varepsilon\mathbb{Z}^d, \quad k(z) \in \{1, \dots, N + M\} \right\}, \quad E_\varepsilon \subset \varepsilon\mathbb{Z}^d \times \{1, \dots, N + M\} \tag{42}$$

with a metric that coincides with the metric in $\varepsilon\mathbb{Z}^d$ for the first component of $(z, k(z)) \in E_\varepsilon$. As in Sect. 3 we denote by $\mathcal{B}(E_\varepsilon)$ the space of bounded functions on E_ε and introduce the transition operator T_ε of random walk $X_\varepsilon(t) = (\widehat{X}_\varepsilon(t), k(\widehat{X}_\varepsilon(t)))$ on E_ε as follows:

$$(T_\varepsilon f)(z, k(z)) = \sum_{w \in \varepsilon\mathbb{Z}^d} p_\varepsilon(z, w) f(w, k(w)), \quad f \in \mathcal{B}(E_\varepsilon). \tag{43}$$

Then T_ε is the contraction on $\mathcal{B}(E_\varepsilon)$.

To construct the limit semigroup we denote $E = \mathbb{R}^d \times \{1, \dots, N + M\}$, and $C_0(E)$ stands for the Banach space of continuous functions vanishing at infinity. A function $F = F(z, k) \in C_0(E)$ can be represented as a vector function

$$F(z, k) = \{f_k(z) \in C_0(\mathbb{R}^d), k = 1, \dots, N + M\}.$$

We introduce the operator

$$L F(z, k) = \sum_{j=1}^N (\Theta^j, \nabla \nabla f_j(z)) \mathbf{1}_{\{k=j\}} + L_A F(z, k), \tag{44}$$

where $\Theta^1, \dots, \Theta^N$ are positive definite matrices defined below in formula (49), and L_A is a generator of the following Markov jump process

$$L_A F(z, k) = \lambda(k) \sum_{\substack{j=1 \\ j \neq k}}^{N+M} \mu_{kj} (f_j(z) - f_k(z)). \tag{45}$$

Here the parameters $\lambda(k)$ and μ_{kj} are determined as follows: first we define transition intensities

$$\alpha_{kj} = \begin{cases} \frac{1}{|B_k|} \sum_{x \in B_k} \sum_{y \in B_j^\#} v(x, y), & \text{if } k, j = 1, \dots, N, k \neq j; \\ \frac{1}{|B_k|} \sum_{x \in B_k} \sum_{y \in \{x_{j-N}\}^\#} v(x, y), & \text{if } k = 1, \dots, N, j = N + 1, \dots, N + M; \\ \sum_{x \in B_j^\#} v(x_{k-N}, x), & \text{if } k = N + 1, \dots, N + M, j = 1, \dots, N; \\ \sum_{y \in \{x_{j-N}\}^\#} v(x_{k-N}, y), & \text{if } k, j = N + 1, \dots, N + M, k \neq j. \end{cases}$$

and then set for each $k, j = 1, \dots, N + M$

$$\lambda(k) = \sum_{i=1}^{N+M} \alpha_{ki}, \quad \mu_{kj} = \frac{\alpha_{kj}}{\lambda(k)}. \tag{46}$$

The operator L is defined on the core

$$D = \{(f_1, \dots, f_{N+M}), f_j \in C_0^\infty(\mathbb{R}^d) \text{ for } j = 1, \dots, N; f_j \in C_0(\mathbb{R}^d), \text{ for } j = N + 1, \dots, N + M\}, \tag{47}$$

which is a dense set in $C_0(E)$. As in Sect. 3 one can check that the operator L on $C_0(E)$ satisfies the positive maximum principle, and the operator $\lambda - L$ is invertible for sufficiently large λ . Then by the Hille-Yosida theorem the closure of L is a generator of a strongly continuous, positive, contraction semigroup $T(t)$ on $C_0(E)$.

In this framework the mapping $\pi_\varepsilon : C_0(E) \mapsto l_0^\infty(E_\varepsilon)$ is defined as follows:

$$(\pi_\varepsilon F)(z, k(z)) = \begin{cases} f_1(z), & \text{if } z \in \varepsilon B_1^\sharp, k(z) = 1; \\ \dots & \\ f_N(z), & \text{if } z \in \varepsilon B_N^\sharp, k(z) = N; \\ f_{N+1}(z), & \text{if } z \in \varepsilon \{x_1\}^\sharp, k(z) = N + 1; \\ \dots & \\ f_{N+M}(z), & \text{if } z \in \varepsilon \{x_M\}^\sharp, k(z) = N + M. \end{cases} \tag{48}$$

It remains to define matrices Θ^j that appeared in (44). In fact, for each $j = 1, \dots, N$, the matrix Θ^j coincides with the effective diffusion matrix of the random walk on B_j^\sharp with transition matrix P^0 . We denote the restriction of P^0 on B_j by P_j^0 and recall of the definition of the effective diffusion matrix. To this end we consider, for each $j = 1, \dots, N$, the equation

$$\sum_{\xi \in \Lambda_y} p_\xi(y) \left(\xi + (h^j(y + \xi) - h^j(y)) \right) = 0, \quad y \in B_j^\sharp,$$

where Λ_y be a finite set of $\xi \in \mathbb{Z}^d$ such that $P^0(y, y + \xi) \neq 0$. Observe that this equation coincides with Eq. (58). Therefore, in the same way as in the proof of Proposition 3 in Appendix one can show that it has a periodic solution $h^j(y)$ which is unique up to an additive constant. We set

$$\Theta^j = \frac{1}{|B_j|} \sum_{y \in B_j} \sum_{\xi \in \Lambda_y} p_\xi(y) \xi \otimes \left(\frac{1}{2} \xi + h^j(y + \xi) \right). \tag{49}$$

By Proposition 7 matrices $\Theta^j, j = 1, \dots, N$ are positive definite.

Theorem 6 *Let $T(t)$ be a strongly continuous, positive, contraction semigroup on $C_0(E)$ with generator L defined by (44)–(46), (49), and T_ε be the linear operator on $l_0^\infty(E_\varepsilon)$ defined by (43).*

Then for every $F \in C_0(E)$

$$T_\varepsilon \left[\frac{t}{\varepsilon^2} \right] \pi_\varepsilon F \rightarrow T(t)F \text{ for all } t \geq 0$$

as $\varepsilon \rightarrow 0$.

For any initial distribution $\nu \in \mathcal{P}(E)$ there exists a Markov process $\mathcal{X}(t)$ corresponding to the semigroup $T(t) : C_0(E) \rightarrow C_0(E)$ with generator L and sample paths in $D_E[0, \infty)$. Moreover, if the initial distributions $\nu_\varepsilon \in \mathcal{P}(E_\varepsilon)$ of the processes X_ε converge weakly, as $\varepsilon \rightarrow 0$, to $\nu \in \mathcal{P}(E)$, then

$$X_\varepsilon(t) \Rightarrow \mathcal{X}(t) \text{ in } D_E[0, \infty) \text{ as } \varepsilon \rightarrow 0.$$

Proof The proof of this Theorem follows the line of the proof of Theorems 1 and 5. We leave it to the reader. □

The limit process $\mathcal{X}(t)$ can be described in the following way. Its second component is a Markov jump process with $N + M$ states whose intensities and transition probabilities are given in (46). The first component that evolves in \mathbb{R}^d remains still when the second one takes on values in $\{N + 1, N + M\}$, and it shows a diffusive behaviour with the covariance Θ^j when the second component is equal to $j, j = 1, \dots, N$.

Acknowledgements The authors would like to thank the anonymous Referees for very useful remarks.

Appendix: Proofs of the Propositions

Proof of Proposition 4 From (33) we obtain the following system of uncoupled equations for the functions $q_j\left(\frac{z}{\varepsilon}\right)$, $z \in \varepsilon B^\sharp$, and for constants α_{0j} :

$$\begin{aligned} & \left((T_\varepsilon^0 - I)q_j\left(\frac{z}{\varepsilon}\right) \right) (f_0(z) - f_j(z)) + \sum_{w \in \varepsilon\{x_j\}^\sharp} v_\varepsilon(z, w)(f_j(z) - f_0(z)) \\ & = \alpha_{0j}(f_j(z) - f_0(z)), \quad j = 1, \dots, M. \end{aligned}$$

Then, for every $j = 1, \dots, M$, the function $q_j\left(\frac{z}{\varepsilon}\right)$ satisfies the equation

$$(T_\varepsilon^0 - I)q_j\left(\frac{z}{\varepsilon}\right) = \sum_{w \in \varepsilon\{x_j\}^\sharp} v_\varepsilon(z, w) - \alpha_{0j}\mathbf{1}_B\left(\frac{z}{\varepsilon}\right), \quad z \in \varepsilon B^\sharp, \tag{50}$$

which is equivalent to the following equation on B^\sharp :

$$(P^0 - I)q_j(x) = \sum_{y \in \{x_j\}^\sharp} v(x, y) - \alpha_{0j}\mathbf{1}_B(x), \quad x \in B^\sharp, \tag{51}$$

where $\mathbf{1}_B(x)$ is the indicator function of B^\sharp , and $q_j(x)$ is Y -periodic. Using the Fredholm theorem we conclude that Eq. (51) has a unique solution if

$$\sum_{y \in \{x_j\}^\sharp} v(x, y) - \alpha_{0j}\mathbf{1}_B \perp \text{Ker}(P^0 - I)^*. \tag{52}$$

Due to the irreducibility of P^0 on B^\sharp we have $\text{Ker}(P^0 - I)^* = \mathbf{1}_B$. Therefore, the orthogonality condition in (52) implies the following unique choice of constants α_{0j} :

$$\alpha_{0j} = \frac{1}{|B|} \sum_{x \in B} \sum_{y \in \{x_j\}^\sharp} v(x, y) \geq 0, \tag{53}$$

where $|B|$ is the cardinality of the set B . Thus α_{0j} is defined by (53) for every $j = 1, \dots, M$, and Eq. (51) has a unique up to an additive constant solution $q_j(x)$, $x \in B^\sharp$, that is a bounded periodic function on the set B^\sharp . Proposition 4 is proved. \square

Proof of Proposition 3 We say that $y \sim x$, $x, y \in \mathbb{Z}^d$, if $p_0(x, y) \neq 0$. Let Λ_x be a finite set of $\xi \in \mathbb{Z}^d$ such that $x + \xi \sim x$. From now on we use the notation

$$p_0(x, x + \xi) = p_\xi(x) \text{ for all } x, \xi \in \mathbb{Z}^d, \text{ such that } x \sim x + \xi.$$

Then

$$\sum_{\xi \in \Lambda_x} p_\xi(x) = 1,$$

and

$$(T_\varepsilon^0 f)(z) = \sum_{\xi \in \Lambda_{\frac{z}{\varepsilon}}} p_\xi\left(\frac{z}{\varepsilon}\right) f(z + \varepsilon\xi), \quad z \in \varepsilon B^\sharp. \tag{54}$$

Using (29) we get for all $z \in \varepsilon B^\sharp$:

$$L_\varepsilon^0 F_\varepsilon^P(z) = \frac{1}{\varepsilon^2} (T_\varepsilon^0 - I) \left(f_0(z) + \varepsilon \left(\nabla f_0(z), h\left(\frac{z}{\varepsilon}\right) \right) \right) + (T_\varepsilon^0 - I) \left(\nabla \nabla f_0(z), g\left(\frac{z}{\varepsilon}\right) \right). \tag{55}$$

Then the vector function $h\left(\frac{z}{\varepsilon}\right)$ should satisfy the relation

$$\frac{1}{\varepsilon^2}(T_\varepsilon^0 - I)\left(f_0(z) + \varepsilon\left(\nabla f_0(z), h\left(\frac{z}{\varepsilon}\right)\right)\right) = O(1). \tag{56}$$

Using (54) we rearrange the left-hand side of (56) as follows:

$$\begin{aligned} & \frac{1}{\varepsilon^2} \sum_{\xi \in \Lambda_{\frac{z}{\varepsilon}}} p_\xi\left(\frac{z}{\varepsilon}\right) (f_0(z + \varepsilon\xi) - f_0(z)) \\ & + \frac{1}{\varepsilon} \sum_{\xi \in \Lambda_{\frac{z}{\varepsilon}}} p_\xi\left(\frac{z}{\varepsilon}\right) \left(\left(\nabla f_0(z + \varepsilon\xi), h\left(\frac{z}{\varepsilon} + \xi\right) \right) - \left(\nabla f_0(z), h\left(\frac{z}{\varepsilon}\right) \right) \right) \\ & = \frac{1}{\varepsilon} \sum_{\xi \in \Lambda_{\frac{z}{\varepsilon}}} p_\xi\left(\frac{z}{\varepsilon}\right) (\nabla f_0(z), \xi) + \frac{1}{\varepsilon} \sum_{\xi \in \Lambda_{\frac{z}{\varepsilon}}} p_\xi\left(\frac{z}{\varepsilon}\right) \left(\nabla f_0(z), h\left(\frac{z}{\varepsilon} + \xi\right) - h\left(\frac{z}{\varepsilon}\right) \right) + O(1) \\ & = \frac{1}{\varepsilon} \left(\nabla f_0(z), \sum_{\xi \in \Lambda_{\frac{z}{\varepsilon}}} p_\xi\left(\frac{z}{\varepsilon}\right) \left(\xi + h\left(\frac{z}{\varepsilon} + \xi\right) - h\left(\frac{z}{\varepsilon}\right) \right) \right) + O(1). \end{aligned} \tag{57}$$

Thus the periodic vector function $h(x)$ should solve the equation

$$(P^0 - I)(l(x) + h(x)) = 0, \quad x \in B^\sharp, \tag{58}$$

where $l(x) = x$ is the linear function. The solvability condition for Eq. (58) reads

$$((P^0 - I)l, \text{Ker}(P^0 - I)^*) = ((P^0 - I)l, \mathbf{1}_B) = \sum_{x \in B} \sum_{\xi \in \Lambda_x} p_\xi(x)\xi = 0.$$

Since $p_\xi(x) = p_{-\xi}(x + \xi)$, this condition holds true, which implies the existence of the unique, up to an additive constant, periodic solution $h(x)$ of Eq. (58).

We follow the similar reasoning to find an equation for the periodic matrix function $g(x)$, $x \in B^\sharp$. Collecting in (55) all terms of the order $O(1)$ and using relation (58) on the function $h(x)$ we get:

$$\begin{aligned} & \frac{1}{\varepsilon^2} \sum_{\xi \in \Lambda_{\frac{z}{\varepsilon}}} p_\xi\left(\frac{z}{\varepsilon}\right) (f_0(z + \varepsilon\xi) - f_0(z)) \\ & + \frac{1}{\varepsilon} \sum_{\xi \in \Lambda_{\frac{z}{\varepsilon}}} p_\xi\left(\frac{z}{\varepsilon}\right) \left(\left(\nabla f_0(z + \varepsilon\xi), h\left(\frac{z}{\varepsilon} + \xi\right) \right) - \left(\nabla f_0(z), h\left(\frac{z}{\varepsilon}\right) \right) \right) \\ & + \sum_{\xi \in \Lambda_{\frac{z}{\varepsilon}}} p_\xi\left(\frac{z}{\varepsilon}\right) \left(\left(\nabla \nabla f_0(z + \varepsilon\xi), g\left(\frac{z}{\varepsilon} + \xi\right) \right) - \left(\nabla \nabla f_0(z), g\left(\frac{z}{\varepsilon}\right) \right) \right) + O(\varepsilon) \\ & = \frac{1}{\varepsilon} \left(\nabla f_0(z), \sum_{\xi \in \Lambda_{\frac{z}{\varepsilon}}} p_\xi\left(\frac{z}{\varepsilon}\right) \left[\xi + h\left(\frac{z}{\varepsilon} + \xi\right) - h\left(\frac{z}{\varepsilon}\right) \right] \right) \\ & + \left(\nabla \nabla f_0(z), \sum_{\xi \in \Lambda_{\frac{z}{\varepsilon}}} p_\xi\left(\frac{z}{\varepsilon}\right) \left[\frac{1}{2}\xi \otimes \xi + \xi \otimes h\left(\frac{z}{\varepsilon} + \xi\right) + \left(g\left(\frac{z}{\varepsilon} + \xi\right) - g\left(\frac{z}{\varepsilon}\right) \right) \right] \right) + O(\varepsilon) \\ & = \left(\nabla \nabla f_0(z), \sum_{\xi \in \Lambda_{\frac{z}{\varepsilon}}} p_\xi\left(\frac{z}{\varepsilon}\right) \left[\frac{1}{2}\xi \otimes \xi + \xi \otimes h\left(\frac{z}{\varepsilon} + \xi\right) \right] \right) + (P^0 - I)g\left(\frac{z}{\varepsilon}\right) + O(\varepsilon). \end{aligned} \tag{59}$$

Let $\frac{z}{\xi} = y \in B$, and denote by $\Phi(h)$ the following matrix function

$$\Phi(h)(y) = \frac{1}{2} \sum_{\xi \in \Lambda_y} p_\xi(y) \xi \otimes \xi + \sum_{\xi \in \Lambda_y} p_\xi(y) \xi \otimes h(y + \xi), \quad y \in B. \tag{60}$$

In order to ensure the convergence in (32) we should find a matrix Θ and a periodic matrix function $g(y)$ such that

$$\Phi(h)_{km}(y) + (P^0 - I)g_{km}(y) = \Theta_{km}, \tag{61}$$

The solvability condition for (61) reads

$$(-\Phi(h)_{km} + \Theta_{km}, Ker(P^0 - I)^*) = (-\Phi(h)_{km} + \Theta_{km}, \mathbf{1}_B) = 0,$$

thus Θ is uniquely defined as follows:

$$\Theta_{km} = \frac{1}{|B|} \sum_{y \in B} \Phi_{km}(h)(y),$$

and $g(y)$ is a solution of Eq. (61). This solution is uniquely defined up to a constant matrix.

Proposition 7 *The matrix Θ defined by*

$$\Theta = \frac{1}{|B|} \sum_{y \in B} \Phi(h)(y) \quad \text{with} \quad \Phi(h)(y) = \sum_{\xi \in \Lambda_y} p_\xi(y) \xi \otimes \left(\frac{1}{2} \xi + h(y + \xi) \right) \tag{62}$$

is positive definite, i.e. $(\Theta \eta, \eta) > 0 \forall \eta \neq 0$.

Proof Step 1. Here we show that

$$\sum_{\xi \in \Lambda_y} \partial_{-\xi} (a(y) \partial_\xi g(y)) = -2(P^0 - I)g(y), \tag{63}$$

where $\partial_\xi g(y) = g(y + \xi) - g(y)$ for every ξ , and $a(y) = a_{\xi\xi}(y) = p_\xi(y)$ is the diagonal matrix. Using

$$a(y) \partial_\xi g(y) = p_\xi(y)(g(y + \xi) - g(y)) \quad \text{and} \quad p_\xi(y - \xi) = p_{-\xi}(y),$$

we obtain (63):

$$\begin{aligned} \sum_{\xi \in \Lambda_y} \partial_{-\xi} a(y) \partial_\xi g(y) &= \sum_{\xi \in \Lambda_y} (p_\xi(y - \xi)(g(y) - g(y - \xi)) - p_\xi(y)(g(y + \xi) - g(y))) \\ &= \sum_{\xi \in \Lambda_y} (p_{-\xi}(y)(g(y) - g(y - \xi)) - p_\xi(y)(g(y + \xi) - g(y))) \\ &= -2(P^0 - I)g(y). \end{aligned}$$

Step 2. From (58) and (63) it follows that the vector function $l + h$ satisfies the equation

$$\sum_{\xi \in \Lambda_y} \partial_{-\xi} a(y) \partial_\xi (l + h)(y) = 0, \quad y \in B.$$

Consequently, for all $\eta \in \mathbb{R}^d$ we get

$$0 = \left(\left(\sum_{y \in B} h(y) \otimes \sum_{\xi \in \Lambda_y} \partial_{-\xi} a(y) \partial_\xi (l + h)(y) \right) \eta, \eta \right) = \left(\sum_{y \in B} \sum_{\xi \in \Lambda_y} \partial_\xi h(y) \otimes a(y) \partial_\xi (l + h)(y) \eta, \eta \right), \tag{64}$$

where we have denoted

$$\left\{ \sum_{y \in B} \sum_{\xi \in \Lambda_y} \partial_\xi h(y) \otimes a(y) \partial_\xi (l + h)(y) \right\}_{km} = \sum_{y \in B} \sum_{\xi \in \Lambda_y} \partial_\xi h_k(y) a(y) \partial_\xi (l + h)_m(y).$$

Step 3. Let us check that the following quadratic form is positive definite:

$$\left(\sum_{y \in B} \sum_{\xi \in \Lambda_y} \partial_\xi (l + h)(y) \otimes a(y) \partial_\xi (l + h)(y) \eta, \eta \right) > 0 \quad \forall \eta \neq 0. \tag{65}$$

To this end, taking into account the fact that $a(y) = \{p_\xi(y)\}$ is the diagonal matrix, we rearrange the left-hand side of (65) as follows

$$\left(\sum_{y \in B} \sum_{\xi \in \Lambda_y} \partial_\xi (l + h)(y) \otimes a(y) \partial_\xi (l + h)(y) \eta, \eta \right) = \sum_{y \in B} \sum_{\xi \in \Lambda_y} p_\xi(y) \left(\sum_{k=1}^d \partial_\xi (l + h)_k(y) \eta_k \right)^2.$$

Since $l(x) = x$, and h is a periodic function, the expression on the right-hand side is strictly positive for any $\eta \neq 0$.

Subtracting (64) from (65) and using the relation $\partial_\xi l(y) = \xi$ we have

$$\left(\sum_{y \in B} \sum_{\xi \in \Lambda_y} \partial_\xi l(y) \otimes a(y) \partial_\xi (l + h)(y) \eta, \eta \right) = \left(\sum_{y \in B} \sum_{\xi \in \Lambda_y} p_\xi(y) \xi \otimes (\xi + h(y + \xi) - h(y)) \eta, \eta \right) > 0. \tag{66}$$

Observe that

$$\begin{aligned} - \sum_{y \in B} \sum_{\xi \in \Lambda_y} p_\xi(y) \xi \otimes h(y) &= \sum_{y \in B} \sum_{\xi \in \Lambda_y} p_{-\xi}(y + \xi) (-\xi) \otimes h(y) \\ &= \sum_{u \in B} \sum_{\xi \in \Lambda_u} p_\xi(u) \xi \otimes h(u + \xi); \end{aligned}$$

here we have set $u = y + \xi$ and uses the identity $p_\xi(y) = p_{-\xi}(u)$. Finally, the expression in (66) can be written as

$$\begin{aligned} &\left(\sum_{y \in B} \sum_{\xi \in \Lambda_y} p_\xi(y) \xi \otimes (\xi + 2h(y + \xi)) \eta, \eta \right) \\ &= 2 \left(\sum_{y \in B} \sum_{\xi \in \Lambda_y} p_\xi(y) \xi \otimes \left(\frac{1}{2} \xi + h(y + \xi) \right) \eta, \eta \right) \\ &= 2 \left(\sum_{y \in B} \Phi(h)(y) \eta, \eta \right) > 0, \end{aligned}$$

and the positive definiteness of the matrix Θ follows. □

This complete the proof of Proposition 3.

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