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HOMOGENIZATION OF BOUNDARY VALUE PROBLEMS FOR MONOTONE OPERATORS IN PERFORATED DOMAINS WITH RAPIDLY OSCILLATING BOUNDARY CONDITIONS OF FOURIER TYPE

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Dedicated to Vasilii Zhikov on the ocassion of his 70th birthday

We deal with homogenization problem for nonlinear elliptic and parabolic equations in a periodically perforated domain, a nonlinear Fourier boundary conditions being imposed on the perforation border. Under the assumptions that the studied differential equation satisfies monotonicity and 2-growth conditions and that the coefficient of the boundary operator is centered at each level set of unknown function, we show that the problem under consideration admits homogenization and derive the effective model. Bibliography: 24 *titles.*

1 Introduction

This paper addresses the homogenization of the boundary value problem

$$\begin{cases} -\operatorname{div} a(Du_\varepsilon, x/\varepsilon) + \lambda u_\varepsilon = f \quad \text{in } \Omega_\varepsilon, \\ a(Du_\varepsilon, x/\varepsilon)\cdot\nu = 0 \quad \text{on } \partial\Omega, \\ a(Du_\varepsilon, x/\varepsilon)\cdot\nu = g(u_\varepsilon, x/\varepsilon) \quad \text{on } S_\varepsilon, \end{cases} \tag{1.1}$$

where Ω_ε is a bounded periodically perforated domain in $\mathbb{R}^N$ ($N \geqslant 2$) and $\varepsilon > 0$ is a small parameter referred to the perforation period. The boundary of Ω_ε consists of two parts, namely,

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the fixed outer boundary $\partial\Omega$ and the boundary of perforations S_ε. We assume that the domain is not perforated in a small (of order ε) neighborhood of $\partial\Omega$ so that the perforation boundaries S_ε and $\partial\Omega$ are disjoint. The coefficients $a = (a_1, \dots, a_N)$ in the equation and the function g in the boundary condition on S_ε are strongly oscillating (with the period ε) functions. The boundary condition on S_ε includes, as a particular case, the inhomogeneous Neumann boundary condition of the form $a(Du_\varepsilon, x/\varepsilon) \cdot \nu = \alpha(x/\varepsilon)$ and the Fourier one, $a(Du_\varepsilon, x/\varepsilon) \cdot \nu = \beta(u_\varepsilon, x/\varepsilon)u_\varepsilon$. Along with the stationary problem (1.1), we also consider the parabolic problem

$$\begin{cases} \partial_t u_\varepsilon - \operatorname{div} a(Du_\varepsilon, x/\varepsilon) = f \quad \text{in } \Omega_\varepsilon \times \{t > 0\}, \\ a(Du_\varepsilon, x/\varepsilon) \cdot \nu = 0 \quad \text{on } \partial\Omega, \\ a(Du_\varepsilon, x/\varepsilon) \cdot \nu = g(u_\varepsilon, x/\varepsilon) \quad \text{on } S_\varepsilon, \\ u_\varepsilon = \widetilde{u} \quad \text{for } t = 0. \end{cases} \tag{1.2}$$

The linear elliptic equations in perforated domains with the Fourier boundary condition on the boundary of perforations were considered, for example, in [1]–[7]. It was shown that, if the coefficient in the Fourier boundary condition is small (of order ε) or the volume fraction of the holes vanishes at a certain rate as $\varepsilon \to 0$, then the asymptotic behavior of solutions to these equations is described in terms of a homogenized problem with an additional potential. By contrast, if the volume fraction of the holes does not vanish as the period of the structure tends to zero, then the dissipative Fourier boundary condition forces solutions vanish.

In the problem studied in the present paper, the surface measure $|S_\varepsilon|$ tends to infinity as $\varepsilon \to 0$. To compensate this measure grows, we assume that the average of the function $g(u, x/\varepsilon)$ (appearing in the boundary condition on S_ε) over the boundary of each hole is zero for any $u \in \mathbb{R}$.

Previously, linear problems with the same assumptions on the coefficient in the Fourier boundary condition were considered in [8]; related spectral problems were studied in [9, 10]. The corresponding homogenized operator is shown to contain an additional potential, this potential is always negative.

A variational problem closely related to (1.1) for a functional with a bulk energy and a surface term on the perforation boundary was studied in [11] by means of Γ-convergence technique.

In contrast to [11], we do not assume that the problem under consideration can be written in variational form. Instead, we assume the monotonicity of $a(\xi, y)$ and apply here the celebrated two-scale convergence method (cf., for example, [12]–[16]). This allows us to treat boundary value problems that cannot be reduced to the minimization of an energy functional. For instance, such a reduction is not possible in the case of linear function $a(\xi, y)$, $a(\xi, y) = A(y)\xi$, with nonsymmetric matrix A.

Since, in general, the monotonicity assumption on $a(\xi, y)$ does not imply the monotonicity of the problem (1.1) (even for large λ), we are not able to show the uniqueness result for (1.1). Moreover, the existence of a solution of (1.1) holds only for sufficiently large λ (cf. the discussion in [11]), while the parabolic problem (1.2) does have a unique solution under certain assumptions on $a(\xi, y)$ and $g(u, y)$.

The key difficulty in applying the two-scale convergence theory to the homogenization of (1.1) and (1.2) is due to the presence of a highly perturbed surface integral in the weak formulations of the said problems. To pass to the limit in the surface integral, we establish a new result related to the two-scale convergence of traces (cf. Proposition 3.1). Closely related results concerning the two-scale convergence of surface integrals were obtained in [3] and [17].

The main result of this work shows that solutions u_ε of the problem (1.1) converge as $\varepsilon \to 0$ to a solution U_0 of the homogenized problem

$$\begin{cases} \operatorname{div}\, a^*(DU_0, U_0) + b^*(DU_0, U_0) + |Y^*|(f - \lambda U_0) = 0 \quad \text{in } \Omega, \\ a^*(DU_0, U_0) \cdot \nu = g^*(U_0) \cdot \nu \quad \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

The coefficients a^* and b^* are defined in terms of a cell problem (cf. the problem (2.13)) and depend both on the coefficients $a = (a_1, \dots, a_N)$ in the equation in (1.1) and on the function g in the boundary condition on S_ε. It is interesting to observe also that the homogenization of (1.1) leads to the change of the boundary condition on $\partial\Omega$ from the homogeneous Neumann condition to a Fourier type one.

In what concerns the parabolic problem (1.2), we show that solutions u_ε of (1.2) converge as $\varepsilon \to 0$ to a solution U_0 of the homogenized problem

$$\begin{cases} |Y^*|\partial_t U_0 - \operatorname{div}\, a^*(DU_0, U_0) - b^*(DU_0, U_0) = |Y^*| f \quad \text{in } \Omega \times \{t > 0\}, \\ a^*(DU_0, U_0) \cdot \nu = g^*(U_0) \cdot \nu \quad \text{on } \partial\Omega, \\ U_0 = \widetilde{u} \quad \text{for } t = 0. \end{cases} \tag{1.4}$$

The analysis of (1.2) involves the same ideas as that of (1.1) combined with a lower semicontinuity trick already used in the parabolic problems in [18]–[21].

An interesting issue in both parabolic and elliptic frameworks is the uniqueness of a solution of the limit problem. The limit operator, although admits a priory estimates, needs not be monotone even for large values of λ. The main difficulty is due to the fact that the first order term $b^*(Du, u)$ in the limit equation couples the unknown function u and its gradient.

The uniqueness is proved only for small space dimensions and in the case where either $a(\xi, y)$ is linear in ξ or $g(u, y)$ is linear in u. Without these additional assumptions it remains an open problem.

The paper is organized as follows. Section 2 is devoted to problem setup and formulation of the main results. Sections 3–5 deal with the elliptic case. In Section 3, we prove the two-scale convergence result which relies on several technical statements. These technical statements are then justified in Sections 4 and 5. Section 6 considers the parabolic case. Finally, in Section 7, we study the properties of the homogenized problems.

2 Presentation of the Main Results

Let Y be the unit cube $Y = [-1/2, 1/2)^N$ $(N \geqslant 2)$, and let G be an open subset of Y such that $\overline{G} \subset (-1/2, 1/2)^N$, with Lipschitz boundary. We set $Y^* = Y \setminus G$ and $S = \bigcup\limits_{m \in \mathbb{Z}} (\partial G + m)$.

Given a bounded connected open set $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, we consider the perforated domain Ω_ε defined by $\Omega_\varepsilon = \Omega \setminus \bigcup\limits_{m \in I_\varepsilon} (\varepsilon G + m\varepsilon)$, $I_\varepsilon = \{m \in \mathbb{Z}^N ; Y_\varepsilon^{(m)} \subset \Omega\}$, where $Y_\varepsilon^{(m)} = (Y + m)\varepsilon$. We have $\partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon$, where S_ε is the boundary of perforations.

We assume that $a : \mathbb{R}^N \times Y \to \mathbb{R}^N$ and $g : \mathbb{R} \times S \to \mathbb{R}$ satisfy

(i) $a(\xi, y)$ (respectively, $g(u, y)$) is continuous in ξ (respectively, u), i.e., $a \in C(\mathbb{R}^N; L^\infty(Y))$ (respectively, $g \in C(\mathbb{R}; L^\infty(S))$), and is Y-periodic in y,

(ii) there is $\varkappa > 0$ such that

$$(a(\xi, y) - a(\zeta, y)) \cdot (\xi - \zeta) \geqslant \varkappa |\xi - \zeta|^2 \quad \forall\, \xi, \zeta \in \mathbb{R}^N, \tag{2.1}$$

(iii) there are constants $C_1, \ldots, C_8 > 0$ such that

$$-C_1 + C_2|\xi|^2 \leqslant a(\xi, y) \cdot \xi, \quad |a(\xi, y)| \leqslant C_3|\xi| + C_4, \tag{2.2}$$

$$|g(u, y)| \leqslant C_5|u| + C_6, \tag{2.3}$$

$$|g(u, y) - g(v, y)| \leqslant C_7|u - v|, \tag{2.4}$$

$$|g'_u(u, y) - g'_u(v, y)| \leqslant C_8|u - v|(1 + |u| + |v|)^{-1}, \tag{2.5}$$

(iv)

$$\int\limits_{S \cap Y} g(u, y)\, \mathrm{d}\sigma_y = 0 \quad \forall\, u \in \mathbb{R}. \tag{2.6}$$

Let us write (1.1) in an abstract form. For this purpose, we consider the space $X_\varepsilon = W^{1,2}(\Omega_\varepsilon)$ and its dual X_ε^* with respect to the duality pairing $\langle \cdot\,, \cdot \rangle_\varepsilon$ induced by the standard inner product in $L^2(\Omega_\varepsilon)$. Define the operators $\mathcal{A}_\varepsilon$, $\mathcal{G}_\varepsilon : X_\varepsilon \to X_\varepsilon^*$ by

$$\langle \mathcal{A}_\varepsilon(u), v \rangle_\varepsilon = \int\limits_{\Omega_\varepsilon} a(Du, x/\varepsilon) \cdot Dv \mathrm{d}x \quad \forall\, v \in X^\varepsilon (= W^{1,2}(\Omega_\varepsilon))$$

$$\langle \mathcal{G}_\varepsilon(u), v \rangle_\varepsilon = \int\limits_{S_\varepsilon} g(u, x/\varepsilon) v \mathrm{d}\sigma \quad \forall\, v \in X^\varepsilon (= W^{1,2}(\Omega_\varepsilon)). \tag{2.7}$$

In terms of these operators, (1.1) reads

$$\mathcal{A}_\varepsilon(u_\varepsilon) + \lambda u_\varepsilon - \mathcal{G}_\varepsilon(u_\varepsilon) = f.$$

According to assumptions (i)–(iii), the operator $\mathcal{A}_\varepsilon$ is monotone and continuous while $\mathcal{G}_\varepsilon$ is a compact operator. It follows that $\mathcal{F}_\varepsilon(u) = \mathcal{A}_\varepsilon(u) + \lambda u - \mathcal{G}_\varepsilon(u)$ $(\lambda > 0)$ is a bounded continuous and pseudomonotone operator (recall that $\mathcal{F}_\varepsilon :\, X_\varepsilon \to X_\varepsilon^*$ is *pseudomonotone* if $u^{(i)} \to u$ weakly in X_ε and $\limsup\limits_{i \to \infty} \langle \mathcal{F}_\varepsilon(u^{(i)}), u^{(i)} - u \rangle_\varepsilon \leqslant 0$ implies $\langle \mathcal{F}_\varepsilon(u), u - v \rangle_\varepsilon \leqslant \liminf\limits_{i \to \infty} \langle \mathcal{F}_\varepsilon(u^{(i)}), u^{(i)} - v \rangle_\varepsilon$ for all $v \in X_\varepsilon$). Then for any $f \in L^2(\Omega)$ the problem (1.1) has a (possibly not unique) solution $u_\varepsilon \in X_\varepsilon$ when $\varepsilon \leqslant \varepsilon_0$, $\lambda \geqslant \lambda_0$ (where λ_0, $\varepsilon_0 > 0$ are specified in Theorem 2.1 below) by Brezis' theorem (cf., for example, [22, Chapter II]), thanks to the following coercivity result.

Theorem 2.1. *Under assumptions* (i)–(iv)*, there are $\lambda_0, \varepsilon_0 > 0$ such that*

$$\langle \mathcal{A}_\varepsilon u + \lambda u - \mathcal{G}_\varepsilon(u), u \rangle_\varepsilon \geqslant \varkappa_1 \|u\|^2_{X_\varepsilon} - \varkappa_2, \tag{2.8}$$

when $\|u\|_{X_\varepsilon} \geqslant R$, for some $\varkappa_1 > 0$, $\varkappa_2 > 0$ and $R > 0$ independent of $\varepsilon \leqslant \varepsilon_0$, and $\lambda \geqslant \lambda_0$.

Under the above assumptions on the perforated domain Ω_ε, there is a bounded linear extension operator $P_\varepsilon : W^{1,2}(\Omega_\varepsilon) \to W^{1,2}(\Omega)$ ($P_\varepsilon v = v$ in Ω_ε for any $v \in W^{1,2}(\Omega_\varepsilon)$) and

$$\|P_\varepsilon v\|_{W^{1,2}(\Omega)} \leqslant C\|v\|_{W^{1,2}(\Omega_\varepsilon)},$$

$$\|P_\varepsilon v\|_{L^2(\Omega)} \leqslant C\|v\|_{L^2(\Omega_\varepsilon)}$$

with C independent of ε (cf., for example, [23]). We keep the notation u_ε for the solution of (1.1) extended to Ω_ε ($u_\varepsilon = P_\varepsilon u_\varepsilon$) and study the asymptotic behavior of u_ε as $\varepsilon \to 0$.

We formulate the first main result of this paper.

Theorem 2.2. *Assume that conditions* (i)–(iv) *are satisfied and f in* (1.1) *belongs to $L^2(\Omega)$. Let $\lambda_0 > 0$ be as in Theorem* 2.1. *Then for any $\lambda \geqslant \lambda_0$ solutions u_ε of* (1.1) *and their derivatives Du_ε two-scale converge as $\varepsilon \to 0$ (up to extracting a subsequence) to $U_0(x)$ and $DU_0(x) + D_yU_1(x,y)$, where the pair $U_0(x)$, $U_1(x,y)$ is a solution of the two-scale homogenized problem: find $U_0(x) \in W^{1,2}(\Omega)$, $U_1(x,y) \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(Y))$ such that*

$$\begin{aligned}
&\int_\Omega \int_{Y^*} (a(DU_0 + D_yU_1, y) \cdot (D\Phi_0 + D_y\Phi_1)\mathrm{d}y\mathrm{d}x \\
&- \int_\Omega \int_{S\cap Y} (g(U_0,y)\Phi_1(x,y) + g'_u(U_0,y)\Phi_0 U_1(x,y))\mathrm{d}\sigma_y\mathrm{d}x \\
&- \int_\Omega \int_{S\cap Y} D_x(g(U_0,y)\Phi_0)\cdot y\mathrm{d}\sigma_y\mathrm{d}x - \int_\Omega |Y^*|(f - \lambda U_0)\Phi_0\mathrm{d}x = 0
\end{aligned} \tag{2.9}$$

for any $\Phi_0(x) \in W^{1,2}(\Omega)$, $\Phi_1(x,y) \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(Y))$. In particular, u_ε converge weakly in $W^{1,2}(\Omega)$ to a solution U_0 of the homogenized problem (1.3), *where $a^*(\xi,u)$, $b^*(\xi,u)$, and $g^*(u)$ are defined by*

$$a^*(\xi,u) = \int_{Y^*} a(\xi + D_yw, y)\mathrm{d}y, \tag{2.10}$$

$$b^*(\xi,u) = \int_{S\cap Y} g'_u(u,y)w\mathrm{d}\sigma_y, \tag{2.11}$$

$$g^*(u) = \int_{Y^*} g(u,y)y\mathrm{d}\sigma_y \tag{2.12}$$

and $w = w(y;\xi,u)$ is a unique (up to an additive constant) solution of the cell problem

$$\begin{cases} \operatorname{div} a(\xi + D_yw, y) = 0 \quad \text{in } Y^*, \\ a(\xi + D_yw, y)\cdot\nu = g(u,y) \quad \text{on } S\cap Y, \\ w \text{ is } Y\text{-periodic}. \end{cases} \tag{2.13}$$

Remark 2.3. Note that (2.9) defines $U_1(x,y)$ modulo an arbitrary function $\widetilde{U}_1(x,y) \in L^2(\Omega, W^{1,2}_{\mathrm{per}}(Y)$ such that $U_1(x,y) = 0$ for $y \in Y^*$. This is due to the freedom in the particular choice of the extension operators P_ε.

Remark 2.4. The third term in (2.9) is reduced by integrating by parts to the boundary integral

$$\int_\Omega \int_{S\cap Y} D_x(g(U_0,y)\Phi_0)\cdot y\mathrm{d}\sigma_y\mathrm{d}x = \int_{\partial\Omega} \Phi_0 g^*(U_0)\cdot\nu\mathrm{d}\sigma_x,$$

which leads to the boundary condition in (1.3).

Remark 2.5. In the linear case, i.e., when a and g are given by $a(\xi, y) = A(y)\xi$ and $g(u, y) = \alpha(y) + u\beta(y)$, the cell problem (2.13) for w splits into three cell problems for $w^{(1)}$,

$$\begin{cases} \operatorname{div}\left(A(y)(\xi + D_y w^{(1)})\right) = 0 \quad \text{in } Y^*, \\ A(y) D_y w^{(1)} \cdot \nu = -A(y)\xi \cdot \nu \quad \text{on } S \cap Y, \\ w^{(1)} \text{ is } Y\text{-periodic}, \end{cases} \tag{2.14}$$

and $w^{(k)}$ $(k = 2, 3)$,

$$\begin{cases} \operatorname{div}\left(A(y) D_y w^{(k)}\right) = 0 \quad \text{in } Y^*, \\ A(y) D_y w^{(k)} \cdot \nu = \delta_{2k}\beta(y) + \delta_{3k}\alpha(y) \quad \text{on } S \cap Y, \\ w^{(k)} \text{ is } Y\text{-periodic} \end{cases} \tag{2.15}$$

(δ_{ij} is the Kronecker delta) so that $w = w^{(1)} + uw^{(2)} + w^{(3)}$. Then the homogenized equation takes the form

$$\operatorname{div} A^{\text{hom}} DU_0 + B^{\text{hom}} \cdot DU_0 + C^{\text{hom}} U_0 + D^{\text{hom}} + |Y^*|(f - \lambda U_0) = 0,$$

where the homogenized matrix A^{hom} coincides with the classical effective matrix for the Neumann problem in perforated domains,

$$A^{\text{hom}}\xi = \int_{Y^*} A(y)(\xi + D_y w^{(1)})\mathrm{d}y,$$

$$B^{\text{hom}} \cdot \xi = \int_{Y^*} A(y) D_y w^{(2)} \cdot (\xi + D_y w^{(1)})\mathrm{d}y,$$

$$C^{\text{hom}} = \int_{Y^*} A(y) D_y w^{(2)} \cdot D_y w^{(2)}\mathrm{d}y,$$

$$D^{\text{hom}} = \int_{Y^*} A(y) D_y w^{(2)} \cdot D_y w^{(3)}\mathrm{d}y.$$

Note that $B^{\text{hom}} = 0$ in the selfadjoint case (when $A = A^T$).

In the case of the parabolic problem (1.2), we prove that there is a unique solution u_ε and its asymptotic behavior in the leading term is described by the homogenized problem (1.4). Formulating the convergence result, we assume, as above, that u_ε is extended onto the whole domain Ω by means of the extension operator P_ε.

Theorem 2.6. *Assume that conditions* (i)– (iv) *are satisfied. Then, if* $f \in L^2((0,T) \times \Omega)$ *and* $\widetilde{u} \in L^2(\Omega)$, *there is a unique solution of the problem* (1.2) *that converges weakly in* $L^2(0, T; W^{1,2}(\Omega))$ *as* $\varepsilon \to 0$ (*up to extracting a subsequence*) *to a solution* U_0 *of the homogenized problem* (1.4), *where* a^*, b^*, *and* g^* *are defined by* (2.10), (2.11), (2.12), *and* (2.13).

3 Proof of the Convergence Result for the Stationary Problem

From Theorem 2.1 it follows that $\|u_\varepsilon\|_{W^{1,2}(\Omega)} \leqslant C$, where C is independent of ε. Therefore, up to extracting a subsequence,

$$u_\varepsilon \to U_0(x) \text{ two-scale,} \tag{3.1}$$

$$Du_\varepsilon \to DU_0(x) + D_y U_1(x,y) \text{ two-scale.} \tag{3.2}$$

Show that the pair (U_0, U_1) solves (2.9). For this purpose, we chose arbitrary functions $V_0(x) \in C^\infty(\overline{\Omega})$ and $V_1(x,y) \in C^\infty(\overline{\Omega} \times \overline{Y})$, where $V_1(x,y)$ is Y-periodic in y, set $v_\varepsilon = V_0(x) + \varepsilon V_1(x, x/\varepsilon)$, and substitute the test function $w_\varepsilon = u_\varepsilon - v_\varepsilon$ into the weak formulation of (1.1),

$$\int\limits_{\Omega_\varepsilon} (a(Du_\varepsilon, x/\varepsilon) \cdot Dw_\varepsilon + \lambda u_\varepsilon w_\varepsilon) \mathrm{d}x - \int\limits_{S_\varepsilon} g(u_\varepsilon, x/\varepsilon) w_\varepsilon \mathrm{d}\sigma = \int\limits_{\Omega_\varepsilon} f w_\varepsilon \mathrm{d}x. \tag{3.3}$$

By the monotonicity assumption (2.1), from (3.3) we find

$$\begin{aligned}&\int\limits_{\Omega_\varepsilon} (a(Dv_\varepsilon, x/\varepsilon) \cdot D(u_\varepsilon - v_\varepsilon) + \lambda v_\varepsilon (u_\varepsilon - v_\varepsilon)) \mathrm{d}x \\ &\quad - \int\limits_{S_\varepsilon} g(u_\varepsilon, x/\varepsilon)(u_\varepsilon - v_\varepsilon) \mathrm{d}\sigma - \int\limits_{\Omega_\varepsilon} f(u_\varepsilon - v_\varepsilon) \mathrm{d}x \leqslant 0.\end{aligned} \tag{3.4}$$

Since $Dv_\varepsilon = DV_0(x) + D_y V_1(x, x/\varepsilon) + \varepsilon D_x V_1(x, x/\varepsilon)$, it is easy to show, by using (i) and (2.2), that $\chi_\varepsilon a(Dv_\varepsilon, x/\varepsilon) \to \chi(y) a(DV_0(x) + D_y V_1(x,y), y)$ in the strong two-scale sense, where χ_ε and χ are the characteristic functions of Ω_ε and Y^* respectively. This allows us to pass to the limit in the first term on the left-hand side of (3.4):

$$\begin{aligned}&\int\limits_{\Omega_\varepsilon} (a(Dv_\varepsilon, x/\varepsilon) \cdot D(u_\varepsilon - v_\varepsilon) + \lambda v_\varepsilon (u_\varepsilon - v_\varepsilon)) \mathrm{d}x \\ &\to \int\limits_{\Omega} \Bigg(\int\limits_{Y^*} (a(DV_0 + D_y V_1, y) \cdot (DU_0 + D_y U_1 - DV_0 - D_y V_1) + \lambda V_0 (U_0 - V_0)) \mathrm{d}y \Bigg) \mathrm{d}x.\end{aligned} \tag{3.5}$$

The limit transition in the last term on the left-hand side of (3.4) yields

$$\int\limits_{\Omega_\varepsilon} f(u_\varepsilon - v_\varepsilon) \mathrm{d}x \to \int\limits_{\Omega} \Bigg(\int\limits_{Y^*} f(U_0 - V_0) \mathrm{d}y \Bigg) \mathrm{d}x. \tag{3.6}$$

Finally, passing to the limit in the middle term, we get

$$\begin{aligned}\int\limits_{S_\varepsilon} g(u_\varepsilon, x/\varepsilon)(u_\varepsilon - v_\varepsilon) \mathrm{d}\sigma &\to \int\limits_{\Omega} \Bigg(\int\limits_{S\cap Y} g(U_0, y)(D(U_0 - V_0) \cdot y + U_1(x,y) - V_1(x,y)) \mathrm{d}\sigma_y \Bigg) \mathrm{d}x \\ &\quad + \int\limits_{\Omega} \Bigg(\int\limits_{S\cap Y} g'_u(U_0, y)(U_0 - V_0)(DU_0 \cdot y + U_1(x,y)) \mathrm{d}\sigma_y \Bigg) \mathrm{d}x.\end{aligned} \tag{3.7}$$

The most nontrivial point is to obtain (3.7). The proof of (3.7) is presented in Sections 4 and 5 in detail and is based on the following result, which is of an interest itself.

Proposition 3.1. *Assume that* $q(x,y) \in C(\Omega; L^\infty(S))$ *satisfies*

(a) $|q(x,y) - q(x',y)| \leqslant C|x-x'|$ *with* $C>0$ *independent of* $x, x' \in \Omega$ *and* $y \in S$,

(b) $q(x,y)$ *is* Y*-periodic in* $y \in S$,

(c) $\displaystyle\int_{Y\cap S} q(x,y)\mathrm{d}\sigma_y = 0$ *for all* $x \in \Omega$.

Then for any sequence $w_\varepsilon \in W^{1,2}(\Omega)$ *such that*

$$w_\varepsilon(x) \to W_0(x), \quad Dw_\varepsilon(x) \to DW_0(x) + D_yW_1(x,y) \text{ two scale as } \varepsilon \to 0 \tag{3.8}$$

we have

$$\int_{S_\varepsilon} q(x,x/\varepsilon)(w_\varepsilon - \overline{w}_\varepsilon)\mathrm{d}\sigma \to \int_\Omega \int_{Y\cap S} q(x,y)(DW_0 \cdot y + W_1(x,y))\mathrm{d}\sigma_y \mathrm{d}x. \tag{3.9}$$

Hereinafter, $\overline{w}_\varepsilon$ *denotes the piecewise constant function obtained by averaging over* $Y_\varepsilon^{(m)}$:

$$\overline{w}_\varepsilon(x) = \frac{1}{\varepsilon^N}\int_{Y_\varepsilon^{(m)}} w_\varepsilon(y)\mathrm{d}y, \quad x \in Y_\varepsilon^{(m)}. \tag{3.10}$$

Proof of Theorem 2.2. Thus, (3.4)–(3.7) yield

$$\begin{aligned}
&\int_\Omega \left(\int_{Y^*} (a(DV_0 + D_yV_1, y)\cdot(DU_0 + D_yU_1 - DV_0 - D_yV_1) + \lambda V_0(U_0 - V_0))\mathrm{d}y \right)\mathrm{d}x \\
&- \int_\Omega \left(\int_{S\cap Y} g(U_0,y)(D(U_0 - V_0)\cdot y + U_1(x,y) - V_1(x,y))\mathrm{d}\sigma_y \right)\mathrm{d}x \\
&- \int_\Omega \left(\int_{S\cap Y} g'_u(U_0,y)(U_0 - V_0)(DU_0\cdot y + U_1(x,y))\mathrm{d}\sigma_y \right)\mathrm{d}x \\
&- \int_\Omega \left(\int_{Y^*} f(U_0 - V_0)\mathrm{d}y \right)\mathrm{d}x \leqslant 0,
\end{aligned} \tag{3.11}$$

By approximation argument, using (i)–(iv), we see that (3.11) holds for any $V_0 \in W^{1,2}(\Omega)$ and $V_1 \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(Y))$. Choosing $V_0 = U_0 \pm \tau\Phi_0$, $V_1 = U_1 \pm \tau\Phi_1$ ($\tau > 0$), dividing (3.11) by τ, and passing to the limit as $\tau \to 0$, we obtain the two-scale homogenization problem (2.9). ☐

Let us clarify details in the final part of the above proof when passing from smooth V_0 and V_1 to arbitrary functions $V_0 \in W^{1,2}(\Omega)$ and $V_1 \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(Y))$ in (3.11). For the first term on the left-hand side this transition is justified by the Nemytskii theorem (cf., for example, [22, Chapter II]), and it is a trivial task for the last term. The second and third terms, corresponding to the limiting functional $M(U_0, U_1, V_0, V_1)$ in (3.7), require more attention. Let us write

$M(U_0, U_1, V_0, V_1)$ as

$$M(U_0, U_1, V_0, V_1) = \int\limits_{\Omega} (g^*(U_0) \cdot D(U_0 - V_0) + (U_0 - V_0)(g^*)'(U_0) \cdot DU_0)\mathrm{d}x$$
$$+ \int\limits_{\Omega} \int\limits_{Y^*} D_y\Theta(y; U_0) \cdot D_y(U_1(x, y) - V_1(x, y))\mathrm{d}y\mathrm{d}x$$
$$+ \int\limits_{\Omega} \int\limits_{Y^*} (U_0 - V_0) D_y\Theta'_u(y; U_0) \cdot D_y U_1(x, y)\mathrm{d}y\mathrm{d}x, \tag{3.12}$$

where $(g^*)'$ denotes the derivative of g^* and $\Theta(y; u)$ is a solution of the problem

$$\begin{cases} \Delta_y\Theta = 0 \quad \text{in } Y^*, \\ \dfrac{\partial\Theta}{\partial\nu} = g(u, y) \quad \text{on } S \cap Y, \\ \Theta \text{ is } Y\text{-periodic.} \end{cases} \tag{3.13}$$

From assumptions (iii) and (iv) it follows that (3.13) has a unique (modulo an additive constant) solution $\Theta(y; u)$ and Θ depends regularly on the parameter u; more precisely,

$$\|D_y\Theta(\,\cdot\,; u)\|_{L^2(Y^*)} \leqslant C(|u| + 1), \tag{3.14}$$

$$\|D_y\Theta(\,\cdot\,; u) - D_y\Theta(\,\cdot\,; v)\|_{L^2(Y^*)} \leqslant C|u - v|, \tag{3.15}$$

$$\|D_y\Theta'_u(\,\cdot\,; u) - D_y\Theta'_u(\,\cdot\,; v)\|_{L^2(Y^*)} \leqslant C|u - v|(1 + |u| + |v|)^{-1}, \tag{3.16}$$

where C is independent of u and v. All these properties are demonstrated similarly. For example, we show (3.14) by using (2.3), (2.6) and the Poincaré inequality (7.6) in $W^{1,2}_{\mathrm{per}}(Y^*)$ (cf. Section 6),

$$\Big|\int\limits_{Y^*} D_y\Theta \cdot D_y\Theta\mathrm{d}y\Big| = \Bigg|\int\limits_{S\cap Y} g(u, y)\bigg(\Theta - \frac{1}{|Y^*|}\int\limits_{Y^*} \Theta\,\mathrm{d}y\bigg)\,\mathrm{d}y\Bigg| \leqslant C(|u| + 1)\|D_y\Theta\|_{L^2(Y^*)}.$$

The bounds (3.14)–(3.16) in conjunction with assumptions (2.3)–(2.5) imply

Proposition 3.2. *The functional* $M(U_0, U_1, V_0, V_1)$ *defined by* (3.12) (*or, equivalently, by the right-hand side of* (3.7)) *is continuous in the space*

$$W^{1,2}(\Omega) \times L^2(\Omega; W^{1,2}_{\mathrm{per}}(Y^*)) \times W^{1,2}(\Omega) \times L^2(\Omega; W^{1,2}_{\mathrm{per}}(Y^*)).$$

4 Auxiliary Results and Proof of Theorem 2.1

4.1 Some inequalities

Recall the classical inequalities in Sobolev spaces:

$$\int\limits_{S\cap Y} \Big|v - \int\limits_{Y} v\mathrm{d}x\Big|^2 \mathrm{d}\sigma \leqslant C\int\limits_{Y} |Dv|^2\mathrm{d}x \quad \forall\, v \in W^{1,2}(Y) \text{ (Poincaré inequality)}, \tag{4.1}$$

$$\int\limits_{S\cap Y} |v|^2\mathrm{d}\sigma \leqslant C\int\limits_{Y} (|v|^2 + |Dv|^2)\mathrm{d}x \quad \forall\, v \in W^{1,2}(Y) \text{ (trace inequality)}. \tag{4.2}$$

By an easy scaling argument, (4.1), (4.2) lead to the inequalities

$$\int\limits_{S_\varepsilon} |v_\varepsilon - \overline{v}_\varepsilon|^2 \mathrm{d}\sigma \leqslant C\varepsilon \int\limits_{\Omega} |Dv_\varepsilon|^2 \mathrm{d}x, \tag{4.3}$$

$$\int\limits_{S_\varepsilon} |v_\varepsilon|^2 \mathrm{d}\sigma \leqslant C\varepsilon^{-1} \Big(\int\limits_{\Omega} |v_\varepsilon|^2 \mathrm{d}x + \varepsilon^2 \int\limits_{\Omega} |Dv_\varepsilon|^2 \mathrm{d}x \Big) \tag{4.4}$$

for any $v_\varepsilon \in W^{1,2}(\Omega)$, where $\overline{v}_\varepsilon$ stands for the piecewise constant function obtained by averaging over each cell $Y_\varepsilon^{(m)}$ (cf. (3.10)) and C depends only on S. We also will make use of the following inequality, which is a simple consequence of the Jensen inequality: for any $r \geqslant 1$

$$\int\limits_{S_\varepsilon} |\overline{v}_\varepsilon|^r \mathrm{d}\sigma \leqslant C\varepsilon^{-1} \int\limits_{\Omega} |v_\varepsilon|^r \mathrm{d}x, \tag{4.5}$$

where $C > 0$ is independent of r and v_ε.

4.2 An asymptotic representation for surface integral in (3.4)

To pass to the limit as $\varepsilon \to 0$ in the surface integral in (3.4), we use the following assertion.

Lemma 4.1. *Let $u_\varepsilon, w_\varepsilon \in W^{1,2}(\Omega)$. Then*

$$\int\limits_{S_\varepsilon} g(u_\varepsilon, x/\varepsilon) w_\varepsilon \mathrm{d}x = \int\limits_{S_\varepsilon} g(\overline{u}_\varepsilon, x/\varepsilon)(w_\varepsilon - \overline{w}_\varepsilon) \mathrm{d}\sigma + \int\limits_{S_\varepsilon} g'_u(\overline{u}_\varepsilon, x/\varepsilon) \overline{w}_\varepsilon (u_\varepsilon - \overline{u}_\varepsilon) \mathrm{d}\sigma + \varrho_\varepsilon \tag{4.6}$$

and

$$|\varrho_\varepsilon| \leqslant C\big(\varepsilon + (\varepsilon \|w_\varepsilon\|_{L^2(\Omega)})^{2/(N+2)}\big)\big(\|w_\varepsilon\|^2_{W^{1,2}(\Omega)} + \|u_\varepsilon\|^2_{W^{1,2}(\Omega)}\big). \tag{4.7}$$

Proof. We have

$$\begin{aligned} g(u_\varepsilon, x/\varepsilon) w_\varepsilon = {} & g(\overline{u}_\varepsilon, x/\varepsilon)(w_\varepsilon - \overline{w}_\varepsilon) + (g(u_\varepsilon, x/\varepsilon) - (g(\overline{u}_\varepsilon, x/\varepsilon))(w_\varepsilon - \overline{w}_\varepsilon) \\ & + (g(u_\varepsilon, x/\varepsilon) - g(\overline{u}_\varepsilon, x/\varepsilon))\overline{w}_\varepsilon + g(\overline{u}_\varepsilon, x/\varepsilon)\overline{w}_\varepsilon. \end{aligned}$$

Therefore (in view of (2.6)),

$$\begin{aligned} \int\limits_{S_\varepsilon} g(u_\varepsilon, x/\varepsilon) w_\varepsilon \mathrm{d}\sigma = {} & \int\limits_{S_\varepsilon} g(\overline{u}_\varepsilon, x/\varepsilon)(w_\varepsilon - \overline{w}_\varepsilon) \mathrm{d}\sigma + \int\limits_{S_\varepsilon} (g(u_\varepsilon, x/\varepsilon) - (g(\overline{u}_\varepsilon, x/\varepsilon))(w_\varepsilon - \overline{w}_\varepsilon) \mathrm{d}\sigma \\ & + \int\limits_{S_\varepsilon} (g(u_\varepsilon, x/\varepsilon) - g(\overline{u}_\varepsilon, x/\varepsilon))\overline{w}_\varepsilon \mathrm{d}\sigma = I_1 + I_2 + I_3. \end{aligned}$$

The term I_2 gives vanishing contribution as $\varepsilon \to 0$. Indeed, by (2.4) and (4.3),

$$|I_2| \leqslant C \int\limits_{S_\varepsilon} |u_\varepsilon - \overline{u}_\varepsilon||w_\varepsilon - \overline{w}_\varepsilon| \mathrm{d}\sigma \leqslant C\varepsilon \|Du_\varepsilon\|_{L^2(\Omega)} \|Dw_\varepsilon\|_{L^2(\Omega)}. \tag{4.8}$$

The term I_3 can be written as

$$I_3 = \int_0^1 \mathrm{d}t \int_{S_\varepsilon} (g'_u(\overline{u}_\varepsilon + t(u_\varepsilon - \overline{u}_\varepsilon), x/\varepsilon) - g'_u(\overline{u}_\varepsilon, x/\varepsilon))\overline{w}_\varepsilon(u_\varepsilon - \overline{u}_\varepsilon)\mathrm{d}\sigma$$

$$+ \int_{S_\varepsilon} g'_u(\overline{u}_\varepsilon, x/\varepsilon)\overline{w}_\varepsilon(u_\varepsilon - \overline{u}_\varepsilon)\mathrm{d}\sigma = \widetilde{I}_3 + \int_{S_\varepsilon} g'_u(\overline{u}_\varepsilon, x/\varepsilon)\overline{w}_\varepsilon(u_\varepsilon - \overline{u}_\varepsilon)\mathrm{d}\sigma.$$

Using (2.5), we get

$$|\widetilde{I}_3| \leqslant C \sup_{0\leqslant t\leqslant 1} \int_{S_\varepsilon} \frac{t|u_\varepsilon - \overline{u}_\varepsilon|^2|\overline{w}_\varepsilon|}{1 + |\overline{u}_\varepsilon| + |\overline{u}_\varepsilon + t(u_\varepsilon - \overline{u}_\varepsilon)|}\mathrm{d}\sigma,$$

which yields, after applying the Hölder inequality,

$$|\widetilde{I}_3| \leqslant C \sup_{0\leqslant t\leqslant 1} \int_{S_\varepsilon} \frac{t|u_\varepsilon - \overline{u}_\varepsilon|^2|\overline{w}_\varepsilon|}{1 + t|u_\varepsilon - \overline{u}_\varepsilon|}\mathrm{d}\sigma$$

$$\leqslant C\bigg(\int_{S_\varepsilon} |\overline{w}_\varepsilon|^q \mathrm{d}\sigma\bigg)^{1/q} \sup_{0\leqslant t\leqslant 1} \bigg(\int_{S_\varepsilon} |u_\varepsilon - \overline{u}_\varepsilon|^2 \frac{t^{q'}|u_\varepsilon - \overline{u}_\varepsilon|^{2q'-2}}{(1 + t|u_\varepsilon - \overline{u}_\varepsilon|)^{q'}}\mathrm{d}\sigma\bigg)^{1/q'},$$

where $q' = q/(q-1)$ and $q = 2(N+2)/N$. Note that the embedding $W^{1,2}(\Omega) \subset L^q(\Omega)$ is compact; moreover (cf., for example, [24]),

$$\exists\, C > 0 \text{ such that } \|u\|_{L^q(\Omega)} \leqslant C\|u\|^{2/q}_{W^{1,2}(\Omega)}\|u\|^{4/(Nq)}_{L^2(\Omega)} \quad \forall u \in W^{1,2}(\Omega). \tag{4.9}$$

Since $1 < q' < 2$, we have

$$\frac{t^{q'}|u_\varepsilon - \overline{u}_\varepsilon|^{2q'-2}}{(1 + t|u_\varepsilon - \overline{u}_\varepsilon|)^{q'}} \leqslant \frac{t^{2q'-2}|u_\varepsilon - \overline{u}_\varepsilon|^{2q'-2}}{(1 + t|u_\varepsilon - \overline{u}_\varepsilon|)^{2q'-2}} \frac{t^{2-q'}}{(1 + t|u_\varepsilon - \overline{u}_\varepsilon|)^{2-q'}} \leqslant 1$$

for any $0 \leqslant t \leqslant 1$. Therefore, using (4.3), (4.5), and (4.9), we get

$$\begin{aligned}|\widetilde{I}_3| &\leqslant C\varepsilon^{-1/q-1/q'+2/q'}\|w_\varepsilon\|_{L^q(\Omega)}\,\|Du_\varepsilon\|^{2/q'}_{L^2(\Omega)}\\ &\leqslant C\varepsilon^{2/(N+2)}\|w_\varepsilon\|^{2/q}_{W^{1,2}(\Omega)}\|w_\varepsilon\|^{4/(Nq)}_{L^2(\Omega)}\|Du_\varepsilon\|^{2/q'}_{L^2(\Omega)}\\ &\leqslant C(\varepsilon\|w_\varepsilon\|_{L^2(\Omega)})^{2/(N+2)}(\|w_\varepsilon\|^2_{W^{1,2}(\Omega)} + \|Du_\varepsilon\|^2_{L^2(\Omega)}),\end{aligned} \tag{4.10}$$

where we also used the Young inequality. The bounds (4.10) and (4.10) yield (4.7) (since $|\varrho_\varepsilon| \leqslant |I_2| + |\widetilde{I}_3|$). □

The proof of the next technical result is similar to Lemma 4.1 (and is left to the reader).

Lemma 4.2. *If* $u_\varepsilon,\ u^{(1)}_\varepsilon \in W^{1,2}(\Omega),\ v_\varepsilon \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$, *then for* $w_\varepsilon = u_\varepsilon - u^{(1)}_\varepsilon$

$$\bigg|\int_{S_\varepsilon} (g(\overline{u}_\varepsilon, x/\varepsilon) - g(\overline{u}^{(1)}_\varepsilon, x/\varepsilon))(u_\varepsilon - v_\varepsilon - \overline{u}_\varepsilon + \overline{v}_\varepsilon)\mathrm{d}\sigma\bigg| \leqslant C\|w_\varepsilon\|_{L^2(\Omega)}\|D(u_\varepsilon - v_\varepsilon)\|_{L^2(\Omega)},$$

$$\left|\int\limits_{S_\varepsilon}(g'_u(\overline{u}_\varepsilon, x/\varepsilon)\overline{u}_\varepsilon - g'_u(\overline{u}_\varepsilon^{(1)}, x/\varepsilon)\overline{u}_\varepsilon^{(1)})(u_\varepsilon - \overline{u}_\varepsilon)\mathrm{d}\sigma\right| \leqslant C\|w_\varepsilon\|_{L^2(\Omega)}\|Du_\varepsilon\|_{L^2(\Omega)},$$

$$\left|\int\limits_{S_\varepsilon}(g'_u(\overline{u}_\varepsilon, x/\varepsilon) - g'_u(\overline{u}_\varepsilon^{(1)}, x/\varepsilon))\overline{v}_\varepsilon(u_\varepsilon - \overline{u}_\varepsilon)\mathrm{d}\sigma\right| \leqslant C\|w_\varepsilon\|_{L^2(\Omega)}\|v\|_{L^\infty(\Omega)}\|Du_\varepsilon\|_{L^2(\Omega)}.$$

Proof of Theorem 2.1. Assume by contradiction that there are sequences $\varepsilon_k \to 0$, $\lambda_k \to +\infty$, and $u_k \in W^{1,2}(\Omega_{\varepsilon_k})$ such that $\|u_k\|_{X_{\varepsilon_k}} \to \infty$,

$$\langle \mathcal{A}_{\varepsilon_k}(u_k), u_k\rangle_{\varepsilon_k} + \lambda_k\langle u_k, u_k\rangle_{\varepsilon_k} - \langle \mathcal{G}_\varepsilon(u_k), u_k\rangle_{\varepsilon_k} \leqslant \delta_k\|u_k\|^2_{X_{\varepsilon_k}},$$

and $\delta_k \to 0$. In view of the definition of $\mathcal{A}_\varepsilon$ and $\mathcal{G}_\varepsilon$, this implies that

$$\int\limits_{\Omega_{\varepsilon_k}}(a(Dv_k, x/\varepsilon)\cdot Dv_k + \lambda_k|v_k|^2)\mathrm{d}x \leqslant \int\limits_{S_\varepsilon} g(v_k, x/\varepsilon)v_k\mathrm{d}\sigma + \delta_k\|v_k\|^2_{W^{1,2}(\Omega)}\mathrm{d}x,$$

where $v_k = P_{\varepsilon_k}u_k$ is the extension of u_k onto Ω. Using (2.2) and the properties of the extension operator P_ε, for $w_k = v_k/\|v_k\|_{W^{1,2}(\Omega)}$ we get

$$\gamma\int\limits_\Omega |Dw_k|^2\mathrm{d}x + \lambda_k\int\limits_{\Omega_{\varepsilon_k}}|w_k|^2\mathrm{d}x \leqslant \frac{1}{\|v_k\|_{W^{1,2}(\Omega)}}\int\limits_{S_{\varepsilon_k}} g(v_k, x/\varepsilon)w_k\mathrm{d}\sigma + \widetilde{\delta}_k \tag{4.11}$$

with some $\gamma > 0$, where $\widetilde{\delta}_k = \delta_k + C/\|v_k\|^2_{W^{1,2}(\Omega)} \to 0$. Now, write

$$\begin{aligned}\int\limits_{S_{\varepsilon_k}} g(v_k, x/\varepsilon_k)w_k\mathrm{d}\sigma &= \int\limits_{S_{\varepsilon_k}}(g(v_k, x/\varepsilon_k) - g(\overline{v}_k, x/\varepsilon_k))w_k\mathrm{d}\sigma + \int\limits_{S_{\varepsilon_k}} g(\overline{v}_k, x/\varepsilon_k)(w - \overline{w}_k)\mathrm{d}\sigma\\ &= I_1 + I_2,\end{aligned} \tag{4.12}$$

where we used (2.6). By (2.4) and (4.3), we have

$$\begin{aligned}|I_1| &\leqslant C\int\limits_{S_{\varepsilon_k}}|v_k - \overline{v}_k||w_k|\mathrm{d}\sigma \leqslant C\varepsilon_k{}^{1/2}\Bigg(\int\limits_\Omega |Dv_k|^2\mathrm{d}x\Bigg)^{1/2}\Bigg(\int\limits_{S_{\varepsilon_k}}|w_k|^2\mathrm{d}x\Bigg)^{1/2}\\ &\leqslant C\|Dv_k\|_{L^2(\Omega)}(\|w_k\|_{L^2(\Omega)} + \varepsilon_k\|Dw_k\|_{L^2(\Omega)}).\end{aligned} \tag{4.13}$$

Similarly, by (2.3) and (4.5),

$$|I_2| \leqslant C\int\limits_{S_{\varepsilon_k}}|w_k - \overline{w}_k|(|\overline{v}_k| + 1)\mathrm{d}\sigma \leqslant C\|Dw_k\|_{L^2(\Omega)}(\|v_k\|_{L^2(\Omega)} + 1). \tag{4.14}$$

Thus,

$$\gamma\|Dw_k\|^2_{L^2(\Omega)} + \lambda_k\|w_k\|^2_{L^2(\Omega_{\varepsilon_k})} \leqslant C(\|w_k\|_{L^2(\Omega)} + \varepsilon_k) + \widetilde{\delta}_k, \tag{4.15}$$

where we used the fact that $\|w_k\|_{W^{1,2}(\Omega)} = 1$. Therefore, $\|w_k\|^2_{L^2(\Omega_{\varepsilon_k})} \to 0$.

By the compactness of the embedding $W^{1,2}(\Omega) \subset L^2(\Omega)$, up to a subsequence, $w_k \to w$ strongly in $L^2(\Omega)$. On the other hand, according to the structure of perforated domains Ω_ε,

$$\int\limits_{\Omega_{\varepsilon_k}} w_k v \mathrm{d}x \to |Y^*| \int\limits_{\Omega} w v \mathrm{d}x \quad \forall\, v \in L^2(\Omega).$$

Taking $v = w$, we get $w = 0$ (since $\|w_k\|_{L^2(\Omega_{\varepsilon_k})} \to 0$) so that $\|w_k\|_{L^2(\Omega)} \to 0$. Then (4.15) yields $\gamma \|Dw_k\|_{L^2(\Omega)} \to 0$ and, consequently, $\|w_k\|_{W^{1,2}(\Omega)} \to 0$, that is a contradiction. □

As a byproduct of the above proof, from (4.12)–(4.14) we obtain for any $u, v \in W^{1,2}(\Omega)$

$$|\langle \mathcal{G}_\varepsilon(u), v\rangle_\varepsilon| \leqslant C(\|u\|_{W^{1,2}(\Omega)}\|v\|_{L^2(\Omega)} + \|v\|_{W^{1,2}(\Omega)}(\|u\|_{L^2(\Omega)}+1) + \varepsilon \|u\|_{W^{1,2}(\Omega)}\|v\|_{W^{1,2}(\Omega)}), \quad (4.16)$$

where C is independent of ε. In particular,

$$\|\mathcal{G}_\varepsilon(u)\|_{X_\varepsilon^*} \leqslant C(\|u\|_{X_\varepsilon} + 1) \quad \forall\, u \in X_\varepsilon. \quad (4.17)$$

Then, possibly modifying $\varkappa_2$ in (2.8), we have

$$\text{(2.8) holds for all } u_\varepsilon \in X_\varepsilon, \quad (4.18)$$

when $\varepsilon \leqslant \varepsilon_0$ and $\lambda \geqslant \lambda_0$.

5 Limit Transition in the Surface Term and Proof of Proposition 3.1

Proof of Proposition 3.1. Let Ω' be a subdomain of Ω such that $\overline{\Omega'} \subset \Omega$. Define the linear functional b_ε on $W^{1,2}(\Omega)$ by the formula

$$b_\varepsilon w_\varepsilon = \int\limits_{S'_\varepsilon} q(x, x/\varepsilon)(w_\varepsilon - \overline{w}_\varepsilon)\, \mathrm{d}\sigma. \quad (5.1)$$

where $S'_\varepsilon = \bigcup\limits_{m:\, Y_\varepsilon^{(m)} \cap \Omega' \neq \varnothing} S_\varepsilon \cap Y_\varepsilon^{(m)}$. It is clear that $S'_\varepsilon \subset S_\varepsilon$.

Step 1 (weak convergence of b_ε). Let us show that

$$\|b_\varepsilon\| \leqslant C \text{ with } C \text{ independent of } \varepsilon, \quad (5.2)$$

$$b_\varepsilon w \to \int\limits_{\Omega'} \int\limits_{Y \cap S} q(x,y) Dw(x) \cdot y \, \mathrm{d}\sigma_y \mathrm{d}x \quad \text{weakly as } \varepsilon \to 0. \quad (5.3)$$

By (4.3), we have

$$|b_\varepsilon w_\varepsilon| \leqslant C \int\limits_{S'_\varepsilon} |w_\varepsilon - \overline{w}_\varepsilon| \mathrm{d}\sigma \leqslant C\varepsilon^{-1/2} \Big(\int\limits_{S'_\varepsilon} |w_\varepsilon - \overline{w}_\varepsilon|^2 \mathrm{d}\sigma \Big)^{1/2} \leqslant C \|w_\varepsilon\|_{W^{1,2}(\Omega)}.$$

We chose an arbitrary w from the dense (in $W^{1,2}(\Omega)$) set $C^2(\overline{\Omega})$. We have

$$b_\varepsilon w = \sum_m \int\limits_{S'_\varepsilon \cap Y_\varepsilon^{(m)}} q(x, x/\varepsilon)(Dw(x_\varepsilon^{(m)}) \cdot (x - x_\varepsilon^{(m)}) + O(\varepsilon^2))\,\mathrm{d}\sigma$$

$$= \sum_m \int\limits_{S'_\varepsilon \cap Y_\varepsilon^{(m)}} q(x_\varepsilon^{(m)}, x/\varepsilon) Dw(x_\varepsilon^{(m)}) \cdot (x - x_\varepsilon^{(m)})\,\mathrm{d}\sigma + O(\varepsilon)$$

$$= \int\limits_{\Omega'} \int\limits_{Y \cap S} q(x, y) Dw(x) \cdot y\,\mathrm{d}\sigma_y \mathrm{d}x + o(1),$$

where $x_\varepsilon^{(m)}$ is the center of the cell $Y_\varepsilon^{(m)}$. Thus, (5.2) and (5.3) are proved.

Step 2 (proof of (3.9) for w_ε with $\operatorname{supp}(w_\varepsilon) \subset \overline{\Omega'}$). Assume that

$$w_\varepsilon = 0 \quad \text{in } \Omega \setminus \Omega', \text{ (in particular, } w_\varepsilon = 0 \quad \text{on } \partial\Omega'). \tag{5.4}$$

Given $\delta > 0$, let $\{Q_\delta^{(\alpha)}\}$ be an open cover of Ω, $\operatorname{diam} Q_\delta^{(\alpha)} \leqslant \delta$, and let $\{\varphi_\delta^{(\alpha)} \in C^\infty(\mathbb{R}^N)\}$ be a partition of unity such that $\operatorname{supp} \varphi_\delta^{(\alpha)} \subset Q_\delta^{(\alpha)}$, $0 \leqslant \varphi_\delta^{(\alpha)} \leqslant 1$, $\sum\limits_\alpha \varphi_\delta^{(\alpha)} = 1$. Then

$$b_\varepsilon w_\varepsilon = \sum_\alpha \int\limits_{S'_\varepsilon} q(\widehat{x}_\delta^{(\alpha)}, x/\varepsilon)(w_\varepsilon - \overline{w}_\varepsilon)\varphi_\delta^{(\alpha)}\,\mathrm{d}\sigma$$

$$+ \sum_\alpha \int\limits_{S'_\varepsilon} (q(x, x/\varepsilon) - q(\widehat{x}_\delta^{(\alpha)}, x/\varepsilon))(w_\varepsilon - \overline{w}_\varepsilon)\varphi_\delta^{(\alpha)}\,\mathrm{d}\sigma = I_1 + I_2, \tag{5.5}$$

where $\widehat{x}_\delta^{(\alpha)} \in Q_\delta^{(\alpha)}$. By the Lipschitz continuity of $q(x, y)$ in x,

$$|I_2| \leqslant C\delta \sum_\alpha \int\limits_{S'_\varepsilon} |w - \overline{w}_\varepsilon| \varphi_\delta^{(\alpha)}\,\mathrm{d}\sigma = C\delta \int\limits_{S'_\varepsilon} |w - \overline{w}_\varepsilon|\,\mathrm{d}\sigma \leqslant C\delta \|Dw_\varepsilon\|_{L^2(\Omega)}. \tag{5.6}$$

We write the first term I_1 as

$$I_1 = \sum_\alpha \left(\int\limits_{S'_\varepsilon} q(\widehat{x}_\delta^{(\alpha)}, x/\varepsilon) w_\varepsilon \varphi_\delta^{(\alpha)}\,\mathrm{d}\sigma - \int\limits_{S'_\varepsilon} q(\widehat{x}_\delta^{(\alpha)}, x/\varepsilon) \overline{w}_\varepsilon \varphi_\delta^{(\alpha)}\,\mathrm{d}\sigma \right) = \sum_\alpha (\widetilde{I}_1^{(\alpha)} + \widehat{I}_1^{(\alpha)}). \tag{5.7}$$

Note that

$$\int\limits_{S'_\varepsilon \cap Y_\varepsilon^{(m)}} q(\widehat{x}_\delta^{(\alpha)}, x/\varepsilon)\varphi_\delta^{(\alpha)}\,\mathrm{d}\sigma = \varepsilon^N \left(\int\limits_{S \cap Y} q(\widehat{x}_\delta^{(\alpha)}, y) D\varphi_\delta^{(\alpha)}(x_\varepsilon^{(m)}) \cdot y\,\mathrm{d}\sigma_y + O(\varepsilon) \right)$$

(as above, $x_\varepsilon^{(m)}$ denotes the center of the cell $Y_\varepsilon^{(m)}$). Since $\overline{w}_\varepsilon \to W_0(x)$ strongly in $L^2(\Omega)$,

$$\widehat{I}_1^{(\alpha)} \to - \int\limits_{\Omega'} \left(W_0(x) \int\limits_{S \cap Y} q(\widehat{x}_\delta^{(\alpha)}, y) D\varphi_\delta^{(\alpha)}(x) \cdot y\,\mathrm{d}\sigma_y \right) \mathrm{d}x$$

$$= \int\limits_{\Omega'} \left(\varphi_\delta^{(\alpha)}(x) \int\limits_{S \cap Y} q(x_\delta^{(\alpha)}, y) DW_0(x) \cdot y\,\mathrm{d}\sigma_y \right) \mathrm{d}x,$$

where we used the fact that $W_0 = 0$ in $\Omega \setminus \Omega'$. Thus,

$$\sum_\alpha \widehat{I}_1^{(\alpha)} \to \int\limits_{S\cap Y} \left(\sum_\alpha \int\limits_{\Omega'} \varphi_\delta^{(\alpha)}(x) q(\widehat{x}_\delta^{(\alpha)}, y) DW_0(x) \cdot y \mathrm{d}x \right) \mathrm{d}\sigma_y.$$

Therefore,

$$\lim_{\delta\to 0} \lim_{\varepsilon \to 0} \sum_\alpha \widehat{I}_1^{(\alpha)} = \int\limits_{\Omega'} \int\limits_{S\cap Y} q(x,y) DW_0(x) \cdot y \mathrm{d}\sigma_y \mathrm{d}x. \tag{5.8}$$

In order to pass to the limit in $\widetilde{I}_1^{(\alpha)}$ as $\varepsilon \to 0$, we consider the solution θ of the problem

$$\begin{cases} \Delta\theta(y) = 0 \quad \text{in } Y^*, \\ \dfrac{\partial\theta}{\partial\nu} = q(\widehat{x}_\delta^{(\alpha)}, y) \quad \text{on } S \cap Y, \\ \theta \text{ is } Y^*\text{-periodic.} \end{cases} \tag{5.9}$$

By property (c) of $q(x, y)$, there is a unique (up to an additive constant) solution θ of (5.9) and $\theta \in W^{1,2}(Y^*)$. We set $\zeta_\varepsilon(x) = \theta(x/\varepsilon)$. Then $\Delta\zeta_\varepsilon = 0$ in Ω_ε and

$$\varepsilon \frac{\partial\zeta_\varepsilon}{\partial\nu} = q(\widehat{x}_\delta^{(\alpha)}, x/\varepsilon) \quad \text{on } S'_\varepsilon,$$

so that

$$\begin{aligned} \int\limits_{S'_\varepsilon} q(\widehat{x}_\delta^{(\alpha)}, x/\varepsilon) w_\varepsilon \varphi_\delta^{(\alpha)} \mathrm{d}\sigma = \varepsilon \int\limits_{S'_\varepsilon} w_\varepsilon \varphi_\delta^{(\alpha)} \frac{\partial\zeta_\varepsilon}{\partial\nu} \mathrm{d}\sigma &= \varepsilon \int\limits_{\Omega_\varepsilon\cap\Omega'} D(w_\varepsilon\varphi_\delta^{(\alpha)}) \cdot D\zeta_\varepsilon \, \mathrm{d}x \\ &= \int\limits_{\Omega_\varepsilon\cap\Omega'} D(w_\varepsilon\varphi_\delta^{(\alpha)}) \cdot (D\theta)(x/\varepsilon) \, \mathrm{d}x, \end{aligned}$$

where we took into account that $w_\varepsilon = 0$ on $\partial\Omega'$. It is easy to check that

$$D(w_\varepsilon\varphi_\delta^{(\alpha)})(x) \to D(W_0\varphi_\delta^{(\alpha)})(x) + \varphi_\delta^{(\alpha)} D_y W_1(x, y) \quad \text{two-scale.}$$

Therefore,

$$\begin{aligned} \widetilde{I}_1^{(\alpha)} &\to \int\limits_{\Omega'} \left(\int\limits_{Y^*} (D(W_0\varphi_\delta^{(\alpha)}) + \varphi_\delta^{(\alpha)} D_y W_1(x,y)) \cdot (D\theta)(y) \, \mathrm{d}y \right) \mathrm{d}x \\ &= \int\limits_{\Omega'} \varphi_\delta^{(\alpha)} \left(\int\limits_{S\cap Y} W_1(x,y) q(\widehat{x}_\delta^{(\alpha)}, y) \, \mathrm{d}\sigma_y \right) \mathrm{d}x, \end{aligned}$$

where we used (5.9). Thus, taking into account the Lipschitz continuity of $q(x, y)$ in x and passing to the limit as $\delta \to 0$, we get

$$\begin{aligned} \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \sum_\alpha \widetilde{I}_1^{(\alpha)} &= \sum_\alpha \int\limits_{\Omega'} \varphi_\delta^{(\alpha)} \Big(\int\limits_{S\cap Y} W_1(x,y) q(x,y) \, \mathrm{d}\sigma_y \Big) \, \mathrm{d}x \\ &= \int\limits_{\Omega'} \int\limits_{S\cap Y} W_1(x,y) q(x,y) \, \mathrm{d}\sigma_y \mathrm{d}x. \end{aligned} \tag{5.10}$$

Finally, by (5.5)–(5.8) and (5.10), we obtain

$$\int\limits_{S'_\varepsilon} q(x, x/\varepsilon)(w_\varepsilon - \overline{w}_\varepsilon)\mathrm{d}\sigma \to \int\limits_{\Omega'}\int\limits_{Y\cap S} q(x,y)(DW_0\cdot y + W_1(x,y))\mathrm{d}\sigma_y \mathrm{d}x. \tag{5.11}$$

Step 3 (general case). Let (w_ε) be an arbitrary sequence such that $w_\varepsilon \to W_0$ weakly in $W^{1,2}(\Omega)$ and $Dw_\varepsilon \to DW_0(x) + D_yW_1(x,y)$ two-scale. Write

$$w_\varepsilon = (w_\varepsilon - (W_0 + w_\varepsilon^{(1)})) + w_\varepsilon^{(1)} + W_0,$$

where $w_\varepsilon^{(1)}$ is a unique solution of the problem

$$\begin{cases} \Delta w_\varepsilon^{(1)} = 0 & \text{in } \Omega', \\ w_\varepsilon^{(1)} = w_\varepsilon - W_0 & \text{on } \partial\Omega', \end{cases}$$

extended in $\Omega\setminus\Omega'$ by setting $w_\varepsilon^{(1)} = w_\varepsilon - W_0$. Since $w_\varepsilon - W_0 \to 0$ weakly in $H^{1/2}(\partial\Omega')$,

$$w_\varepsilon^{(1)} \to 0 \quad \text{strongly in } W^{1,2}(K) \text{ for any compact set } K\subset\Omega' \tag{5.12}$$

by standard elliptic estimates. This implies, in particular, that $w_\varepsilon^{(1)} \to 0$ and $Dw_\varepsilon^{(1)} \to 0$ two-scale. Moreover, in view of (4.3), for any compact subset K of Ω',

$$\begin{aligned} |b_\varepsilon w_\varepsilon^{(1)}| &\leqslant C \sum_{m:Y_\varepsilon^{(m)}\cap K\neq\varnothing} \int\limits_{Y_\varepsilon^{(m)}} |w_\varepsilon^{(1)} - \overline{w}_\varepsilon^{(1)}|\mathrm{d}\sigma + C \sum_{m:Y_\varepsilon^{(m)}\cap K=\varnothing} \int\limits_{Y_\varepsilon^{(m)}\cap\Omega'} |w_\varepsilon^{(1)} - \overline{w}_\varepsilon^{(1)}|\mathrm{d}\sigma \\ &\leqslant C\Bigg(\int\limits_{K_\delta} |Dw_\varepsilon^{(1)}|^2\mathrm{d}x\Bigg)^{1/2} + C|\Omega'_\delta\setminus K|^{1/2}\Bigg(\int\limits_{\Omega} |Dw_\varepsilon^{(1)}|^2\mathrm{d}x\Bigg)^{1/2} \end{aligned} \tag{5.13}$$

if $\varepsilon \leqslant \delta/N$, where C is independent of ε and δ, K_δ, Ω'_δ are the δ-neighborhoods of K and Ω' respectively, and $\delta > 0$ is arbitrary. (The summation in (5.13) is taken over m such that $Y_\varepsilon^{(m)}\cap\Omega' \neq \varnothing$.) From (5.12) and (5.13) it follows that $b_\varepsilon w_\varepsilon^{(1)} \to 0$ as $\varepsilon\to 0$, while, according to the first and second steps,

$$b_\varepsilon W_0 \to \int\limits_{\Omega'}\int\limits_{Y\cap S} q(x,y)DW_0\cdot y\mathrm{d}\sigma_y\mathrm{d}x,$$

$$b_\varepsilon(w_\varepsilon - (W_0 + w_\varepsilon^{(1)})) \to \int\limits_{\Omega'}\int\limits_{Y\cap S} q(x,y)W_1(x,y)\mathrm{d}\sigma_y\mathrm{d}x.$$

Thus, (5.11) is proved for any sequence (w_ε) such that (3.8) holds.

Final step. We set $\Omega' = \{x\in\Omega; \mathrm{d}ist(x,\partial\Omega) > \delta\}$, where $\delta > 0$. By (4.3), we have

$$\int\limits_{S_\varepsilon\setminus S'_\varepsilon} |w_\varepsilon - \overline{w}_\varepsilon|\mathrm{d}\sigma \leqslant C\frac{\delta^{1/2}}{\varepsilon^{1/2}}\Bigg(\int\limits_{S_\varepsilon\setminus S'_\varepsilon} |w_\varepsilon - \overline{w}_\varepsilon|^2\mathrm{d}\sigma\Bigg)^{1/2} \leqslant C\delta^{1/2}\|w_\varepsilon\|_{W^{1,2}(\Omega)} \tag{5.14}$$

for sufficiently small ε, where C is independent of δ and ε. Therefore, (5.14) combined with (5.11), yields (3.9) for any sequence (w_ε) such that (3.8) holds. □

Proof of (3.7)**.** We approximate U_0 by functions $u_\delta^{(1)} \in C^1(\overline{\Omega})$ ($\delta > 0$) in the strong topology of $L^2(\Omega)$: $\|U_0 - u_\delta^{(1)}\|_{L^2(\Omega)} \leqslant \delta$. By Lemma 4.1, the strong-L^2 convergence of u_ε to U_0, and Lemma 4.2, we have

$$\limsup_{\varepsilon\to 0}\Bigg| \int\limits_{S_\varepsilon} g(u_\varepsilon, x/\varepsilon)(u_\varepsilon - v_\varepsilon)\mathrm{d}\sigma - \int\limits_{S_\varepsilon} g(\overline{u}_\delta^{(1)}, x/\varepsilon)(u_\varepsilon - v_\varepsilon - \overline{u}_\varepsilon + \overline{v}_\varepsilon)\mathrm{d}\sigma$$

$$- \int\limits_{S_\varepsilon} g'_u(\overline{u}_\delta^{(1)}, x/\varepsilon)(\overline{u}_\delta^{(1)} - \overline{v}_\varepsilon)(u_\varepsilon - \overline{u}_\varepsilon)\mathrm{d}\sigma\Bigg| \leqslant C\delta. \tag{5.15}$$

On the other hand, the regularity of $g(u, y)$ in u (the conditions (2.3), (2.4), and (2.5)) implies the point-wise bounds

$$|g(\overline{u}_\delta^{(1)}, x/\varepsilon) - g(u_\delta^{(1)}, x/\varepsilon)| \leqslant C\varepsilon \quad \text{on } S_\varepsilon,$$

$$|g'_u(\overline{u}_\delta^{(1)}, x/\varepsilon)(\overline{u}_\delta^{(1)} - \overline{v}_\varepsilon) - g'_u(u_\delta^{(1)}, x/\varepsilon)(u_\delta^{(1)} - V_0)| \leqslant C\varepsilon \quad \text{on } S_\varepsilon$$

(recall that $v_\varepsilon = V_0(x) + \varepsilon V_1(x, x/\varepsilon)$ and V_0, V_1 are smooth functions), which, by (4.3), lead to

$$\limsup_{\varepsilon\to 0}\Bigg| \int\limits_{S_\varepsilon} (g(\overline{u}_\delta^{(1)}, x/\varepsilon) - g(u_\delta^{(1)}, x/\varepsilon))(u_\varepsilon - v_\varepsilon - \overline{u}_\varepsilon + \overline{v}_\varepsilon)\mathrm{d}\sigma$$

$$+ \int\limits_{S_\varepsilon} (g'_u(\overline{u}_\delta^{(1)}, x/\varepsilon)(\overline{u}_\delta^{(1)} - \overline{v}_\varepsilon) - g'_u(u_\delta^{(1)}, x/\varepsilon)(u_\delta^{(1)} - V_0))(u_\varepsilon - \overline{u}_\varepsilon)\mathrm{d}\sigma\Bigg| = 0. \tag{5.16}$$

Now, applying Proposition 3.1 first with $q(x, y) = g(u_\delta^{(1)}(x), y)$, $w_\varepsilon = u_\varepsilon - v_\varepsilon$, then with $q(x, y) = g(u_\delta^{(1)}(x), y)u_\delta^{(1)}(x)$, $w_\varepsilon = u_\varepsilon$, and, finally, with $q(x, y) = g(u_\delta^{(1)}(x), y)V_0(x)$, $w_\varepsilon = u_\varepsilon$, we get

$$\int\limits_{S_\varepsilon} \big(g(u_\delta^{(1)}, x/\varepsilon)(u_\varepsilon - v_\varepsilon - \overline{u}_\varepsilon + \overline{v}_\varepsilon) + g'_u(u_\delta^{(1)}, x/\varepsilon)(u_\delta^{(1)} - V_0)(u_\varepsilon - \overline{u}_\varepsilon)\big)\mathrm{d}\sigma$$

$$\to \int\limits_{\Omega} \int\limits_{S\cap Y} g(u_\delta^{(1)}, y)(D(U_0 - V_0)\cdot y + U_1(x, y) - V_1(x, y))\mathrm{d}\sigma_y \mathrm{d}x$$

$$+ \int\limits_{\Omega} \int\limits_{S\cap Y} g'_u(u_\delta^{(1)}, y)(u_\delta^{(1)} - V_0)(DU_0\cdot y + U_1(x, y))\mathrm{d}\sigma_y \mathrm{d}x. \tag{5.17}$$

Assuming $\delta \to 0$ in (5.15), (5.16), and (5.17) yields (3.7). □

6 Homogenization of the Parabolic Problem (1.2)

In terms of the operators $\mathcal{A}_\varepsilon$ and $\mathcal{G}_\varepsilon$, the problem (1.2) is as follows:

$$\begin{cases} \partial_t u_\varepsilon(t) + \mathcal{A}_\varepsilon(u_\varepsilon(t)) - \mathcal{G}_\varepsilon(u_\varepsilon(t)) = f(t), \quad t > 0, \\ u_\varepsilon(0) = \widetilde{u}. \end{cases} \tag{6.1}$$

We study the asymptotic behavior of solutions u^ε of (6.1) as $\varepsilon \to 0$ adapting the notion of two-scale convergence to functions depending on the time variable t which is treated as a parameter. Namely, following [11], we say that

$$\begin{gathered}\text{a sequence } v_\varepsilon = v_\varepsilon(x,t), \text{ bounded in } L^2(\Omega\times[0,T]),\\ \text{two-scale converges to } V_0(x,y,t) \text{ if}\\ \int\limits_0^T\int\limits_\Omega v_\varepsilon\phi(x,x/\varepsilon,t)\,\mathrm{d}x\mathrm{d}t \to \int\limits_0^T\int\limits_Y\int\limits_\Omega V_0\phi(x,y,t)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t,\\ \text{for any } Y\text{-periodic in } y \text{ function } \phi(x,y,t)\in C^\infty(\Omega\times Y\times[0,T]).\end{gathered} \tag{6.2}$$

The basic properties of the convergence (6.2) are similar to those of the standard two-scale convergence. Namely, any bounded in $L^2(\Omega\times[0,T])$ sequence has a subsequence converging in the sense of (6.2); if $\|v_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega))} \leqslant C$ then, up to extracting a subsequence, v_ε and Dv_ε converge in the sense of (6.2) to V_0 and $DV_0(x,t)+D_yV_1(x,y,t)$ correspondingly, where $V_0\in L^2(0,T;W^{1,2}(\Omega))$ and $V_1\in L^2([0,T]\times\Omega;W^{1,2}_{\mathrm{per}}(Y))$. Note, however, that (6.2) does not imply, in general, that $v_\varepsilon(\,\cdot\,,t)$ converges in two-scale sense for a.e. $t\in[0,T]$, but rather

$$\int\limits_\alpha^\beta v_\varepsilon\mathrm{d}t \to \int\limits_\alpha^\beta V_0\mathrm{d}t \quad \text{two scale for all } 0\leqslant\alpha<\beta\leqslant T.$$

6.1 Well-posedness of the problem (6.1)

Given $T>0$, let us show that the problem (6.1) has a unique solution on the time interval $[0,T]$. For this purpose, we first note that the operator $\mathcal{A}_\varepsilon(u)-\mathcal{G}_\varepsilon(u)+\widetilde{\lambda}u$ becomes monotone if one chooses a suitable $\widetilde{\lambda}>0$ (depending on ε). Indeed, using (2.4), we get

$$\begin{aligned}\langle\mathcal{G}_\varepsilon(u)-\mathcal{G}_\varepsilon(v),u-v\rangle_\varepsilon &\leqslant C\int\limits_{S_\varepsilon}|u-v|^2\mathrm{d}\sigma\\ &\leqslant \varkappa/2\|D(u-v)\|^2_{L^2(\Omega_\varepsilon)}+\Gamma_\varepsilon\|u-v\|^2_{L^2(\Omega_\varepsilon)} \quad \forall\, u,v\in W^{1,2}(\Omega_\varepsilon),\end{aligned} \tag{6.3}$$

where $\varkappa$ is the constant appearing in (2.1) and Γ_ε is independent of u_ε and v_ε (the last inequality in (6.3) is due to the compactness of the trace operator $T_\varepsilon : W^{1,2}(\Omega_\varepsilon)\to L^2(S_\varepsilon)$, $T_\varepsilon w=$ trace of w on S_ε). Then, setting $\widetilde{\lambda}=\Gamma_\varepsilon+1$ and using (2.1) and (6.3), it is easy to verify that

$$\text{the operator } u\mapsto\mathcal{A}_\varepsilon(u)-\mathcal{G}_\varepsilon(u)+\widetilde{\lambda}u \text{ is monotone.} \tag{6.4}$$

Changing the unknown $v_\varepsilon=e^{-\widetilde{\lambda}t}u_\varepsilon$, we reduce the problem (6.1) to the evolution problem for the equation

$$\partial_t v_\varepsilon(t)+\widetilde{\mathcal{A}}_\varepsilon(v_\varepsilon(t),t)-\widetilde{\mathcal{G}}_\varepsilon(v_\varepsilon(t),t)+\widetilde{\lambda}v_\varepsilon=e^{-\widetilde{\lambda}t}f(t),\quad t>0,$$

with the initial condition $v_\varepsilon(0)=\widetilde{u}$, where $\widetilde{\mathcal{A}}_\varepsilon : v\mapsto e^{-\widetilde{\lambda}t}\mathcal{A}_\varepsilon(e^{\widetilde{\lambda}t}v)$ and $\widetilde{\mathcal{G}}_\varepsilon : v\mapsto e^{-\widetilde{\lambda}t}\mathcal{G}_\varepsilon(e^{\widetilde{\lambda}t}v)$. By the standard theory of parabolic problems for monotone operators (cf., for example, [22]), from (6.4), (2.1), and (6.3) it follows that the latter problem has a unique solution on $[0,T]$ as far as $f\in L^2([0,T];X^*_\varepsilon)$ and $\widetilde{u}\in L^2(\Omega)$.

6.2 Uniform a priori bounds

Let us show that for any $T > 0$ the solution u_ε of (6.1) satisfies the following bounds for sufficiently small ε:

$$\|\partial_t u_\varepsilon\|^2_{L^2(0,T;X^*_\varepsilon)}, \|u_\varepsilon\|^2_{L^2(0,T;X_\varepsilon)} \leqslant C(\langle \widetilde{u}, \widetilde{u}\rangle_\varepsilon + \|f\|^2_{L^2(0,T;X^*_\varepsilon)} + 1) \tag{6.5}$$

with a constant C independent of ε. Let ε_0, λ_0 be as in Theorem 2.1. By (6.1), for $\varepsilon \leqslant \varepsilon_0$

$$\begin{aligned}\langle u_\varepsilon(t), u_\varepsilon(t)\rangle_\varepsilon + 2\int_0^t \langle \mathcal{A}_\varepsilon(u_\varepsilon(\tau)) + \mathcal{G}_\varepsilon(u_\varepsilon(\tau)) + \lambda_0 u_\varepsilon(\tau), u_\varepsilon(\tau)\rangle_\varepsilon \mathrm{d}\tau \\ = \langle \widetilde{u}, \widetilde{u}\rangle_\varepsilon + 2\int_0^t \langle f(\tau) + \lambda_0 u_\varepsilon(\tau), u_\varepsilon(\tau)\rangle_\varepsilon \mathrm{d}\tau.\end{aligned} \tag{6.6}$$

Then (6.6), combined with (4.18), yields

$$\begin{aligned}\langle u_\varepsilon(T'), u_\varepsilon(T')\rangle_\varepsilon + 2\varkappa_1\|u_\varepsilon\|^2_{L^2(0,T';X_\varepsilon)} \leqslant \langle \widetilde{u}, \widetilde{u}\rangle_\varepsilon + \|f\|_{L^2(0,T';X^*_\varepsilon)}\|u_\varepsilon\|_{L^2(0,T';X_\varepsilon)} \\ + 2T'\varkappa_2 + 2\lambda_0\int_0^{T'} \langle u_\varepsilon(t), u_\varepsilon(t)\rangle_\varepsilon \mathrm{d}t \quad \forall\, 0 \leqslant T' \leqslant T.\end{aligned} \tag{6.7}$$

Therefore,

$$\langle u_\varepsilon(T'), u_\varepsilon(T')\rangle_\varepsilon \leqslant e^{2\lambda_0 T'}(\langle \widetilde{u}, \widetilde{u}\rangle_\varepsilon + \frac{1}{\varkappa_1}\|f\|^2_{L^2(0,T';X^*_\varepsilon)} + 2T'\varkappa_2). \tag{6.8}$$

Combined with (6.7), this implies the second bound in (6.5); while

$$\|\partial_t u_\varepsilon\|_{L^2(0,T;X^*_\varepsilon)} \leqslant \|\mathcal{A}_\varepsilon(u_\varepsilon)\|_{L^2(0,T;X^*_\varepsilon)} + \|\mathcal{G}_\varepsilon(u_\varepsilon)\|_{L^2(0,T;X^*_\varepsilon)} + \|f\|_{L^2(0,T;X^*_\varepsilon)},$$

and thus the first bound in (6.5) is a consequence of the second one and (4.17).

6.3 Homogenization of the problem (6.1)

Let u_ε be continued in x variable onto Ω by using the extension operator P_ε. Then the resulting function, still denoted u_ε, satisfies

$$\begin{aligned}&\|u_\varepsilon(t)\|_{L^2(\Omega)} \leqslant C \quad \forall\, t \in [0, T], \\ &\|u_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega))} \leqslant C,\end{aligned} \tag{6.9}$$

with a constant C independent of ε. This implies that, up to extracting a subsequence,

$$u_\varepsilon \to U_0(x,t) \text{ two-scale (in the sense of (6.2)) and weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{6.10}$$

$$D_x u_\varepsilon \to D_x U_0(x,t) + D_y U_1(x,y,t) \text{ two-scale (in the sense of (6.2))}, \tag{6.11}$$

where $U_0 \in L^2(0,T;W^{1,2}(\Omega))$ and $U_1 \in L^2(0,T;L^2(\Omega;W^{1,2}_{\text{per}}(Y)))$. Furthermore, if we set $\widehat{u}_\varepsilon = u_\varepsilon$ for $x \in \Omega_\varepsilon$ and $\widehat{u}_\varepsilon = 0$ for $x \in \Omega \setminus \Omega_\varepsilon$, then (6.10) yields that $\widehat{u}_\varepsilon \to |Y^*|U_0(x,t)$ weakly in $L^2(0,T;L^2(\Omega))$.

Let $X = W^{1,2}(\Omega)$, and let X^* be its dual with respect to the duality pairing

$$\langle u, v\rangle = |Y^*| \int\limits_{\Omega} uv \mathrm{d}x.$$

Show that $U_0 \in W^{1,2}(0,T;X^*)$ and $\widehat{u}_\varepsilon(t) \to |Y^*|U_0(t)$ weakly in $L^2(\Omega)$ for all $0 \leqslant t \leqslant T$. From (6.10) we have, for any $\phi \in X$ and $\varphi \in C_0^\infty([0,T])$,

$$\int\limits_0^T \langle \partial_t u_\varepsilon, \phi\rangle_\varepsilon \varphi(t)\mathrm{d}t = -\int\limits_0^T \langle u_\varepsilon, \phi\rangle_\varepsilon \varphi'(t)\mathrm{d}t \to -\int\limits_0^T \langle U_0, \phi\rangle \varphi'(t)\mathrm{d}t. \tag{6.12}$$

On the other hand, using (6.5), we get

$$\left|\int\limits_0^T \langle \partial_t u_\varepsilon, \phi\rangle_\varepsilon \varphi(t)\mathrm{d}t\right|^2 \leqslant C\int\limits_0^T \|\phi\|_{X_\varepsilon}^2 |\varphi(t)|^2 \mathrm{d}t \leqslant C\|\varphi\phi\|^2_{L^2(0,T;X)}. \tag{6.13}$$

Then (6.12) and (6.13) show that $U_0 \in W^{1,2}(0,T;X^*)$. By (6.8), the norms $\|\widehat{u}_\varepsilon(t)\|_{L^2(\Omega)}$ are uniformly in $0 < \varepsilon \leqslant \varepsilon_0$ and $t \in [0,T]$ bounded. Thus, to prove that $\widehat{u}_\varepsilon(t) \to |Y^*|U_0(t)$ weakly in $L^2(\Omega)$ for every $t \in [0,T]$, it suffices to show that

$$\langle u_\varepsilon(t), \phi\rangle_\varepsilon \to \langle U_0(t), \phi\rangle \quad \forall\, \phi \in X. \tag{6.14}$$

By the first bound in (6.5),

$$|\langle u_\varepsilon(t) - u_\varepsilon(t'), \phi\rangle_\varepsilon| \leqslant C|t-t'|^{1/2}\|\phi\|_X.$$

On the other hand, (6.14) holds in the sense of weak star convergence in $L^\infty(0,T)$ since $\widehat{u}_\varepsilon \to |Y^*|U_0(x,t)$ weakly in $L^2(0,T;L^2(\Omega))$. Thus, (6.14) holds for any $t \in [0,T]$, so that $\widehat{u}_\varepsilon(t) \to |Y^*|U_0(t)$ weakly in $L^2(\Omega)$ for all $t \in [0,T]$. In particular,

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \langle u_\varepsilon(T), u_\varepsilon(T)\rangle_\varepsilon \\
&\quad = \liminf_{\varepsilon\to 0} \int\limits_{\Omega_\varepsilon} \left((u_\varepsilon(T) - U_0(T))^2 - U_0^2(T)\right)\mathrm{d}x + 2\lim_{\varepsilon\to 0}\int\limits_{\Omega} \widehat{u}_\varepsilon(T)U_0(T)\,\mathrm{d}x \\
&\quad = \liminf_{\varepsilon\to 0} \int\limits_{\Omega_\varepsilon} (u_\varepsilon(T) - U_0(T))^2\,\mathrm{d}x + \langle U_0(T), U_0(T)\rangle \geqslant \langle U_0(T), U_0(T)\rangle
\end{aligned} \tag{6.15}$$

and, clearly,

$$\langle u_\varepsilon(T), v_\varepsilon\rangle_\varepsilon \to \langle U_0(T), V_0\rangle \text{ for any sequence } v_\varepsilon \to V_0 \text{ strongly in } L^2(\Omega). \tag{6.16}$$

Lemma 6.1. *If (u_ε) is a (sub)sequence of solutions of (6.1) such that (6.10) holds, then*

$$\|u_\varepsilon - U_0\|_{L^2(\Omega\times[0,T])} \to 0 \text{ as } \varepsilon \to 0. \tag{6.17}$$

Proof. By (6.10), it suffices to establish the (relative) compactness of (u_ε) in $L^2(\Omega\times[0,T])$. This is achieved by constructing a sequence of compact sets K_k $(k=1,2,\dots)$ in $L^2(\Omega\times[0,T])$ such that

$$\lim_{k\to\infty}\limsup_{\varepsilon\to 0}\ \mathrm{dist}_{L^2(\Omega\times[0,T])}(u_\varepsilon,K_k)=0.$$

Let $0=\omega_\varepsilon^{(0)}<\omega_\varepsilon^{(1)}\leqslant\ldots\leqslant\omega_\varepsilon^{(j)}\leqslant\ldots$ be the spectrum of the Neumann eigenvalue problem

$$\begin{cases} -\Delta\phi=\omega\phi \quad \text{in } \Omega_\varepsilon,\\ \dfrac{\partial\phi}{\partial\nu}=0 \quad \text{on } \partial\Omega_\varepsilon. \end{cases}$$

The eigenfunctions $\phi_\varepsilon^{(j)}$ can be chosen to form an orthogonal basis for $L^2(\Omega_\varepsilon)$. Then

$$u_\varepsilon(t)=\sum_{j=0}^{\infty} f_\varepsilon^{(j)}(t)P_\varepsilon\phi_\varepsilon^{(j)},$$

where $f_\varepsilon^{(j)}(t)=\langle u_\varepsilon(t),\phi_\varepsilon^{(j)}\rangle_\varepsilon$. Moreover, $\phi_\varepsilon^{(j)}/(\omega_\varepsilon^{(j)}+1)^{1/2}$ form an orthonormal basis for $X_\varepsilon(=W^{1,2}(\Omega_\varepsilon)$. Hence

$$\sum_{j=0}^{\infty}(1+\omega_\varepsilon^{(j)})\int_0^T|f_\varepsilon^{(j)}(t)|^2\mathrm{d}t=\|u_\varepsilon\|^2_{L^2(0,T;X_\varepsilon)}\leqslant\|u_\varepsilon\|^2_{L^2(0,T;X)}\leqslant C. \tag{6.18}$$

It is well known that $\omega_\varepsilon^{(k)}\to\omega^{(k)}$ as $\varepsilon\to 0$, where $0=\omega^{(0)}<\omega^{(1)}\leqslant\ldots\leqslant\omega^{(j)}\leqslant\ldots$ is the discrete spectrum of a homogenized problem. By the first bound in (6.5), we have

$$|f_\varepsilon^{(j)}(t)-f_\varepsilon^{(j)}(t')|\leqslant C|t-t'|^{1/2}\|\phi_\varepsilon^{(j)}\|_{X_\varepsilon}=C|t-t'|^{1/2}(1+\omega_\varepsilon^{(j)})^{1/2}$$

for all $t,t'\in[0,T]$. It follows that, for every k fixed, the sequence

$$(u_\varepsilon^{(k)}:=\sum_{j=0}^{k} f_\varepsilon^{(j)}(t)P_\varepsilon\phi_\varepsilon^{(j)})$$

is in a bounded closed subset K_k of $C^{1/2}([0,T];X)$. It is clear that K_k is a compact set in $L^2(\Omega\times[0,T])$. On the other hand, due to the properties of the extension operator P_ε,

$$\|u_\varepsilon-u_\varepsilon^{(k)}\|^2_{L^2(\Omega\times[0,T])}\leqslant C\int_0^T\|u_\varepsilon-u_\varepsilon^{(k)}\|^2_{L^2(\Omega_\varepsilon)}\mathrm{d}t=C\sum_{j=k+1}^{\infty}\int_0^T|f_\varepsilon^{(j)}(t)|^2\mathrm{d}t.$$

Therefore, in view of (6.18),

$$\begin{aligned}\limsup_{\varepsilon\to 0}\ \mathrm{dist}_{L^2(\Omega\times[0,T])}(u_\varepsilon,K_k)&\leqslant\limsup_{\varepsilon\to 0}\|u_\varepsilon-u_\varepsilon^{(k)}\|_{L^2(\Omega\times[0,T])}\\&\leqslant C/\omega^{(k+1)}\to 0\quad\text{as } k\to\infty.\end{aligned}$$

□

Now, we set $V_0(x,t) \in C^\infty(\overline{\Omega} \times [0,T])$ and $V_1(x,y,t) \in C^\infty(\overline{\Omega} \times \overline{Y} \times [0,T])$ with $V_1(x,y,t)$ being Y-periodic in y. Setting $v_\varepsilon = V_0(x,t) + \varepsilon V_1(x, x/\varepsilon, t)$ an, using the test function $w_\varepsilon = u_\varepsilon - v_\varepsilon$ in (6.1), we obtain

$$\frac{1}{2}\langle u_\varepsilon(T), u_\varepsilon(T)\rangle_\varepsilon - \frac{1}{2}\langle \widetilde{u}, \widetilde{u}\rangle_\varepsilon - \langle u_\varepsilon(T), v_\varepsilon(T)\rangle_\varepsilon + \langle \widetilde{u}, v_\varepsilon(0)\rangle_\varepsilon + \int_0^T \langle u_\varepsilon(t), \partial_t v_\varepsilon(t)\rangle_\varepsilon \mathrm{d}t$$
$$+ \int_0^T \langle \mathcal{A}_\varepsilon(u_\varepsilon(t)), w_\varepsilon(t)\rangle_\varepsilon \mathrm{d}t - \int_0^T \langle \mathcal{G}_\varepsilon(u_\varepsilon(t)), w_\varepsilon(t)\rangle_\varepsilon \mathrm{d}t = \int_0^T \langle f(t), w_\varepsilon(t)\rangle_\varepsilon \mathrm{d}t. \tag{6.19}$$

Using (6.10) and (6.15), (6.16), we can take $\liminf_{\varepsilon\to 0}$ for various terms in (6.19) to get

$$\liminf_{\varepsilon\to 0}\Big(\frac{1}{2}\langle u_\varepsilon(T), u_\varepsilon(T)\rangle_\varepsilon - \frac{1}{2}\langle \widetilde{u}, \widetilde{u}\rangle_\varepsilon - \langle u_\varepsilon(T), v_\varepsilon(T)\rangle_\varepsilon + \langle \widetilde{u}, v_\varepsilon(0)\rangle_\varepsilon$$
$$+ \int_0^T (\langle u_\varepsilon(t), \partial_t v_\varepsilon(t)\rangle_\varepsilon - \langle f(t), w_\varepsilon(t)\rangle_\varepsilon)\mathrm{d}t\Big)$$
$$\geqslant \frac{1}{2}\langle U_0(T), U_0(T)\rangle - \frac{1}{2}\langle \widetilde{u}, \widetilde{u}\rangle - \langle U_0(T), V_0(T)\rangle + \langle \widetilde{u}, V_0(0)\rangle$$
$$+ \int_0^T (\langle U_0(t), \partial_t V_0(t)\rangle - \langle f(t), U_0(t) - V_0(t)\rangle)\mathrm{d}t. \tag{6.20}$$

By (6.11), we also have

$$\lim_{\varepsilon\to 0}\int_0^T \langle \mathcal{A}_\varepsilon(v_\varepsilon(t)), w_\varepsilon(t)\rangle_\varepsilon \mathrm{d}t$$
$$= \int_0^T \int_\Omega \int_{Y^*} a(D_x V_0 + D_y V_1, y) \cdot (D_x U_0 + D_y U_1 - D_x V_0 - D_y V_1)\mathrm{d}y\mathrm{d}x\mathrm{d}t. \tag{6.21}$$

Let us show that

$$\int_0^T \langle \mathcal{G}_\varepsilon(u_\varepsilon), u_\varepsilon - v_\varepsilon\rangle_\varepsilon \mathrm{d}t \to \int_0^T M(U_0, U_1, V_0, V_1)\mathrm{d}t \text{ as } \varepsilon \to 0, \tag{6.22}$$

where $M(U_0, U_1, V_0, V_1)$ is given by (3.12) (or, equivalently, by the right-hand side of (3.7)). The proof of (6.22) follows closely the arguments at the end of Section 5 (proof of (3.7)). In place of Proposition 3.1, we make use now of

Proposition 6.2. *Assume that $q(t,x,y) \in C([0,T] \times \Omega; L^\infty(S))$ satisfies*

(a) $|q(t,x,y) - q(t',x',y)| \leqslant C(|x-x'| + |t-t'|)$ *with $C > 0$ independent of $x, x' \in \Omega$, $t, t' \in [0,T]$ and $y \in S$,*

(b) *$q(t,x,y)$ is Y-periodic in $y \in S$,*

(c) $\int\limits_{Y\cap S} q(t,x,y)\mathrm{d}\sigma_y = 0$ *for all* $x \in \Omega$, $t \in [0,T]$.

Then, given $w_\varepsilon \in L^2(0,T;W^{1,2}(\Omega))$ *such that* $w_\varepsilon \to W_0$, $D_x w_\varepsilon(x,t) \to D_x W_0(x,t) + D_y W_1(x,y,t)$ *two scale (in the sense of* (6.2)*) as* $\varepsilon \to 0$, *we have*

$$\int\limits_0^T \int\limits_{S_\varepsilon} q(t,x,x/\varepsilon)(w_\varepsilon - \overline{w}_\varepsilon)\mathrm{d}\sigma\mathrm{d}t \to \int\limits_0^T \int\limits_\Omega \int\limits_{Y\cap S} q(t,x,y)(D_x W_0 \cdot y + W_1)\mathrm{d}\sigma_y\mathrm{d}x\mathrm{d}t. \tag{6.23}$$

Proof. We set $0 = t_0^{(n)} < \cdots < t_j^{(n)} = Tj/n < \cdots < t_n^{(n)} = T$ and $\Delta_j^{(n)} = (t_{j-1}^{(n)}, t_j^{(n)})$. Using (4.3) and the Lipschitz continuity of $q(t,x,y)$ in t, we obtain

$$\begin{aligned}\int\limits_0^T \int\limits_{S_\varepsilon} q(t,x,x/\varepsilon)(w_\varepsilon - \overline{w}_\varepsilon)\mathrm{d}\sigma\mathrm{d}t &= \sum_{j=1}^n \int\limits_{\Delta_j^{(n)}} \int\limits_{S_\varepsilon} q(t,x,x/\varepsilon)(w_\varepsilon - \overline{w}_\varepsilon)\mathrm{d}\sigma\mathrm{d}t \\ &= \sum_{j=1}^n \int\limits_{S_\varepsilon} q(t_j^{(n)},x,x/\varepsilon) \int\limits_{\Delta_j^{(n)}} (w_\varepsilon - \overline{w}_\varepsilon)\mathrm{d}t\mathrm{d}\sigma + r_\varepsilon^{(n)}\end{aligned} \tag{6.24}$$

with

$$|r_\varepsilon^{(n)}| \leqslant \frac{C}{n} \int\limits_0^T \int\limits_{S_\varepsilon} |w_\varepsilon - \overline{w}_\varepsilon|\mathrm{d}\sigma\mathrm{d}t \leqslant \frac{C}{n} \int\limits_0^T \|w_\varepsilon\|_{W^{1,2}(\Omega)}\mathrm{d}t. \tag{6.25}$$

Setting

$$W_\varepsilon = \int\limits_{\Delta_j^{(n)}} w_\varepsilon \mathrm{d}t$$

and applying Proposition 3.1, we get

$$\lim_{\varepsilon\to 0} \int\limits_{S_\varepsilon} q(t_j^{(n)},x,x/\varepsilon)(W_\varepsilon - \overline{W}_\varepsilon)\mathrm{d}t\mathrm{d}\sigma = \int\limits_{\Delta_j^{(n)}} \int\limits_\Omega \int\limits_{Y\cap S} q(t_j^{(n)},x,y)(D_x W_0 \cdot y + W_1)\mathrm{d}\sigma_y\mathrm{d}x\mathrm{d}t. \tag{6.26}$$

If we pass to the limit (along a subsequence) as $\varepsilon \to 0$ in (6.24) and send n to ∞ in the resulting relation, then, by (6.25) and (6.26), we obtain (6.23). $\square$

Proof of (6.22) (continued). By Lemma 4.1 and (6.9), we have

$$\begin{aligned}\int\limits_0^T \langle \mathcal{G}_\varepsilon(u_\varepsilon), u_\varepsilon - v_\varepsilon\rangle_\varepsilon \mathrm{d}t &= \int\limits_0^T \int\limits_{S_\varepsilon} g(\overline{u}_\varepsilon, x/\varepsilon)(u_\varepsilon - v_\varepsilon - \overline{u}_\varepsilon + \overline{v}_\varepsilon)\mathrm{d}\sigma\mathrm{d}t \\ &+ \int\limits_0^T \int\limits_{S_\varepsilon} g'_u(\overline{u}_\varepsilon, x/\varepsilon)(\overline{u}_\varepsilon - \overline{v}_\varepsilon)(u_\varepsilon - \overline{u}_\varepsilon)\mathrm{d}\sigma\mathrm{d}t + O(\varepsilon^{2/(N+2)}).\end{aligned}$$

Then, assuming that $u_\delta^{(1)} \in C^1(\overline{\Omega} \times [0,T])$ is such that $\|U_0 - u_\delta^{(1)}\|_{L^2(\Omega\times[0,T])} \leqslant \delta$ and using Lemma 4.2, Lemma 6.1 (convergence of u_ε to U_0 in $L^2(\Omega \times [0,T])$), continuity properties of $g(u,y)$ and $g'_u(u,y)$ in u (the conditions (2.3), (2.4), and (2.5)), (4.3), and the second bound in (6.9), we obtain

$$\lim_{\varepsilon\to 0}\int_0^T \langle \mathcal{G}_\varepsilon(u_\varepsilon), u_\varepsilon - v_\varepsilon\rangle_\varepsilon \mathrm{d}t = \lim_{\varepsilon\to 0}\int_0^T\int_{S_\varepsilon}\Big(g(u_\delta^{(1)}, x/\varepsilon)(u_\varepsilon - v_\varepsilon - \overline{u}_\varepsilon + \overline{v}_\varepsilon)$$
$$+ g'_u(u_\delta^{(1)}, x/\varepsilon)(u_\delta^{(1)} - V_0)(u_\varepsilon - \overline{u}_\varepsilon)\Big)\mathrm{d}\sigma\mathrm{d}t + O(\delta) \qquad (6.27)$$

provided that the limits exist. Using Proposition 6.2, we identify the limits on the right-hand side of (6.27) and then obtain (6.22) by passing to the limit $\delta \to 0$. □

Now, by the monotonicity of the operator $\mathcal{A}_\varepsilon(u)$, we can take $\liminf_{\varepsilon\to 0}$ in (6.19) to obtain, by virtue of (6.20), (6.21), and (6.22),

$$\int_0^T (\langle \partial_t U_0(t), U_0(t) - V_0(t)\rangle - \langle f(t), U_0(t) - V_0(t)\rangle)\mathrm{d}t$$
$$+ \int_0^T\int_\Omega\int_{Y^*} a(D_x V_0 + D_y V_1, y)\cdot(D_x U_0 + D_y U_1 - D_x V_0 - D_y V_1)\mathrm{d}y\mathrm{d}x\mathrm{d}t$$
$$- \int_0^T M(U_0, U_1, V_0, V_1)\mathrm{d}t \leqslant 0. \qquad (6.28)$$

This inequality is shown for any $V_0(x,t) \in C^\infty(\overline{\Omega} \times [0,T])$, $V_1(x,y,t) \in C^\infty(\overline{\Omega} \times \overline{Y} \times [0,T])$ (Y-periodic in y) since an approximation argument it still holds for any $V_0 \in L^2(0,T; W^{1,2}(\Omega))$, $V_1 \in L^2(0,T; L^2(\Omega; W^{1,2}_{\mathrm{per}}(Y)))$. We set $V_0 = U_0$, $V_1 = U_1 \pm \delta\phi(x,t)w(y)$, where $w \in W^{1,2}_{\mathrm{per}}(Y)$, $\phi \in C^\infty(\overline{\Omega} \times [0,T])$, $\delta > 0$ are arbitrary, divide (6.28) by δ, and pass to the limit as $\delta \to 0$:

$$\int_0^T\int_\Omega\left(\int_{Y^*} a(D_x U_0 + D_y U_1, y)\cdot D_y w\mathrm{d}y - \int_{S\cap Y} g(U_0, y)w\mathrm{d}\sigma_y\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t = 0. \qquad (6.29)$$

This means that U_1 solves (2.13) with $u = U_0$ and $\xi = D_x U_0$ for almost all $(x,t) \in \Omega \times [0,T]$. Now, we set $V_0 = U_0 \pm \delta\Phi(x,t)$ and $V_1 = U_1$, where $\Phi \in C^\infty(\overline{\Omega} \times [0,T])$ and $\delta > 0$ are arbitrary, divide (6.28) by δ, and pass to the limit as $\delta \to 0$. As a result, we obtain

$$|Y^*|\int_0^T\int_\Omega \partial_t U_0(x,\tau)\Phi(x,\tau)\mathrm{d}x\mathrm{d}\tau$$
$$+ \int_0^T\int_\Omega (a^*(D_x U_0, U_0)\cdot D_x\Phi - b^*(D_x U_0, U_0)\Phi - \mathrm{div}\,_x(g^*(U_0)\Phi))\mathrm{d}x\mathrm{d}\tau$$
$$= |Y^*|\int_0^T\int_\Omega f(x,t)\Phi(x,\tau)\mathrm{d}x\mathrm{d}\tau, \qquad (6.30)$$

which yields (1.4).

7 Properties of the Homogenized Problem

Define the operators $\mathcal{A}^*, \mathcal{B}^*, \mathcal{T}^* : X \to X^*$ by the formulas

$$\mathcal{B}^*(u) = b^*(Du, u),$$

$$\langle \mathcal{A}^*(u), v\rangle = \int_\Omega a^*(Du, u) \cdot Dv \mathrm{d}x \quad \forall\, v \in X,$$

$$\langle \mathcal{T}^*(u), v\rangle = \int_{\partial\Omega} g^*(u) \cdot \nu\, v \mathrm{d}\sigma = \int_\Omega \operatorname{div}(g^*(u)v)\mathrm{d}x \quad \forall\, v \in X.$$

In terms of the operator $\mathcal{F}^*(u) = \mathcal{A}^*(u) - \mathcal{B}^*(u) - \mathcal{T}^*(u)$, the problems (1.3) and (1.4) read

$$\mathcal{F}^*(u) + \lambda u = f, \tag{7.1}$$

$$\begin{cases} \partial_t u + \mathcal{F}^*(u) = f, & t > 0, \\ u = \widetilde{u}, & t = 0. \end{cases} \tag{7.2}$$

According to Theorem 2.2, there is a solution (obtained as the limit of solutions of (1.1)) of (7.1) for every $f \in L^2(\Omega)$. Similarly, by Theorem 2.6, the problem (7.2) has a solution on the time interval $[0, T]$ when $f \in L^2(\Omega \times [0, T])$ and $\widetilde{u} \in L^2(\Omega)$. The solvability of the problems (7.1) and (7.2) can be proved for more general f; namely, we can assume merely $f \in X^*$ and $f \in L^2(0, T; X^*)$ in (7.1) and (7.2) respectively. However, we focus on the uniqueness results.

7.1 Properties of a^* and b^*

Lemma 7.1. *The functions a^* and b^* given by* (2.10) *and* (2.11) *are continuous. Moreover, there are constants $\gamma, \alpha, r > 0$ and C such that*

$$a^*(\xi, u) \cdot \xi \geqslant \gamma|\xi|^2 - C(|u|^2 + 1) \text{ and } |a^*(\xi, u)| \leqslant C(|\xi| + |u| + 1), \tag{7.3}$$

$$(a^*(\xi, u) - a^*(\zeta, v)) \cdot (\xi - \zeta) \geqslant \alpha|\xi - \zeta|^2 - r(u - v)^2, \tag{7.4}$$

$|b^*(\xi, u)| \leqslant C(|\xi| + |u| + 1)$, *and*

$$\begin{aligned}(b^*(\xi, u) - b^*(\zeta, v))(v - u) \leqslant{} & \frac{1}{4}(a^*(\xi, u) - a^*(\zeta, v)) \cdot (\xi - \zeta)) \\ & + C\big(|u - v|^2 + |u - v|^2(|\xi| + |u| + 1)/(1 + |u - v|)\big).\end{aligned} \tag{7.5}$$

The proof of this lemma is based on the study of properties of the solutions $w(y; \xi, u)$ of the problem (2.13). We make use of the following well known results:

$$\int_{S\cap Y} \Big|w - \frac{1}{|Y^*|}\int_{Y^*} w \mathrm{d}x\Big|^2 \mathrm{d}\sigma \leqslant C \int_{Y^*} |Dw|^2 \,\mathrm{d}x, \tag{7.6}$$

$$\int_{Y^*} |D_y w + \xi|^2 \,\mathrm{d}y \geqslant \rho|\xi|^2, \quad \rho > 0, \tag{7.7}$$

for all $\xi \in \mathbb{R}^N$ and $w \in W^{1,2}_{\mathrm{per}}(Y^*)$, where C and ρ are independent of w and ξ.

Lemma 7.2. *For any $\xi \in \mathbb{R}^N$ and $u \in \mathbb{R}$ there is a unique (modulo an additive constant) solution $w(y;\xi,u)$ of the problem* (2.13) *and*

(a) $\displaystyle\int_{Y^*} |D_y w(y;\xi,u)|^2 \mathrm{d}y \leqslant C(|\xi|^2 + |u|^2 + 1)$,

(b) $a^*(\xi,u)\cdot\xi \geqslant \gamma|\xi|^2 - C(|u|\,|\xi| + |u|^2 + 1)$ *(with $\gamma > 0$)*,

(c) *there are $\alpha, \beta > 0$ and r such that, for any $\xi, \zeta \in \mathbb{R}^N$ and $u, v \in \mathbb{R}$*

$$(a^*(\xi,u) - a^*(\zeta,v))\cdot(\xi-\zeta) \geqslant \alpha|\xi-\zeta|^2 - r(u-v)^2 + \beta\int_{Y^*}|D\widehat{w}|^2\mathrm{d}y,$$

where $\widehat{w} = w(y;\xi,u) - w(y;\zeta,v)$,

(d) $w(y;\zeta,v) \to w(y;\xi,u)$ *strongly in $W^{1,2}_{\mathrm{per}}(Y^*)\setminus\mathbb{R}$ as $\zeta\to\xi$ and $v\to u$.*

Proof. The existence of a unique solution of (2.13) in $W^{1,2}_{\mathrm{per}}(Y^*)\setminus\mathbb{R}$ easily follows from assumptions (i)–(iii) and (vi) on the functions a and g. To show (a), we derive from (2.13) by integrating by parts

$$\int_{Y^*} a(\xi+Dw,y)\cdot(\xi+Dw)\mathrm{d}y = \int_{S\cap Y} g(u,y)w\mathrm{d}\sigma + \int_{Y^*} a(\xi+Dw,y)\cdot\xi\mathrm{d}y. \tag{7.8}$$

Applying the Poincaré inequality (7.6) and using (2.6), (2.3), for any $k > 0$ we obtain

$$\begin{aligned}\int_{Y^*} a(\xi+Dw,y)\cdot(\xi+Dw)\mathrm{d}y &\leqslant C(|u|+1)\|Dw\|_{L^2(Y^*)} + C|\xi|\,\|\xi+Dw\|_{L^2(Y^*)}\\ &\leqslant C(|u|+1)(\|\xi+Dw\|_{L^2(Y^*)}+|\xi|) + C|\xi|\,\|\xi+Dw\|_{L^2(Y^*)}\\ &\leqslant k((|u|+1)^2 + \frac{C}{k}(|\xi|^2+\|\xi+Dw\|^2_{L^2(Y^*)}),\end{aligned} \tag{7.9}$$

where C is independent of k, u, and ξ. Choosing k in (7.9) large enough and using (2.2), we get

$$\int_{Y^*}|\xi+Dw|^2\mathrm{d}y \leqslant C(|u|^2+|\xi|^2+1),$$

which, in turn, implies (a).

Using (7.7) on the left-hand side of (7.8) and (7.6) in conjunction with (2.3), (2.6) in the first term on the right-hand side, we easily derive (b).

In order to show (c), we use (2.13) to get by integrating by parts

$$\begin{aligned}(a^*(\xi,u) - a^*(\zeta,v))\cdot(\xi-\zeta) &= \int_{S\cap Y}(g(v,y)-g(u,y))\widehat{w}\mathrm{d}\sigma\\ &+\int_{Y^*}(a(\xi+D_yw(y;\xi,u)) - a(\zeta+D_yw(y;\zeta,v)))\cdot(\xi-\zeta+D_y\widehat{w})\mathrm{d}y.\end{aligned} \tag{7.10}$$

Taking into account (2.4), (2.6) and applying (7.6), we can estimate the first term I_1 on the right-hand side of (7.10) as follows:

$$|I_1| \leqslant k|u-v|^2 + \frac{C}{k}\int\limits_{Y^*}|D\widehat{w}|^2\mathrm{d}y \quad \text{for any } r>0, \tag{7.11}$$

where C is independent of k, ξ, ζ, u, and v. In view of (2.1) and (7.7), we have the following lower bound for the second term I_2 in (7.10):

$$I_2 \geqslant (1-\delta)\varkappa\rho|\xi-\zeta|^2 + \delta\varkappa\int\limits_{Y^*}|\xi-\zeta+D_y\widehat{w}|^2\mathrm{d}y$$

with $0<\delta<1$ to be chosen later. On the other hand, by the elementary inequality $a^2 \leqslant 2(a+b)^2+2b^2$,

$$\int\limits_{Y^*}|D_y\widehat{w}|^2\mathrm{d}y \leqslant 2\int\limits_{Y^*}|\xi-\zeta+D_y\widehat{w}|^2\mathrm{d}y + 2|\xi-\zeta|^2.$$

Thus,

$$I_2 \geqslant \varkappa(\rho-\delta(\rho+1))|\xi-\zeta|^2 + \frac{\delta\varkappa}{2}\int\limits_{Y^*}|D_y\widehat{w}|^2\mathrm{d}y.$$

Choose $0<\delta<1$ so that $\rho-\delta(\varkappa+1)>0$ and set $k=4C/(\delta\varkappa)$ (where C is the constant appearing in (7.11)). We thus obtain (b) with $\alpha=\varkappa(\rho-\delta(\rho+1))>0$ and $\beta=(\delta\varkappa)/4>0$.

Finally, statement (d) is a direct consequence of (a) and (c). □

Proof of Lemma 7.1. According to Lemma 7.2, it suffices to show (7.5). We set $\widehat{w} = w(y;\xi,u)-w(y;\zeta,v)$. Using (7.6) and assumptions (i), (iii), (iv) on g, we have

$$\begin{aligned}
&(b^*(\xi,u)-b^*(\zeta,v))(v-u)\\
&= (v-u)\int\limits_{S\cap Y} g'_u(v,y)\widehat{w}\mathrm{d}\sigma_y + (v-u)\int\limits_{S\cap Y}(g'_u(u,y)-g'_u(v,y))w(y;\xi,u)\mathrm{d}\sigma_y\\
&\leqslant C|u-v|\|D\widehat{w}\|_{L^2(Y^*)} + C|u-v|^2\|Dw(\,\cdot\,;\xi,u)\|_{L^2(Y^*)}/(1+|u|+|v|).
\end{aligned} \tag{7.12}$$

Then statements (a) and (c) of Lemma 7.2 yield (7.5). □

Remark 7.3. In the case where the function $g(u,y)$ is linear in u, the bound (7.5) simplifies to the following one:

$$(b^*(\xi,u)-b^*(\zeta,v))(v-u) \leqslant \frac{1}{4}(a^*(\xi,u)-a^*(\zeta,v))\cdot(\xi-\zeta)) + C|u-v|^2.$$

Let us consider the particular case where $a(\xi,y)$ is linear in ξ, i.e., a is given by $a(\xi,y)=A(y)\xi$ with $A\in L^\infty(Y;\mathbb{R}^{N\times N})$, $A(y)\xi\cdot\xi \geqslant \varkappa|\xi|^2$ ($\varkappa>0$) for all $\xi\in\mathbb{R}^N$, $y\in Y$. Then we can write the solution of (2.13) as the sum $w(y;\xi,u)=w^{(1)}(y;\xi)+\widetilde{w}(y;u)$ with $w^{(1)}$ solving (2.13) and $\widetilde{w}$ being a unique (up to an additive constant) solution of the problem

$$\begin{cases}
\operatorname{div}(A(y)D_y\widetilde{w}) = 0 \quad \text{in } Y^*,\\
A(y)D_y\widetilde{w}\cdot\nu = g(u,y) \quad \text{on } S\cap Y,\\
\widetilde{w} \text{ is } Y\text{-periodic}.
\end{cases} \tag{7.13}$$

Note that $w^{(1)}(y;\xi)$ depends linearly on ξ. Also,

$$\|\widetilde{w}(y;u)\|_{W^{1,2}(Y^*)\setminus\mathbb{R}} \leqslant C(|u|+1),$$
$$\|\widetilde{w}(y;u)-\widetilde{w}(y;v\|_{W^{1,2}(Y^*)\setminus\mathbb{R}} \leqslant C|u-v|,$$
$$\|\widetilde{w}'_u(y;u)-\widetilde{w}'_u(y;v)\|_{W^{1,2)}(Y^*)\setminus\mathbb{R}} \leqslant C|u-v|/(1+|u|+|v|),$$

where C is independent of u and v. The proof of these bounds is analogous to that of (3.14)–(3.16). Thus,

$$b^*(\xi,u) = \frac{\partial}{\partial u}\int\limits_{Y^*} A(y)D_y\widetilde{w}(y;u)\cdot D_y w^{(1)}(y;\xi)\mathrm{d}y$$
$$+\int\limits_{Y^*} A(y)D_y\widetilde{w}'_u(y;u)\cdot D_y\widetilde{w}(y;u)\mathrm{d}y = H'(u)\cdot\xi + h(u) \tag{7.14}$$

with H and h such that $|H(u)-H(v)|\leqslant C|u-v|$ and $|h(u)-h(v)|\leqslant C|u-v|$.

7.2 Uniqueness results for the problem (7.1)

In the particular cases, where the dimension of the space $N\leqslant 3$ or $a(\xi,y)$ is linear in ξ, or $g(u,y)$ is linear in u, we show that the problem (7.1) cannot have two distinct solutions for sufficiently large λ.

The following inequality will be used to estimate the expressions involving traces on $\partial\Omega$. For every $\delta>0$ there is Λ_δ such that

$$\int\limits_{\partial\Omega} |w|^2\mathrm{d}\sigma \leqslant \delta\|Dw\|^2_{L^2(\Omega)} + \Lambda_\delta\|w\|_{L^2(\Omega)} \quad \forall\, w\in W^{1,2}(\Omega). \tag{7.15}$$

This inequality is a consequence of the compactness of the trace operator $T_{\partial\Omega}: W^{1,2}(\Omega)\to L^2(\partial\Omega)$, $T_{\partial\Omega}u=$ trace of u on $\partial\Omega$. By the Lipschitz continuity of $g(u,y)$ in the variable u, the inequality (7.15) implies

$$|\langle\mathcal{T}^*(u)-\mathcal{T}^*(v),u-v\rangle| \leqslant \frac{\alpha}{4}\|u-v\|^2_X + C\|u-v\|^2_{L^2(\Omega)}, \tag{7.16}$$

where $\alpha>0$ is the same as in (7.4).

Let u and v be solutions of (7.1).

Case I: *$g(u,y)$ is linear in u.* Using Lemma 7.1, Remark 7.3, and (7.16), we get

$$\langle\mathcal{F}^*(u)-\mathcal{F}^*(v)+\lambda(u-v),u-v\rangle \geqslant \frac{\alpha}{4}\|u-v\|^2_X + (\lambda-\widehat{\lambda}_0)\|u-v\|^2_{L^2(\Omega)} \tag{7.17}$$

with $\widehat{\lambda}_0$ independent of λ. It follows that $u=v$ if $\lambda\geqslant\widehat{\lambda}_0$.

Case II: *$a(\xi, y)$ is linear in ξ.* According to (7.14), we have

$$\langle B^*(u) - \mathcal{B}^*(v)), v - u\rangle = |Y^*| \int_\Omega (u - v)(\operatorname{div}(H(u) - H(v)) + h(u) - h(v))\mathrm{d}x$$
$$= |Y^*| \int_\Omega (D(v - u) \cdot (H(u) - H(v)) + (u - v)(h(u) - h(v)))\mathrm{d}x$$
$$+ |Y^*| \int_{\partial\Omega} (u - v)(H(u) - H(v)) \cdot \nu \,\mathrm{d}\sigma \leqslant \frac{\alpha}{4}\|u - v\|_X^2 + C\|u - v\|_{L^2(\Omega)}^2,$$

where we used (7.15). This inequality and Lemma 7.1 yield (7.17) (with possibly another constant $\widehat{\lambda}_0$).

Case III: *the space dimension N is equal to two or three.* It is well known that for these space dimensions $X(= W^{1,2}(\Omega))$ is compactly embedded into $L^4(\Omega)$; moreover,

$$\|w\|_{L^4(\Omega)}^2 \leqslant C\delta\|w\|_X^2 + C\delta^{-N/(4-N)}\|w\|_{L^2(\Omega)}^2$$

for all $w \in X$ and $\delta > 0$, where C is independent of $\delta > 0$ and w (cf., for example, [24]). Using this inequality, Lemma 7.1, and (7.16), we easily show that

$$\langle \mathcal{F}^*(u) - \mathcal{F}^*(v), u - v\rangle \geqslant \frac{\alpha}{4}\|u - v\|_X^2 - C(\delta\|u - v\|_X^2 + \delta^{-N/(4-N)}\|u - v\|_{L^2(\Omega)}^2)(\|u\|_X + 1) \quad \forall\, \delta > 0. \tag{7.18}$$

On the other hand, Lemma 7.1 and the very definition of $\mathcal{T}^*(u)$ imply that for every $w \in X$

$$\langle \mathcal{A}^*(w), w\rangle \geqslant \gamma\|w\|_X^2 - C(\|w\|_{L^2(\Omega)}^2 + 1),$$
$$|\langle \mathcal{B}^*(w), w\rangle| \leqslant C(\|w\|_X + \|w\|_{L^2(\Omega)} + 1)\|w\|_{L^2(\Omega)},$$
$$|\langle \mathcal{T}^*(w), w\rangle| \leqslant C\|w\|_X\|w\|_{L^2(\Omega)}.$$

Therefore, there is $\widetilde{\lambda}_0$ such that

$$\langle F^*(u), u\rangle \geqslant \frac{\gamma}{2}\|u\|_X^2 - \widetilde{\lambda}_0\langle u, u\rangle.$$

Hence for $\lambda \geqslant \widetilde{\lambda}_0$ we have the a priori bound

$$\|u\|_X \leqslant C(\|f\|_{X^*} + 1)$$

with C independent of u, f, and $\lambda \geqslant \widetilde{\lambda}_0$. Thus, u and v being solutions of (7.1), the estimate (7.18) yields

$$\frac{\alpha}{4}\|u - v\|_X^2 + \lambda\|u - v\|_{L^2(\Omega)}^2 \leqslant C(\|f\|_{X^*} + 1)(\delta\|u - v\|_X^2 + \delta^{-N/(4-N)}\|u - v\|_{L^2(\Omega)}^2),$$

and, setting $\delta = \alpha/(8C((\|f\|_{X^*} + 2))$, we get $u = v$ as far as $\lambda \geqslant \widehat{\lambda}_0$ $(= \max\{\widetilde{\lambda}_0, C(\|f\|_{X^*} + 1)\delta^{-N/(4-N)}\})$ ($\widehat{\lambda}_0$ can be chosen independent of f if $N = 2$).

7.3 Uniqueness results for the problem (7.2)

Given $T > 0$, we show that the problem (7.2) cannot have two distinct solutions u and v on the time interval $[0, T]$ if $a(\xi, y)$ is linear in ξ or $g(u, y)$ is linear in u. Indeed, $w = u - v$ satisfies

$$\partial_t \langle w(t), w(t)\rangle + 2\langle \mathcal{F}^*(u(t)) - \mathcal{F}^*(v(t)), u(t) - v(t)\rangle = 0, \quad 0 < t < T,$$

and $w(0) = 0$, while (7.17) yields

$$-2\langle \mathcal{F}^*(u(t)) - \mathcal{F}^*(v(t)), u(t) - v(t)\rangle \leqslant C\langle w(t), w(t)\rangle, \quad 0 < t < T.$$

Therefore, $e^{-Ct}\|w(t)\|^2_{L^2(\Omega)} \leqslant 0$ so that $w \equiv 0$.

In the case where the space dimension is equal to two, we also have a uniqueness result. Note that we have at least one solution $u \in L^2(0, T; X)$ of (7.2). Then, if v is another solution, we set $w = u - v$, $R(t) = \langle w(t), w(t)\rangle$ and derive, using (7.18) with $\delta = \alpha/(8C((\|u\|_X + 1))$,

$$R'(t) - CR(t)(\|u(t)\|_X + 1)^2 \leqslant 0, \quad 0 < t < T, \qquad \text{and} \;\; R(0) = 0.$$

This implies that

$$R(t) \exp\left\{ - C \int\limits_0^t (\|u(\tau)\|_X + 1)^2 \, \mathrm{d}\tau \right\} \leqslant 0$$

and, consequently, $R \equiv 0$, i.e., $u = v$.

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