Journal of Mathematical Sciences, Vol. 169, No. 2, 2010

HOMOGENIZATION OF THE SPECTRAL DIRICHLET PROBLEM FOR A SYSTEM OF DIFFERENTIAL EQUATIONS WITH RAPIDLY OSCILLATING COEFFICIENTS AND CHANGING SIGN DENSITY

S. A. Nazarov *

Institute of Problems in Mechanical Engineering of the Russian Academy of Sciences
61, Bolshoi pr. V.O., St. Petersburg 199178, Russia
serna@snark.ipme.ru,srgnazarov@yahoo.co.uk

A. L. Pyatnitskii

The Lebedev Physical Institute of the Russian Academy of Sciences
53, Leninskiy pr., Moscow 119333, Russia

Narvik Institute of Technology
N-8501 Narvik, Norway
andrey@sci.lebedev.ru

UDC 517.956.8

We study the asymptotic behavior of the spectrum of the Dirichlet problem for a formally selfadjoint elliptic system of differential equations with rapidly oscillating coefficients and changing sign density ρ. Since the factor ρ at the spectral parameter changes sign, the problem possesses two – positive and negative – infinitely large sequences of eigenvalues. Their asymptotic structure essentially depends on whether the mean $\overline{\rho}$ over the periodicity cell vanishes. In particular, in the case $\overline{\rho} = 0$, the homogenized problem becomes a quadratic pencil. Bibliography: 20 titles.

1. Statements of the Problem and Description of the Results

1. Spectral problem. Let Ω be a domain in the Euclidean space $\mathbb{R}^n$ with the smooth boundary $\partial\Omega$ of class $C^{2,\delta}$, $\delta \in (0,1)$ and compact closure $\overline{\Omega} = \Omega \cup \partial\Omega$. We consider spectral Dirichlet problem for the following formally selfadjoint system of second order differential equations (althoug we deal with a family of boundary value problems with parameter $\varepsilon \in (0, \varepsilon_0]$ but we often mention it as a single problem under the assumption that ε is small and fixed)

$$\mathscr{L}(\varepsilon^{-1}x, \nabla_x)u^\varepsilon(x) = \lambda^\varepsilon \rho(\varepsilon^{-1}x)u^\varepsilon(x), \quad x \in \Omega, \tag{1.1}$$

$$u^\varepsilon(x) = 0, \quad x \in \partial\Omega, \tag{1.2}$$

* To whom the correspondence should be addressed.

Translated from *Problems in Mathematical Analysis* **48**, July 2010, pp. 75–107

where λ^ε is the spectral parameter, ∇_x is the gradient, $u^\varepsilon = (u_1^\varepsilon, \dots, u_k^\varepsilon)^\top$ is a vector-valued function (a column, i.e., the symbol $\top$ means transposition) and $\varepsilon > 0$ is a small parameter. The real-valued density ρ and complex-valued coefficients of the matrix differential operator $\mathscr{L}$ are periodic[1)] functions of "fast" variables

$$y = (y_1, \dots, y_n)^\top = \varepsilon^{-1} x = (\varepsilon^{-1} x_1, \dots, \varepsilon^{-1} x_n)^\top. \tag{1.3}$$

Let us describe the structure of the $(k \times k)$–matrix $\mathscr{L}(y, \nabla_y)$ of the differential operator in detail. Let $\mathscr{D}(\nabla_y)$ be the $(K \times k)$–matrix of homogeneous first order differential operators with constant complex-valued coefficients, and let $\mathscr{A}$ be a Hermitian positive definite $(K \times K)$–matrix-valued function for all $x \in \Omega$. Suppose that the matrix $\mathscr{D}(\xi)$ is *algebraically complete* [**1**], i.e., there exists a natural number $\sigma_{\mathscr{D}}$ such that for any row $P(\xi) = (P_1(\xi), \dots, P_k(\xi))$ of homogeneous polynomials of degree $\sigma \geqslant \sigma_{\mathscr{D}}$ in the variables $\xi = (\xi_1, \dots, \xi_n)^\top$ there is a row of polynomials $Q(\xi) = (Q_1(\xi), \dots, Q_K(\xi))$ such that the following identity holds:

$$P(\xi) = Q(\xi)\mathscr{D}(\xi), \quad \xi \in \mathbb{R}^n. \tag{1.4}$$

In this case, the formally selfadjoint differential matrix operator

$$\mathscr{L}(y, \nabla_y) = \overline{\mathscr{D}(-\nabla_y)}^\top \mathscr{A}(y) \mathscr{D}(\nabla_y) \tag{1.5}$$

is called [**1**] *formally positive* and is elliptic (cf. [**1**]–[**3**]). Furthermore, it possesses the *polynomial property* [**2, 3**]

$$a(u, u; \Xi) = 0, \;\; u \in C^1(\overline{\Xi})^k \quad \Leftrightarrow \quad u \in \mathscr{P}\Big|_{\Xi}, \tag{1.6}$$

where Ξ is an arbitrary domain in $\mathbb{R}^n$, $\mathscr{P}$ is a finite-dimensional lineal of vector polynomials and a is the sesquilinear form

$$a(u, v; \Xi) = (\mathscr{A}\mathscr{D}(\nabla_x)u, \mathscr{D}(\nabla_x)v)_\Xi, \tag{1.7}$$

where $(\, , \,)_\Xi$ denotes the inner product in the Lebesgue space[2)] $L_2(\Xi)^K$. Based on (1.4), it is easy to check that

$$\mathscr{P} = \left\{ p = (p_1, \dots, p_k)^\top : \; \mathscr{D}(\nabla_x)p(y) = 0 \in \mathbb{C}^K, y \in \mathbb{R}^n \right\}, \tag{1.8}$$

and the degrees of scalar polynomials p_j do not exceed $\sigma_{\mathscr{D}} - 1$ (cf. [**3**] for details).

We set

$$\mathscr{A}^\varepsilon(x) = \mathscr{A}(x/\varepsilon), \qquad \mathscr{L}^\varepsilon(x, \nabla_x) = \mathscr{L}(\varepsilon^{-1}x, \nabla_x)$$

and

$$a^\varepsilon(u, v; \Xi) = (\mathscr{A}^\varepsilon \mathscr{D}(\nabla_x)u, \mathscr{D}(\nabla_x)v)_\Xi. \tag{1.9}$$

The variational setting of the problem (1.1), (1.2) is to find a nontrivial vector-valued function u^ε in the Sobolev space $\mathring{H}^1(\Omega)^k$ (the symbol $^\circ$ indicates that the Dirichlet condition (1.2) is used) and numbers $\lambda^\varepsilon \in \mathbb{C}$ for which the following integral identity holds (cf. [**4**]):

$$a^\varepsilon(u^\varepsilon, v; \Omega) = \lambda^\varepsilon (\rho^\varepsilon u^\varepsilon, v)_\Omega, \quad v \in \mathring{H}^1(\Omega)^k, \tag{1.10}$$

---

1) Throughout the paper, without loss of generality we assume that the period in each of the variables y_j, $j = 1, 2, \dots, n$, is equal to 1.

2) The superscript K indicates the number of components of a vector-valued function, but we omit it in the notation of inner products and norms.

where ρ^ε denotes the function $x \mapsto \rho(\varepsilon^{-1}x)$. An eigenvalue λ^ε and the corresponding vector-valued eigenfunction u^ε form an *eigenpair* $\{u^\varepsilon, \lambda^\varepsilon\}$ – a solution to the spectral problem (1.1), (1.2) (or (1.10)).

In the general statement (1.10), it suffices to assume that the entries of the matrix $\mathscr{A}$ and the density ρ are bounded measurable functions. However, to justify the homogenization procedure, we need the higher smoothness of coefficients:

$$\mathscr{A} \in C^{1,\alpha}_{\rm per}(\overline{S})^{K\times K},$$

where $S = (0,1)^n$ is the cubic periodicity cell and $C^{1,\alpha}_{\rm per}(\overline{S})$, $\alpha \in (0,1)$, is the Hölder class of periodic functions on S, equipped with the norm

$$\|v; C^{1,\alpha}_{\rm per}(\overline{S})\| = \sup_{y\in S}\left(|v(y)| + |\nabla_y v(y)|\right) + \sup_{y,\eta\in S}\left(|y-\eta|^{-\alpha}|\nabla_y v(y) - \nabla_\eta v(\eta)|\right).$$

The density ρ is assumed to be bounded and measurable, i.e.,

$$\rho \in L_\infty(\Omega), \qquad \|\rho; L_\infty(S)\| = \operatorname{ess\,sup}\left\{|\rho(y)| : \, y \in S\right\}.$$

We also need the space $W^1_{\infty,\rm per}(S)$ of periodic functions v equipped with the norm

$$\|v; W^1_{\infty,\rm per}(S)\| = \|v; L_\infty(S)\| + \|\nabla_x v; L_\infty(S)\|.$$

The main goal of this paper is to derive and justify asymptotic formulas for spectral pairs of the problem under consideration.

2. Some special problems in mathematical physics. We consider several examples of systems of differential equations that possess the above-listed properties.

Example 1.1. Let $k = 1$, $K = n$ and $\mathscr{D}(\nabla_x) = \nabla_x$. Then $\mathscr{L}(y, \nabla_y) = -\nabla_x^\top \mathscr{A}(y)\nabla_y$ is a scalar divergence operator, $\sigma_{\mathscr{D}} = 1$, and $\mathscr{P} = \mathbb{C}$. □

Example 1.2. In the matrix form (not tensor form; cf., for example, [**5, 6**]), the operator $\mathscr{L}$ of the three-dimensional ($n = 3$) system of linearized equations of elasticity governing strains of an anisotropic inhomogeneous body (composite) is formed by the rapidly oscillating matrix $\mathscr{A}(\varepsilon^{-1}x)$ of elastic moduli and the (6×3)-matrix $\mathscr{D}$:

$$\mathscr{D}(\xi)^\top = \begin{pmatrix} \xi_1 & 0 & 0 & 0 & 2^{-1/2}\xi_3 & 2^{1/2}\xi_2 \\ 0 & \xi_2 & 0 & 2^{-1/2}\xi_3 & 0 & 2^{-1/2}\xi_1 \\ 0 & 0 & \xi_3 & 2^{-1/2}\xi_2 & 2^{-1/2}\xi_1 & 0 \end{pmatrix}. \tag{1.11}$$

It is easy to verify the algebraic completeness (1.1) of the matrix; in particular, $k = 3$, $K = 6$ and $\sigma_{\mathscr{D}} = 2$. The polynomial property (1.6) contains the lineal

$$\mathscr{P} = \{p(x) = d(x)a : \, a = (a_1, a_2, a_3, a_4, a_5, a_6)^\top \in \mathbb{R}^6\},$$

$$d(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 2^{-1/2}x_3 & -2^{1/2}x_2 \\ 0 & 1 & 0 & -2^{-1/2}x_3 & 0 & 2^{-1/2}x_1 \\ 0 & 0 & 1 & 2^{-1/2}x_2 & -2^{-1/2}x_1 & 0 \end{pmatrix}.$$

The vectors $d(x)a$ are rigid displacements, translations ($a_4 = a_5 = a_6 = 0$) and rotations ($a_1 = a_2 = a_3 = 0$). Owing to the factors $2^{-1/2}$, the strain column

$$\mathscr{D}(\nabla_x)u = \Big(\varepsilon_{11}(u), \varepsilon_{22}(u), \varepsilon_{33}(u), 2^{1/2}\varepsilon_{23}(u), 2^{1/2}\varepsilon_{31}(u), 2^{1/2}\varepsilon_{12}(u)\Big)^\top \tag{1.12}$$

(cf. [**6**, Chapter 2]) has the same natural norm as the strain tensor $(\varepsilon_{jk}(u))_{j,k=1}^3$ with Cartesian components

$$\varepsilon_{jk}(u) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right).$$

The column $\mathscr{A}\mathscr{D}(\nabla_x)u$ of structure similar to (1.12) contains the components of stress tensor. The matrix $\mathscr{A}$ connecting the stress and strain columns is called the rigidity matrix or the Hooke matrix. According to the nature of an elastic medium, this matrix is symmetric and positive definite. □

Example 1.3. Suppose that $n = 3$, $k = 4$, $K = 9$,

$$\mathscr{D}(\xi)^\top = \begin{pmatrix} \xi_1 & 0 & 0 & 0 & 2^{-1/2}\xi_3 & 2^{-1/2}\xi_2 & 0 & 0 & 0 \\ 0 & \xi_2 & 0 & 2^{-1/2}\xi_3 & 0 & 2^{-1/2}\xi_1 & 0 & 0 & 0 \\ 0 & 0 & \xi_3 & 2^{-1/2}\xi_2 & 2^{-1/2}\xi_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 \end{pmatrix}, \tag{1.13}$$

and the matrix $\mathscr{A}$ has the form

$$\mathscr{A} = \begin{pmatrix} \mathscr{A}_{(1,1)} & \mathscr{A}_{(1,2)} \\ \mathscr{A}_{(2,1)} & -\mathscr{A}_{(2,2)} \end{pmatrix}, \tag{1.14}$$

moreover, $\mathscr{A}_{(1,1)}$ and $\mathscr{A}_{(2,2)}$ are positive definite matrix-valued functions of size 6×6 and 3×3 and $\mathscr{A}_{(1,2)} = \mathscr{A}_{(2,1)}^\top$ is a (6×3)–matrix. The differential operator (1.5) is used for describing a piezoelectric medium with rapidly oscillating properties: the first three elements of u are elastic displacements, and the fourth one is the electric potential taken with the opposite sign (cf., for example, [**7, 8**]). The matrix (1.13) is obtained by adding the row $(0, 0, 0, 0, 0, 0, \xi_1, \xi_2, \xi_3)$ to the (3×6)–matrix (1.11) and, consequently, possesses the property (1.6). At the same time, the matrix (1.14) does not possess this property because of the "wrong" sign at the right lower block. In particular, the operator (1.5) is not formally positive. At the same time, the statement of the problem about harmonic oscillations of a piezoelectric medium involves the system (1.1) with the right-hand side

$$\lambda^\varepsilon R(\varepsilon^{-1}x)u^\varepsilon(x),$$

where $R(y) = \operatorname{diag}\{\rho(y), \rho(y), \rho(y), 0\}$ is a diagonal matrix. In other words, the fourth row of the system does not contain the spectral parameter. It is shown in [**3**, Example 1.13] and [**9**], how the spectral problem is reduced to the form admitting an analysis by methods used in this paper. □

3. Operator statement of the problem. By the Gording inequality (cf. Lemma 2.1 below) and the positive definiteness of the matrix $\mathscr{A}$, the Hermitian sesquilinear form (1.9) can be taken for the inner product in the Hilbert space $\mathscr{H} = \mathring{H}^1(\Omega)^k$:

$$\langle u, v\rangle = (\mathscr{A}^\varepsilon \mathscr{D}(\nabla_x)u, \mathscr{D}(\nabla_x)v)_\Omega. \tag{1.15}$$

In this space, we introduce a linear compact symmetric (consequently, selfadjoint) operator $\mathscr{K}^\varepsilon$ by the formula

$$\langle \mathscr{K}^\varepsilon u, v\rangle = (\rho^\varepsilon u, v)_\Omega, \quad u, v \in \mathscr{H}. \tag{1.16}$$

Replacing the spectral parameter

$$\mu^\varepsilon = (\lambda^\varepsilon)^{-1} \tag{1.17}$$

we pass from the problem (1.10) to the abstract spectral equation

$$\mathscr{K}^\varepsilon u^\varepsilon = \mu^\varepsilon u^\varepsilon \in \mathscr{H}.$$

If ρ is a nontrivial nonnegative real function, then the operator $\mathscr{K}^\varepsilon$ is positive and, by [**10**, Theorem 9.2.1], its spectrum is concentrated on the segment $[0, k^\varepsilon]$ of the real axis; moreover, the point $\mu = 0$ belongs to the essential spectrum, whereas the half-interval $(0, k^\varepsilon]$ contains the discrete spectrum; here, k^ε denotes the norm of the operator $\mathscr{K}^\varepsilon$.

In this paper, we study the problem (1.10) with changing sign density, i.e.,

$$\operatorname*{ess\,sup}_{y\in(0,1)^n} \rho(y) > 0, \qquad \operatorname*{ess\,inf}_{y\in(0,1)^n} \rho(y) < 0. \tag{1.18}$$

Proposition 1.4. *Under the assumption* (1.18), *the spectrum of the operator* $\mathscr{K}^\varepsilon$ *is contained in* $[-k^\varepsilon, k^\varepsilon]$; *moreover, the point* $\mu = 0$ *belongs to its essential spectrum and* $[-k^\varepsilon, 0) \cup (0, k^\varepsilon]$ *contains the discrete spectrum consisting of the following two infinitely small sequences of eigenvalues, positive and negative*:

$$\mu^\varepsilon_{+,1} \geqslant \mu^\varepsilon_{+,2} \geqslant \ldots \geqslant \mu^\varepsilon_{+,j} \geqslant \cdots \to +0, \tag{1.19}$$

$$\mu^\varepsilon_{-,1} \leqslant \mu^\varepsilon_{-,2} \leqslant \ldots \leqslant \mu^\varepsilon_{-,j} \leqslant \cdots \to -0, \tag{1.20}$$

where the eigenvalues are enumerated with their multiplicity taken into account.

Proof. Since $\mathscr{K}^\varepsilon$ is a compact operator, its essential spectrum coincides with the point $\mu = 0$ [**10**, Theorem 9.2.1]. We show that positive and negative eigenvalues form infinite sequences which necessarily converge to zero. For this purpose, we apply the minimum principle. In particular, we have

$$\mu^\varepsilon_{-1} = \min_{\langle u,u\rangle=1} (\rho^\varepsilon u, u), \tag{1.21}$$

where the minimum is taken over all vector-valued functions $u \in \mathring{H}^1(\Omega)^k$ normalized by the equality $\langle u, u\rangle = 1$. By (1.18), we have

$$\mu^\varepsilon_{-1} < 0.$$

Indeed, it suffices to substitute into the right-hand side of (1.21) a smoothed characteristic function of the set $\{x \in \Omega : \rho^\varepsilon(x) < 0\}$, normalized and multiplied by the number column $a \in \mathbb{C}^k$, $|a| = 1$. Certainly, the support of the kernel of the smoothnig operator should be taken small (recall that if the diameter of support decreases, then the smoothed functions converge to the original function in the class L_2). As is known, μ^ε_{-1} is a point of the discrete spectrum of the operator $\mathscr{K}^\varepsilon$ and there is a vector-valued function $u^\varepsilon_{(-1)} \in \mathring{H}^1(\Omega)^k$ such that

$$\mathscr{K}^\varepsilon u^\varepsilon_{(-1)} = \mu^\varepsilon_{-1} u^\varepsilon_{(-1)}$$

and the minimum in (1.21) is attained at this function. Now, using again the minimum principle, we find

$$\mu_{-2}^{\varepsilon} = \min_{\langle u,u\rangle=1,\ \langle u,u^{\varepsilon}_{(-1)}\rangle=0} (\rho^{\varepsilon} u, u),$$

where the minimum is taken over all vector-valued functions $u \in \overset{\circ}{H}{}^1(\Omega)^k$ such that $\langle u, u\rangle = 1$ and $\langle u, u^{\varepsilon}_{(-1)}\rangle = 0$. It is easy to check that for any collection of functions $u_{(1)}, \dots, u_{(m)} \in \overset{\circ}{H}{}^1(\Omega)^k$ the linear set

$$\Big\{u \in \overset{\circ}{H}{}^1(\Omega)^k \ :\ \langle u, u_{k)}\rangle = 0,\ k = 1, \dots, m;\ (\rho^{\varepsilon} u, u) < 0\Big\}$$

is nonempty. Therefore,

$$\mu_{-2}^{\varepsilon} < 0.$$

Continuing the procedure, we find a sequence of eigenvalues

$$\mu_{-j}^{\varepsilon} < 0$$

and the corresponding vector-valued eigenfunctions

$$u^{\varepsilon}_{(-j)} \in \overset{\circ}{H}{}^1(\Omega)^k, \quad j = 1, 2, \dots.$$

By construction,

$$\mu_{-1}^{\varepsilon} \leqslant \mu_{-2}^{\varepsilon} \leqslant \mu_{-3}^{\varepsilon} \cdots$$

and

$$\mu_{-j}^{\varepsilon} < 0 \quad \text{for all } j.$$

By the compactness of the operator $\mathscr{K}^{\varepsilon}$,

$$\lim_{j\to\infty} \mu_{-j}^{\varepsilon} = 0$$

for every $\varepsilon > 0$.

The existence of an infinite sequence of positive eigenvalues is established in a similar way. For example, the operator $\mathscr{K}^{\varepsilon}$ is replaced with the operator $-\mathscr{K}^{\varepsilon}$. □

Proposition 1.4 and the relation (1.17) between the spectral parameters yield the following assertion.

Proposition 1.5. *In the case* (1.18), *the problem* (1.1), (1.2) (*more exactly, its variational setting* (1.10)) *has a discrete spectrum splitting into two infinitely large sequences*

$$0 < \lambda_{+1}^{\varepsilon} \leqslant \lambda_{+2}^{\varepsilon} \leqslant \ldots \leqslant \lambda_{+j}^{\varepsilon} \leqslant \cdots \to +\infty, \tag{1.22}$$

$$0 > \lambda_{-1}^{\varepsilon} \geqslant \lambda_{-2}^{\varepsilon} \geqslant \ldots \geqslant \lambda_{-j}^{\varepsilon} \geqslant \cdots \to -\infty. \tag{1.23}$$

The corresponding vector-valued eigenfunctions

$$u^{\varepsilon}_{(j)} \in \overset{\circ}{H}{}^1(\Omega)^k$$

can be subject to the orthogonality and normalization condition

$$\langle u^{\varepsilon}_{(j)}, u^{\varepsilon}_{(l)}\rangle = \delta_{j,l}, \quad j, l \in \mathbb{N} = \{1, 2, \dots\}, \tag{1.24}$$

where $\langle\, ,\, \rangle$ *is the inner product* (1.15) *and* $\delta_{j,l}$ *is the Kronecker symbol.*

4. Structure of the paper and description of the results. In Section 2, we present several auxiliary results. Formal asymptotic expansions of spectral pairs for the problem (1.1), (1.2) are constructed in Section 3 and are justified in Section 4.

To imagine the asymptotic behavior of eigenvalues $\lambda^\varepsilon_{\pm j}$ as $\varepsilon \to +0$, it suffices to consider the first two eigenvalues $\lambda^\varepsilon_{\pm 1}$ of the sequences (1.22), (1.23) since they are the endpoints of the *main interval* Υ^ε where the problem is uniquely solvable, and this interval is free from the spectrum and a priori contains the point $\lambda = 0$ because the operator (1.5) is formally positive. It turns out that the size and location of Υ^ε depend on the mean of the density ρ over the periodicity cell $S = (0,1)^n$

$$\overline{\rho} = \int\limits_S \rho(y)\,dy. \tag{1.25}$$

Thus, if $\overline{\rho} = 0$, the length of the unique solvability interval is of order $O(\varepsilon^{-1})$, and this interval is symmetric with respect to the point $\lambda = 0$, which corresponds to the following asymptotic formulas for eigenvalues:

$$\lambda^\varepsilon_{\pm 1} = \varepsilon^{-1}(\beta_{\pm 1} + o(1)), \tag{1.26}$$

where $\pm\beta_{\pm 1} > 0$ (cf. Subsection 3.2 and Theorem 4.1). In the case $\overline{\rho} > 0$, the length of Υ^ε is of order $O(\varepsilon^{-2})$ and the interval is considerably displaced towards $-\infty$, which is determined by the following relations for eigenvalues:

$$\lambda^\varepsilon_{+1} = \lambda^0_{+1} + o(1),\ \lambda^0_{+1} > 0, \quad \lambda^\varepsilon_{-1} \leqslant -c\varepsilon^{-2} \tag{1.27}$$

(cf. Subsection 3.1 and Theorems 4.4 and 4.6). Certainly, for $\overline{\rho} < 0$ the roles of the eigenvalues λ^ε_{+1} and λ^ε_{-1} in (1.27) are exchanged and the interval Υ^ε is displaced towards $+\infty$.

Formulas (1.27) and (1.26) are determined by the asymptotic ansätze for eigenpairs of the problem (1.1), (1.2). In the case $\overline{\rho} > 0$, procedures for constructing and justifying asymptotic expansions for the eigenvalues λ^ε_{+j} is similar to known procedures of the homogenization theory (cf., for example, [**11**]–[**13**] for scalar problems and [**14, 15**] for elasticity systems of equations). Thus, the second term of the ansatz for vector-valued eigenfunctions is the standard asymptotic corrector $\varepsilon N(\varepsilon^{-1}x)\mathscr{D}(\nabla_x)u^0_{+j}(x)$ (N is a periodic solution to the system (3.7)) and, by the estimate (4.36), the eigenvalues λ^ε_{+j} converge to the eigenvalues λ^0_{+j} of the homogenized problem with density (1.25) and differential operator

$$\mathbf{L}(\nabla_x) = \overline{\mathscr{D}(-\nabla_x)}^\top \mathbf{A}\mathscr{D}(\nabla_x), \tag{1.28}$$

where $\mathbf{A}$ is the constant Hermitian positive definite $(K \times K)$–matrix defined by formula (3.10) below for $\mathscr{A}$ and N.

Turning to the case $\overline{\rho} = 0$ we should revise even the main asymptotic ansätze; namely, formula (1.26) will involve a large factor ε^{-1} and the asymptotic corrector

$$\varepsilon\left(N(\varepsilon^{-1}x)\mathscr{D}(\nabla_x)u^0_{\pm j}(x) + \beta_{\pm j}N^0(\varepsilon^{-1}x)u^0_{\pm j}(x)\right)$$

will obtain an additional term that contains the spectral parameter of the homogenized problem and the periodic solution N^0 to the system of differential equations (3.17) with the right-hand side $\rho\mathbb{I}$ on the periodicity cell. The homogenized problem will be also modified: the following quadratic pencil appears:

$$\beta \ \mapsto\ \mathbf{L}(\nabla_x) - \beta\mathbf{s}\mathscr{D}(-\nabla_x) + \beta\overline{\mathscr{D}(\nabla_x)}^\top\mathbf{s}^\top - \beta^2\mathbf{m},$$

where $\mathbf{L}(\nabla_x)$ is the differential operator (1.28), $\mathbf{s}$ is a $(k \times K)$–matrix, and $\mathbf{m}$ is a $(k \times k)$–matrix (cf. formulas (3.20) and (3.19)). The Dirichlet problem for this pencil is studied in Subsection 3.2 (cf. Theorem 3.4). In particular, we show that this problem has a real discrete spectrum with two condensation points $\pm\infty$. We emphasize that the justification of asymptotics in the case $\overline{\rho} = 0$ also requires new ideas (cf. Subsections 4.1–4.3).

In the case $\overline{\rho} > 0$, the question on the asymptotic structure of the negative part (positive for $\overline{\rho} < 0$) of the spectrum is still open. Theorem 4.6 yields an upper estimate for the eigenvalues $\lambda^{\varepsilon}_{-j}$ (cf. the second relation in (1.26)); we attempt to construct asymptotics of these eigenvalues in Subsection 3.3. However, this attempt turns out to be unsuccessful because of the impossibility of studying the spectrum of a formally homogenized problem.

2. Auxiliaries

1. The Korn inequality and the Gording inequality. To study the spectral problem (1.10), we will use the following known assertions. Their proof is simple, and we reproduce it for the sake of convenience.

Lemma 2.1. *For any vector-valued function $v \in \overset{o}{H}{}^1(\Omega)^k$ the Gording inequality holds*:

$$\|v; H^1(\Omega)\| \leqslant c_{\mathscr{D}} \|\mathscr{D}(\nabla_x)v; L_2(\Omega)\|, \tag{2.1}$$

where the constant $c_{\mathscr{D}}$ depends only on the matrix $\mathscr{D}$.

Proof. We extend v by zero from Ω to the entire space $\mathbb{R}^n$. Using the Fourier transform $v(x) \mapsto \widehat{v}(\xi)$ and the Parseval equality, we find

$$\|\mathscr{D}(\nabla_x)v; L_2(\Omega)\|^2 = C \int\limits_{\mathbb{R}^n} |\mathscr{D}(\xi)\widehat{v}(\xi)|^2 \, d\xi. \tag{2.2}$$

Let us check that

$$|\mathscr{D}(\xi)a|^2 \geqslant c_{\mathscr{D}} |\xi|^2 |a|^2, \quad a \in \mathbb{C}^k, \ \xi \in \mathbb{R}^n, \quad c_{\mathscr{D}} > 0. \tag{2.3}$$

Then we obtain (2.1) by using the Friedrichs inequality with formulas (2.2), (2.3) and applying the inverse Fourier transform.

Assume that the inequality (2.3) fails and there exist nonzero $a^0 \in \mathbb{C}^k$ and $\xi^0 \in \mathbb{R}^n$ such that

$$\mathscr{D}(\xi^0)a^0 = 0.$$

We set $P(\xi) = (a^0)^\top |\xi|^{2\sigma_{\mathscr{D}}}$ in (1.4) and multiply from the right by the column a^0. For $\xi = \xi^0$ we have

$$|a^0|^2 |\xi^0|^{2\sigma_{\mathscr{D}}} = Q(\xi^0)\mathscr{D}(\xi^0)a^0 = 0,$$

i.e., either $a^0 = 0$ or $\xi^0 = 0$. The obtained contradiction proves the inequality (2.3). □

We consider the problem on a periodicity cell[3)] in the cube $S = (0,1)^n$:

$$a(U, V; S) = F(V), \quad V \in H^1_{\text{per}}(S)^k, \tag{2.4}$$

where $H^1_{\text{per}}(S)$ is the subspace of $H^1(S)$ obtained as the closure of the lineal of y-periodic functions and F is a linear functional on the subspace $H^1_{\text{per}}(S)^k$ of periodic vector-valued functions.

To study the problem (2.4), we need the following assertion [**1**, Theorem 3.7.6].

[3)] Using a suitable affine transformation of coordinates, it is possible to transform an arbitrary cell to a cubic cell. We assume that the necessary replacement were already made for (1.2), (1.3).

Lemma 2.2. *For any vector-valued functions $v \in H^1(S)^k$ the following generalized Korn inequality holds*:

$$\|\nabla_y V; L_2(S)\| \leqslant c\,(\|\mathscr{D}(\nabla_y)V; L_2(S)\| + \|V; L_2(S)\|), \tag{2.5}$$

where the constant c depends only on the matrix $\mathscr{D}$.

Proposition 2.3. 1) *Let $F \in \big(H^1_{\mathrm{per}}(S)^k\big)^*$ be a linear functional on the space $H^1_{\mathrm{per}}(S)^k$ that degenerates on constant vector-valued functions, i.e.,*

$$F(V) = 0, \quad V \in \mathbb{C}^k. \tag{2.6}$$

Then the problem (2.4) *has a solution $U \in H^1_{\mathrm{per}}(S)^k$, determined up to a constant summand in $\mathbb{C}^k$. Since the solution U satisfies the orthogonality condition*

$$\int\limits_S U(y)\,dy = 0 \in \mathbb{C}^k, \tag{2.7}$$

it is unique and satisfies the estimate

$$\|U; H^1(S)\| \leqslant c\,\Big\|F; \Big(H^1_{\mathrm{per}}(S)^k\Big)^*\Big\|. \tag{2.8}$$

2) *If the functional F is given by the equality*

$$F(V) = (\mathbf{F}, V)_S$$

and the vector-valued function $\mathbf{F} \in L_2(S)^k$ has zero mean over S, then the solution U mentioned in item 1) *belongs to the space $H^2_{\mathrm{per}}(S)^k$ and satisfies the estimate*

$$\|U; H^2(S)\| \leqslant c\|\mathbf{F}; L_2(S)\|. \tag{2.9}$$

3) *If $\mathbf{F} \in L_\infty(S)^k$ and the assumptions of item* 2) *are satisfied, then*

$$U \in W^1_{\infty,\mathrm{per}}(S)^k$$

and

$$\|U; W^1_{\infty,\mathrm{per}}(S)\| \leqslant c\|\mathbf{F}; L_\infty(S)\|. \tag{2.10}$$

Proof. 1) We consider the auxiliary problem

$$a(U, V; S) + \tau(U, V)_S = F(V), \quad V \in H^1_{\mathrm{per}}(S)^k, \tag{2.11}$$

under the assumption that $\tau \geqslant 0$. By the inequality (2.5), the left-hand side of (2.11) with $\tau > 0$ is the inner product in the space $H^1_{\mathrm{per}}(S)^k$. Therefore, by the Riesz theorem on representation of linear functionals in a Hilbert space, the problem (2.11) is uniquely solvable for $\tau > 0$. Since the embedding $H^1_{\mathrm{per}}(S) \subset L_2(S)$ is compact, the Fredholm alternative holds for the problem (2.11)–(2.4) with $\tau = 0$. Any solution U to the homogeneous ($F = 0$) problem (2.4) annuls the form (1.7), i.e., it belongs to the lineal of polynomials (1.8) in view of (1.6) and becomes a constant column in view of periodicity. The solvability condition (2.6) is the condition of orthogonality of the right-hand side and solutions to the homogeneous problem, whereas the condition (2.7) allows us to avoid the arbitrariness in the choice of a solution.

2) Let $h \in \mathbb{R}^n$. The function

$$\Delta_h U(y) = h^{-1}(U(y + h) - U(y))$$

satisfies the integral identity

$$(\mathscr{A}\mathscr{D}(\nabla_y)\Delta_h U, \mathscr{D}(\nabla_y)V)_S + (\Delta_h \mathscr{A}\mathscr{D}(\nabla_y)U(\cdot + h), \mathscr{D}(\nabla_y)V)_S = (\Delta_h \mathbf{F}, V)_S = (\mathbf{F}, \Delta_{-h}V)_S.$$

Since

$$|\Delta_h \mathscr{A}(y)| \leqslant \|\mathscr{A}; C^1(\overline{S})\|$$

and

$$\|\Delta_{-h}V; L_2(S)\| \leqslant \|\nabla_x V; L_2(S)\|,$$

the estimate (2.8) shows that

$$\|\Delta_h V; H^1(S)\| \leqslant c\,(\|\mathscr{D}(\nabla_y)U; L_2(S)\| + \|\mathbf{F}; L_2(S)\|),$$

where c is independent of h. It remains to pass to the limit of $h = \hbar e_j$, where $e_1, \dots, e_n$ is the standard basis for $\mathbb{R}^n$, as $\hbar \to 0$.

3) We first assume that

$$\mathscr{A} \in C^\infty_{\text{per}}(\overline{S})^{N\times N}$$

and

$$\mathbf{F} \in C^\infty_{\text{per}}(\overline{S})^k.$$

Consequently,

$$U \in C^\infty_{\text{per}}(\overline{S})^k$$

because the operator $\mathscr{L}$ is elliptic. We fix a point $y^0 \in S$ and denote by $\chi^0 \in C^\infty_c(\mathbb{R}^n)$ a cut-off function such that $\chi^0 = 1$ on the cube $\mathbb{Q}^0_{2\delta}$ with center y^0 and edge 4δ and $\chi^0 = 0$ outside the cube $\mathbb{Q}^0_{4\delta}$. For $U^0 = \chi^0 U$ we write out the following system of equations:

$$\begin{aligned} \mathscr{L}^0(\nabla_y)U^0(y) &:= \overline{\mathscr{D}(-\nabla_y)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_y)U^0(y) \\ &= \chi^0(y)\mathbf{F}(y) + \overline{\mathscr{D}(-\nabla_y)}^\top \big(\mathscr{A}(y^0) - \mathscr{A}(y)\big)\mathscr{D}(\nabla_y)U^0(y) \\ &\qquad + [\mathscr{L}(y, \nabla_y), \chi(y)]\,U(y) \\ &=: F^0(y) =: F^1(y) + F^2(y) + F^3(y), \quad y \in S. \end{aligned} \tag{2.12}$$

We note that vector-valued functions U^0 and F^0 can be smoothly extended by zero to the entire space $\mathbb{R}^n$.

Let Φ be the fundamental matrix for the operator $\mathscr{L}^0$ in $\mathbb{R}^n$. As is known (cf., for example, [**16**]), it has the form

$$\Phi(y) = \begin{cases} |y|^{2-n}\Phi'(|y|^{-1}y), & n \geqslant 3, \\ C\ln|y| + \Phi'(|y|^{-1}y), & n = 2, \end{cases} \tag{2.13}$$

where $C \in \mathbb{C}^{k\times k}$ and Φ' are smooth $(k\times k)$–matrix-valued functions on the sphere $\mathbb{S}^{n-1}$. Using the fundamental matrix, we can compute the solution U^0 and its derivatives at the point y^0:

$$\begin{aligned} U^0(y^0) &= \int\limits_{\mathbb{R}^n} \Phi(y^0 - y)F^0(y)\,dy, \\ \frac{\partial U^0}{\partial y_j}(y^0) &= -\int\limits_{\mathbb{R}^n} \frac{\partial \Phi}{\partial y_j}(y^0 - y)F^0(y)\,dy. \end{aligned} \tag{2.14}$$

Denote by I_0^p and I_j^p the integrals on the right-hand sides of (2.14) after the representation of F^0 via F^p by formula (2.12). By (2.13), the following estimates hold:

$$|I_0^1| + |I_j^1| + |I_0^2| \leqslant c\|F; L_\infty(S)\| \int_0^{\sqrt{n}} (r^{2-n} + r^{1-n}) r^{n-1}\, dr \leqslant c\|F; L_\infty(S)\|. \tag{2.15}$$

The support of the component F^3 lies in the set $\overline{\mathbb{Q}_{4\delta}^0} \setminus \mathbb{Q}_{2\delta}^0$, where $r = |y - y^0| \geqslant \delta$. Consequently,

$$|I_0^3| + |I_j^3| \leqslant c\|U; H^2(S)\|. \tag{2.16}$$

Integrating by parts and taking the commutator with a cut-off function, we find

$$I_j^2 = \int_{\mathbb{R}^n} \overline{\Big(\mathscr{D}(-\nabla_y) \frac{\partial \Phi}{\partial y_j}(y^0 - y)\Big)}^\top \chi^0(y) \mathscr{A}(y) \mathscr{D}(-\nabla_y) U(y)\, dy$$
$$+ \int_{\mathbb{R}^n} \overline{\Big(\mathscr{D}(-\nabla_y) \frac{\partial \Phi}{\partial y_j}(y^0 - y)\Big)}^\top \mathscr{A}(y) \left[\mathscr{D}(-\nabla_y), \chi^0(y)\right] U(y)\, dy.$$

We can estimate the last integral by using the same arguments as in the proof of the inequality (2.16). Thus, from the inequality

$$|\mathscr{A}(y) - \mathscr{A}(y^0)| \leqslant c|y - y^0|^\alpha$$

which is a consequence of the inclusion $\mathscr{A} \in C^{1,\alpha}_{\rm per}(\overline{S})^{N\times N}$, we find

$$|I_j^2| \leqslant c\,\|U; W^1_\infty(S)\| \int_0^{4\delta} r^n r^\alpha r^{n-1}\, dr \leqslant c\delta^\alpha \|U; W^1_\infty(S)\|. \tag{2.17}$$

Combining the relations (2.15), (2.16), and (2.17), we arrive at the inequality

$$|U(y^0)| + |\nabla_y U(y^0)| \leqslant c\,(\|\mathbf{F}; L_\infty(S)\| + \|U; H^2(S)\| + \delta^\alpha \|U; W^1_\infty(S)\|).$$

Choosing $\delta > 0$ small enough, taking into account (2.9), and computing the supremum with respect to $y^0 \in S$, we obtain the required estimate (2.10). It remains to get rid of the requirement of superfluous smoothness. □

We also need the following consequence of the Korn inequality.

Lemma 2.4. *For any vector-valued function $U \in H^1(S)^k$ the following inequality holds:*

$$\|U; H^1(S)\|^2 \leqslant c\,(\|\mathscr{D}(\nabla_x)U; L_2(S)\|^2 + |\mathscr{X}(U)|^2), \tag{2.18}$$

where $\mathscr{X}$ is the column of functionals on the space $H^1(S)^k$ such that

$$\begin{aligned} &\mathscr{X}(tU) = t\mathscr{X}(U), \quad t > 0, \\ &U^m \longrightarrow U \quad \text{weakly in } H^1(S)^k \text{ as } m \to \infty \quad \Rightarrow \quad \mathscr{X}(U^m) \to \mathscr{X}(U), \\ &\mathscr{X}(p) = 0 \text{ for } p \in \mathscr{P} \quad \Rightarrow \quad p = 0. \end{aligned} \tag{2.19}$$

The constant c in (2.18) *depends on $\mathscr{D}$ and $\mathscr{X}$, but is independent of U.*

Proof. By (2.5), it suffices to check the inequality

$$\|U; L_2(S)\|^2 \leqslant c\,(\|\mathscr{D}(\nabla_x)U; L_2(S)\|^2 + |\mathscr{X}(U)|^2),$$

Assume that it fails and there is a sequence $\{U^m\}$ in $H^1(S)^k$ such that

$$\begin{aligned} &\|U^m; L_2(S)\| = 1, \\ &\|\mathscr{D}(\nabla_x)U^m; L_2(S)\| \to 0 \quad \text{as } m \to \infty. \\ &|\mathscr{X}(U)| \to 0 \quad \text{as } m \to \infty, \end{aligned} \tag{2.20}$$

By the estimate (2.5), the norms $\|\nabla_x U^m; L_2(S)\|$ are uniformly bounded, i.e., there is a subsequence $\{U^{m_q}\}$ that converges weakly in $H^1(S)^k$ and strongly in $L_2(S)^k$ to a vector-valued function $U^\infty \in H^1(S)^k$. Since $\mathscr{D}U^m \to 0$ strongly in $L_2(S)^K$, we have $\mathscr{D}U^\infty = 0$ and, consequently, the polynomial property guarantees the inclusion $U^\infty \in \mathscr{P}$. At the same time, $\mathscr{X}(U^m) \to 0 = \mathscr{X}(U^\infty)$, i.e., $U^\infty = 0$ because of (2.19). This conclusion contradicts the first formula in (2.20), and, consequently, the inequality (2.18) is valid. □

2. Estimates for homogenization error. To treat residuals caused by the formal asymptotic construction, we need some auxiliary inequalities. Although these inequalities are known, we prove them below for the sake of convenience.

The following simple inequalities are direct consequences of the one-dimensional Hardy inequality and can be easily checked (cf., for example, [**6**, Lemma 1.2.4]).

Lemma 2.5. *For $Y^0 \in \overset{\circ}{H}{}^1(\Omega)$ and $Y \in H^1(\Omega)$ the following estimates hold:*

$$\varepsilon^{-1}\|Y^0; L_2(\Theta_{2h\varepsilon})\| \leqslant c\|\mathbf{r}^{-1}Y^0; L_2(\Omega)\| \leqslant c\|Y^0; H^1(\Omega)\|, \tag{2.21}$$

$$\|Y; L_2(\Theta_{2h\varepsilon})\| \leqslant c\varepsilon^{1/2}\|Y; H^1(\Omega)\|, \tag{2.22}$$

where the constant c is independent of $\varepsilon \in (0,1]$, $\mathbf{r}$ is the distance to the boundary $\partial\Omega$, and $\Theta_{h\varepsilon}$ is the intersection of Ω and an $(h\varepsilon)$-neighborhood of its boundary $\partial\Omega$, $h > 0$ is fixed.

The construction of leading terms of asymptotic expansions of eigenfunctions leads to the violation of the boundary condition. Therefore, these terms will be multiplied by cut-off functions. We denote by X_ε a cut-off function that vanishes in the $(h\varepsilon)$-neighborhood of $\partial\Omega$ and is equal to 1 outside the $(2h\varepsilon)$-neighborhood. It is clear that we can satisfy the conditions

$$0 \leqslant X_\varepsilon(x) \leqslant 1, \quad |\nabla_x X_\varepsilon(x)| \leqslant c\varepsilon^{-1}. \tag{2.23}$$

We assume that $h > \sqrt{n}$, i.e., any periodicity cell

$$S_\varepsilon^\alpha = \{x :\, x_j - \varepsilon\alpha_j \in (0,\varepsilon), j = 1, \dots, n\}, \tag{2.24}$$

where $X_\varepsilon = 1$, lies in the domain Ω; here, $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\alpha_j \in \mathbb{Z}$ is an integer.

Proposition 2.6. *Let $\overline{Z}$ be the mean of a function Z over the periodicity cell $S = (0,1)^n$, and let either the inclusions*

$$Y \in H^1(\Omega), \quad Z \in L_2(S) \tag{2.25}$$

or the inclusions

$$Y \in W^1_\infty(\Omega), \quad Z \in L_1(S) \tag{2.26}$$

hold. Then

$$\left| \int\limits_\Omega X_\varepsilon(x) Y(x) Z\left(\frac{x}{\varepsilon}\right) dx - \overline{Z} \int\limits_\Omega Y(x)\, dx \right| \leqslant c\varepsilon P, \tag{2.27}$$

where P is the product of the norms of functions (2.25) *or* (2.26)*, and the constant c depends neither these functions nor the parameter $\varepsilon \in (0, 1]$.*

Proof. We begin with the case (2.25). Denote by $\Sigma^\varepsilon(\Omega)$ the union of cells intersecting Ω and by Σ^ε_X the union of those cells in $\Sigma^\varepsilon(\Omega)$ where $X_\varepsilon = 1$. It is clear that the numbers $\sigma^\varepsilon(\Omega)$ and σ^ε_{1-X} of cells forming $\Sigma^\varepsilon(\Omega)$ and $(\Sigma^\varepsilon(\Omega) \setminus \Sigma^\varepsilon_X)$ do not exceed $c\varepsilon^n$ and $c\varepsilon^{n-1}$ respectively. Therefore, by (2.22), we have

$$\left| \int\limits_\Omega X_\varepsilon(x) Y(x) Z\left(\frac{x}{\varepsilon}\right) dx - \int\limits_{\Sigma^\varepsilon_X} Z\left(\frac{x}{\varepsilon}\right) Y(x)\, dx \right|$$

$$\leqslant c\|Y; L_2(\Sigma^\varepsilon(\Omega) \setminus \Sigma^\varepsilon_X)\| (\varepsilon^{n-1} \|Z; L_2(S_\varepsilon)\|^2)^{1/2} \leqslant c\varepsilon P. \tag{2.28}$$

It remains to consider the integral over the set Σ^ε_X; the symbol $\sum$ will mean the sum over its cells S^α_ε. We have

$$\int\limits_{\Sigma^\varepsilon_X} Y(x) Z\left(\frac{x}{\varepsilon}\right) dx = \sum \left(\int\limits_{S^\alpha_\varepsilon} Z\left(\frac{x}{\varepsilon}\right) \overline{Y}^\alpha\, dx + \int\limits_{S^\alpha_\varepsilon} Z\left(\frac{x}{\varepsilon}\right) \left(Y(x) - \overline{Y}^\alpha\right) dx \right)$$

$$= \sum \left(\overline{Z} \int\limits_{S^\alpha_\varepsilon} Y(x)\, dx + \overline{Z} \int\limits_{S^\alpha_\varepsilon} (\overline{Y}^\alpha - Y(x))\, dx + \int\limits_{S^\alpha_\varepsilon} Z\left(\frac{x}{\varepsilon}\right) (Y(x) - \overline{Y}^\alpha)\, dx \right),$$

where $\overline{Y}^\alpha$ denotes the mean of Y over the cell S^α_ε. Using the Poincaré inequality on the small set S^α_ε

$$\int\limits_{S^\alpha_\varepsilon} \left|Y(x) - \overline{Y}^\alpha\right|^2 dx \leqslant c\varepsilon^2 \int\limits_{S^\alpha_\varepsilon} \left|\nabla_x \left(Y(x) - \overline{Y}^\alpha\right)\right|^2 dx = c\varepsilon^2 \int\limits_{S^\alpha_\varepsilon} |\nabla_x Y(x)|^2\, dx, \tag{2.29}$$

we find

$$\left| \int\limits_{\Sigma^\varepsilon_X} Y(x) Z\left(\frac{x}{\varepsilon}\right) dx - \overline{Z} \int\limits_{\Sigma^\varepsilon_X} Y(x)\, dx \right| = \left| \int\limits_{\Sigma^\varepsilon_X} (Y(x) - \overline{Y}^\alpha) \Big(Z\Big(\frac{x}{\varepsilon}\Big) - \overline{Z} \Big) dx \right|$$

$$\leqslant c\varepsilon \|\nabla_x Y(x); L_2(\Sigma^\varepsilon_X)\|\, \Big\|Z\Big(\frac{\cdot}{\varepsilon}\Big); L_2(\Sigma^\varepsilon_X)\Big\| \leqslant c\varepsilon P.$$

Arguing in the same way as in the proof of (2.28), we see that the integrals over the set Σ^ε_X on the left-hand side becomes integrals over the entire domain Ω.

The case (2.26) is simpler since it is not necessary to use the Cauchy–Bunyakovskii inequality. An additional small factor appears because of the relation

$$|Y(\mathbf{x}) - \overline{Y}^\alpha| \leqslant c\varepsilon \sup_{x \in S^\alpha_\varepsilon \subset \Omega} |\nabla_x Y(x)| \leqslant c\varepsilon \|Y; W^1_\infty(S)\|, \quad \mathbf{x} \in S^\alpha_\varepsilon.$$

but not the Poincaré inequality (2.29). □

3. Approximate solutions of the abstract spectral equation. The following assertion is known as a "lemma about almost eigenvalues and eigenvectors." The proof of this assertion can be found in [**17**] (cf. also [**10**]).

Lemma 2.7. *Suppose that a nonzero real number $\mathscr{M}$ and a vector-valued function $\mathscr{U} \in \mathscr{H}$ are such that*

$$\|\mathscr{U}\,; \mathscr{H}\| = 1, \quad \delta := \|\mathscr{K}^\varepsilon \mathscr{U} - \mathscr{M}\mathscr{U}\,; \mathscr{H}\| < |\mathscr{M}|. \tag{2.30}$$

Then there exists an eigenvalue μ_l^ε of the operator $\mathscr{K}^\varepsilon$ such that

$$|\mu_l^\varepsilon - \mathscr{M}| \leqslant \delta.$$

Furthermore, for any $\delta_1 \in (\delta, |\mathscr{M}|)$ there are a_j^ε such that

$$\Big\|\mathscr{U} - \sum a_j^\varepsilon u_q^\varepsilon; \mathscr{H}\Big\| \leqslant 2\,\frac{\delta}{\delta_1}$$

where the sum is taken over all the eigenvalues (1.19), (1.20) *of the operator $\mathscr{K}^\varepsilon$ lying in $[\mathscr{M} - \delta_1, \mathscr{M} + \delta_1]$ and $u_j^\varepsilon \in \mathscr{H}$ are the corresponding eigenvectors satisfying the condition* (1.24); *moreover, the coefficients a_j are normalized by the condition*

$$\sum |a_j^\varepsilon|^2 = 1.$$

The proof of the following simple algebraic assertion can be found, for example, in [**6**, Lemma 7.1.7].

Lemma 2.8. *If a $(\varkappa \times \varkappa)$–matrix a is "almost unitary" i.e.,*

$$\|a^* a - \mathbb{I}_\varkappa; \mathbb{C}^\varkappa \to \mathbb{C}^\varkappa\| = \theta \in (0, 1),$$

then there exists a unitary matrix b such that

$$\|ab - \mathbb{I}_\varkappa; \mathbb{C}^\varkappa \to \mathbb{C}^\varkappa\| \leqslant \theta.$$

Here, $\mathbb{I}_\varkappa$ is the identity $(\varkappa \times \varkappa)$–matrix.

3. Formal Asymptotic Analysis

1. Homogenized problem in the case $\overline{\rho} > 0$. As usual, to provide a formal asymptotic analysis, we assume that all the data of the problem under consideration are sufficiently smooth.

We consider the following asymptotic ansätze, used in the homogenization theory:

$$u^\varepsilon(x) = u^0(x) + \varepsilon N(\varepsilon^{-1}x)\mathscr{D}(\nabla_x)u^0(x) + \varepsilon^2 w(\varepsilon^{-1}x) + \dots \tag{3.1}$$

$$\lambda^\varepsilon = \lambda^0 + \dots, \tag{3.2}$$

where N is the asymptotic corrector, i.e., in our case, it is a function of size $k \times K$, periodic in the fast variables (1.3), whereas a number λ^0 and vector-functions u^0, w should be found. We substitute the ansätze (3.1) and (3.2) into the system (1.1) and apply the chain rule:

$$\mathscr{D}(\nabla_x)w(\varepsilon^{-1}x, x) = \big(\varepsilon^{-1}\mathscr{D}(\nabla_y)w(y, x) + \mathscr{D}(\nabla_x)w(y, x)\big)\Big|_{y=\varepsilon^{-1}x}. \tag{3.3}$$

Collecting coefficients at the same powers of ε, we obtain the following systems of differential equations on the periodicity cell $S = (0,1)^n \ni y$ with parameter $x \in \Omega$:

$$\mathscr{L}(y,\nabla_y)N(y)\mathscr{D}(\nabla_x)u^0(x) = \overline{\mathscr{D}(\nabla_y)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_x)u^0(x), \quad y \in S; \tag{3.4}$$

$$\begin{aligned} \mathscr{L}(y,\nabla_y)w(y,x) = \overline{\mathscr{D}(\nabla_x)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_x)u^0(x) + \lambda^0\rho(y)u^0(x) \\ + \overline{\mathscr{D}(\nabla_x)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_y)N(y)\mathscr{D}(\nabla_x)u^0(x) \\ + \overline{\mathscr{D}(\nabla_y)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_x)N(y)\mathscr{D}(\nabla_x)u^0(x), \quad y \in S. \end{aligned} \tag{3.5}$$

These systems are completed with the periodicity condition with respect to y, which is not given explicitly. In other words, solutions to the problem on the cell $S = (0,1)^n$ are always looked for in the class $H^1_{per}(S)^k$. We emphasize that in formulas (3.4) and (3.5), the variables x and y are assumed to be independent.

Since the matrix $\mathscr{A}$ is periodic, the following equality holds:

$$\int_S \overline{\mathscr{D}(\nabla_y)}^\top \mathscr{A}(y)\,dy = 0 \in \mathbb{C}^{k\times K}. \tag{3.6}$$

From (3.4) we obtain the problem for asymptotic corrector

$$\mathscr{L}(y,\nabla_y)N(y) = \overline{\mathscr{D}(\nabla_y)}^\top \mathscr{A}(y), \quad y \in S. \tag{3.7}$$

By Proposition 2.3, this problem has a unique periodic solution $N \in H^1_{\text{per}}(S)^{k\times K}$ satisfying the orthogonality condition (2.7).

As in the case (3.6), the mean of the last term in (3.5) over the cell vanishes. The solvability condition (2.6) for the problem (3.5) takes the form

$$\overline{\mathscr{D}(-\nabla_x)}^\top \int_S (\mathscr{A}(y) + \mathscr{A}(y)\mathscr{D}(\nabla_y)N(y))\,dy\,\mathscr{D}(\nabla_x)u^0(x) = \lambda^0 \int_S \rho(y)\,dy\,u^0(x)$$

or

$$\mathbf{L}(\nabla_x)u^0(x) = \lambda^0\overline{\rho}u^0(x), \quad x \in \Omega, \tag{3.8}$$

where $\overline{\rho}$ is the mean (1.25) of the density ρ and $\mathbf{L}$ is the differential operator (1.28) with matrix

$$\mathbf{A} = \int_S (\mathscr{A}(y) + \mathscr{A}(y)\mathscr{D}(\nabla_y)N(y))\,dy. \tag{3.9}$$

Lemma 3.1. *The $(K\times K)$–matrix* (3.9) *is Hermitian and positive definite.*

Proof. Taking into account the problem (3.7) for the corrector N and integrating by parts, we find

$$\mathbf{A} = \int\limits_S (\mathscr{A}(y) + \mathscr{A}(y)\mathscr{D}(\nabla_y)N(y) + (\overline{\mathscr{D}(\nabla_y)N(y)})^\top \mathscr{A}(y) + \overline{N(y)}^\top \mathscr{D}(\nabla_y)\mathscr{A}(y))\, dy$$

$$= \int\limits_S (\mathscr{A}(y) + \mathscr{A}(y)\mathscr{D}(\nabla_y)N(y) + (\overline{\mathscr{D}(\nabla_y)N(y)})^\top \mathscr{A}(y) + \overline{N(y)}^\top \mathscr{L}(y, \nabla_y)N(y))\, dy$$

$$= \int\limits_S (\mathscr{A}(y) + \mathscr{A}(y)\mathscr{D}(\nabla_y)N(y) + (\overline{\mathscr{D}(\nabla_y)N(y)})^\top \mathscr{A}(y) + (\overline{\mathscr{D}(\nabla_y)N(y)})^\top \mathscr{A}(y)\mathscr{D}(\nabla_y)N(y))dy$$

$$= \int\limits_S (\overline{\mathscr{D}(\nabla_y)N(y) + \mathbb{I}_K})^\top \mathscr{A}(y)(\mathscr{D}(\nabla_y)N(y) + \mathbb{I}_K)\, dy, \tag{3.10}$$

where $\mathbb{I}_K$ is the identity $(K \times K)$–matrix. Now, it is obvious that the matrix $\mathbf{A}$ is Hermitian. The matrix (3.10) is nonnegative definite because the periodic matrix $\mathscr{A}$ is positive definite. We assume that $\overline{\xi}^\top \mathbf{A}\xi = 0$ for some column $\xi \in \mathbb{C}^K$. Then

$$0 = \int\limits_S (\overline{\mathscr{D}(\nabla_y)N(y)\xi + \xi})^\top \mathscr{A}(y)(\mathscr{D}(\nabla_y)N(y)\xi + \xi)\, dy \geqslant c_{\mathscr{A}} \|\mathscr{D}(\nabla_y)N\xi + \xi; L_2(S)\|^2, \quad c_A > 0.$$

Hence

$$\mathscr{D}(\nabla_y)N(y)\xi + \xi = 0.$$

Integrating over the cell S and using a formula similar to (3.6), from the last relation we find $\xi = 0$, which is required for confirming the positive definiteness of the matrix $\mathbf{A}$. □

We complete the system (3.8) with the Dirichlet condition

$$u^0(x) = 0, \quad x \in \partial\Omega \tag{3.11}$$

coming from the initial problem (1.1), (1.2). Since $\mathbf{L}$ is a formally positive operator and $\overline{\rho} > 0$ by assumption, the following assertion is obvious.

Proposition 3.2. *The problem* (3.8), (3.11) *in the variational form*

$$\mathbf{a}^0(u^0, v; \Omega) := (\mathbf{A}\mathscr{D}(\nabla_x)u^0, \mathscr{D}(\nabla_x)v)_\Omega = \lambda^0 \overline{\rho}(u^0, v)_\Omega, \quad v \in \overset{o}{H}{}^1(\Omega)^k, \tag{3.12}$$

has an infinitely large sequence of eigenvalues

$$0 < \lambda_1^{0+} \leqslant \lambda_2^{0+} \leqslant \ldots \leqslant \lambda_j^{0+} \leqslant \cdots \to +\infty, \tag{3.13}$$

and the corresponding vector-valued eigenfunctions $u_{(j)}^{0+} \in \overset{o}{H}{}^1(\Omega)^k$ *satisfy the orthogonality and normalization conditions*

$$\overline{\rho} \left(u_{(j)}^{0+}, u_{(l)}^{0+} \right)_\Omega = \delta_{j,l}, \quad j, l = 1, 2, \ldots \tag{3.14}$$

Remark 3.3. Since the boundary $\partial\Omega$ is of class $C^{2,\delta}$, the vector-valued eigenfunctions $u_{(j)}^{0+}$ belong to the Hölder space $C^{2,\delta_*}(\Omega)^k$ for any $\delta_* \in (0, \delta)$ and are infinitely smooth inside the domain Ω. In particular, they belong to the class $C^2(\overline{\Omega})^k$, i.e., they are twice continuously differentiable up to the boundary. □

2. Homogenized problem in the case $\overline{\rho} = 0$. We change the asymptotic ansätze (3.1) and (3.2) and represent the eigenpair $\{\lambda^\varepsilon, u^\varepsilon\}$ of the problem (1.1), (1.2) in the form

$$u^\varepsilon(x) = u^0(x) + \varepsilon\left(N(\varepsilon^{-1}x)\mathscr{D}(\nabla_x)u^0(x) + \beta N^0(\varepsilon^{-1}x)u^0(x)\right) + \varepsilon^2 w(\varepsilon^{-1}x, x) + \ldots, \tag{3.15}$$

$$\lambda^\varepsilon = \varepsilon^{-1}\beta + \ldots \tag{3.16}$$

Here, u^0, N, and w have the same sense as above, and β is a new spectral parameter that appears explicitly in the ansatz (3.15) for vector-valued eigenfunctions. Furthermore, the additional asymptotic corrector N^0 is a periodic $(k \times k)$–matrix-valued function with zero mean over the cell S that satisfies the system of differential equations

$$\mathscr{L}(y, \nabla_y)N^0(y) = \rho(y)\mathbb{I}_k, \quad y \in S. \tag{3.17}$$

By Proposition 2.3, the problem (3.17) is solvable because of the requirement $\overline{\rho} = 0$ which is satisfied in this subsection. We write the variational statement of the problem (3.17):

$$(\mathscr{A}\mathscr{D}(\nabla_y)N^0, \mathscr{D}(\nabla_y)V)_S = (\rho, V)_S, \quad V \in H^1_{\mathrm{per}}(S)^k.$$

We substitute the asymptotic ansätze (3.15) and (3.16) into the system (1.1). The total coefficient at ε^{-1} vanishes by the definition of asymptotic correctors. Indeed, by (3.7) and (3.17), we have

$$\begin{aligned}&\overline{\mathscr{D}(\nabla_y)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_x)u^0(x) + \beta\rho(y)u^0(x)\\ &+ \overline{\mathscr{D}(\nabla_y)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_y)\big(N(y)\mathscr{D}(\nabla_x)u^0(x) + \beta N^0(y)u^0(x)\big) = 0.\end{aligned}$$

The problem for the third term w of the ansatz (3.15) has the form

$$\begin{aligned}\mathscr{L}(y, \nabla_y)w(y,x) = {}&\overline{\mathscr{D}(\nabla_x)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_x)u^0(x)\\ &+ \beta\rho(y)\big(N(y)\mathscr{D}(\nabla_x)u^0(x) + \beta N^0(y)u^0(x)\big)\\ &+ \overline{\mathscr{D}(\nabla_x)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_y)\big(N(y)\mathscr{D}(\nabla_x)u^0(x) + \beta N^0(y)u^0(x)\big)\\ &+ \overline{\mathscr{D}(\nabla_y)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_x)\big(N(y)\mathscr{D}(\nabla_x)u^0(x) + \beta N^0(y)u^0(x)\big), \quad y \in S.\end{aligned} \tag{3.18}$$

The last term on the right-hand side has zero mean over the cell S. Furthermore,

$$\begin{aligned}\mathbf{m} := \int\limits_S \rho(y)N^0(y)\,dy &= -\int\limits_S \overline{\big(\overline{\mathscr{D}(\nabla_y)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_y)N^0(y)\big)}^\top N^0(y)\,dy\\ &= \int\limits_S \big(\overline{\mathscr{D}(\nabla_y)N^0(y)}\big)^\top \mathscr{A}(y)\mathscr{D}(\nabla_y)N^0(y)\,dy.\end{aligned} \tag{3.19}$$

It is clear that $\mathbf{m}$ is a Hermitian positive definite $(k \times k)$–matrix.

We continue calculations:

$$\mathbf{s} := \int_S \rho(y) N(y)\, dy = -\int_S \overline{(\overline{\mathscr{D}(\nabla_y)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_y)N^0(y))}^\top N(y)\, dy$$

$$= \int_S \overline{(\mathscr{D}(\nabla_y)N^0(y))}^\top \mathscr{A}(y)\mathscr{D}(\nabla_y)\, dy \in \mathbb{C}^{k\times k},$$

$$\int_S \mathscr{A}(y)\mathscr{D}(\nabla_y)N^0(y)\, dy = -\int_S (\overline{\mathscr{D}(\nabla_y)}^\top \mathscr{A}(y))^\top N^0(y)\, dy \tag{3.20}$$

$$= \int_S (\overline{\overline{\mathscr{D}(\nabla_y)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_y)N(y)})^\top N^0(y)\, dy$$

$$= -\int_S (\overline{\mathscr{D}(\nabla_y)N(y)})^\top \mathscr{A}(y)\mathscr{D}(\nabla_y)N^0(y)\, dy = -\overline{\mathbf{s}}^\top.$$

Thus, according to the definitions (1.28) and (3.9), the solvability condition (2.6) for the problem (3.18) takes the form of a quadratic pencil

$$\mathbf{L}(\nabla_x)u^0(x) - \beta\mathbf{s}\mathscr{D}(\nabla_x)u^0(x) + \beta\overline{\mathscr{D}(\nabla_x)}^\top \overline{\mathbf{s}}^\top u^0(x) - \beta^2\mathbf{m}u^0(x) = 0, \quad x \in \Omega. \tag{3.21}$$

The system of differential equations (3.21) is equipped with the Dirichlet boundary condition (3.11). The variational statement of the problem (3.21), (3.11) is to find a nontrivial vector-valued function $u^0 \in \mathring{H}^1(\Omega)^k$ and a number $\beta \in \mathbb{C}$ satisfying the integral identity [**4**]

$$\mathbf{a}(u^0, v; \Omega) - \beta\mathbf{b}(u^0, v; \Omega) - \beta^2(\mathbf{m}u^0, v)_\Omega = 0, \quad v \in \mathring{H}^1(\Omega)^k, \tag{3.22}$$

where $\mathbf{a}$ is the form on the right-hand side of (3.12) and $\mathbf{b}$ is a Hermitian sesquilinear form

$$\mathbf{b}(u^0, v; \Omega) = (\mathbf{s}\mathscr{D}(\nabla_x)u^0, v)_\Omega + (u^0, \mathbf{s}\mathscr{D}(\nabla_x)v)_\Omega. \tag{3.23}$$

By the Riesz theorem on representation of linear functionals in a Hilbert space, with the problem (3.22) we can associate an operator $\mathfrak{A}(\beta)$ in the space $\mathring{H}^1(\Omega)^k$ that is a quadratic function of the spectral parameter $\beta \in \mathbb{C}$.

Theorem 3.4. *The pencil $\beta \mapsto \mathfrak{A}(\beta)$ has a discrete spectrum. The eigenvalues are algebraically simple (there are no adjoint vectors) and form two – positive and negative – infinitely large sequences*

$$0 < \beta_{+1} \leqslant \beta_{+2} \leqslant \dots \beta_{+j} \leqslant\to \dots + \infty, \tag{3.24}$$

$$0 > \beta_{-1} \geqslant \beta_{-2} \geqslant \dots \beta_{-j} \geqslant \dots \to -\infty. \tag{3.25}$$

The corresponding vector-valued eigenfunctions $u^0_{\pm j} \in \mathring{H}^1(\Omega)^k$ can be subject to the orthogonality and normalization conditions

$$\mathbf{a}(u^0_{(j)}, u^0_{(l)}; \Omega) + \beta_j\beta_l(\mathbf{m}u^0_{(j)}, u^0_{(l)})_\Omega = \delta_{j,l}, \quad j, l = \pm 1, \pm 2, \dots \tag{3.26}$$

Proof. Since $\mathfrak{A}(\beta^1) - \mathfrak{A}(\beta^2)$ with any $\beta^1, \beta^2 \in \mathbb{C}$ is a compact operator in $\mathring{H}^1(\Omega)^k$ (the obstruction form $\mathbf{a}$, vanishes), the spectrum is discrete in accordance with [**18**, Theorem 1.5.1].

Let $\beta \in \mathbb{C}$, $u^0 \in \mathring{H}^1(\Omega)^k \setminus \{0\}$ be an eigenpair of the problem (3.22). Taking u^0 for a test function and separating the real part from the imaginary part, we find

$$\mathbf{a}(u^0, u^0; \Omega) - \mathrm{Re}\beta\mathbf{b}(u^0, u^0; \Omega) - ((\mathrm{Re}\beta)^2 - (\mathrm{Im}\,\beta)^2)(\mathbf{m}u^0, u^0)_\Omega = 0,$$

$$-\mathrm{Im}\,\beta\mathbf{b}(u^0, u^0; \Omega) - 2\,\mathrm{Im}\,\beta\mathrm{Re}\beta(\mathbf{m}u^0, u^0)_\Omega = 0.$$

In the case $\mathrm{Im}\,\beta \neq 0$, this implies

$$\begin{aligned} 0 &= \mathbf{a}(u^0, u^0; \Omega) + 2(\mathrm{Re}\beta)^2(\mathbf{m}u^0, u^0)_\Omega + ((\mathrm{Im}\,\beta)^2 - (\mathrm{Re}\beta)^2)(\mathbf{m}u^0, u^0)_\Omega \\ &= \mathbf{a}(u^0, u^0; \Omega) + |\beta|^2(\mathbf{m}u^0, u^0)_\Omega > 0. \end{aligned} \tag{3.27}$$

We took into account that both forms on the right-hand side are positive definite. The obtained contradiction means that $\mathrm{Im}\,\beta = 0$ and the eigenvalues are real and are different from zero.

Now, we check that there are no adjoint vectors, which are found from the problem

$$\mathbf{a}(u^1, v; \Omega) - \beta\mathbf{b}(u^1, v; \Omega) - \beta^2(\mathbf{m}u^1, v)_\Omega = \mathbf{b}(u^0, v; \Omega) + 2\beta(\mathbf{m}u^0, v)_\Omega, \quad v \in H^1(\Omega)^k. \tag{3.28}$$

Explanation. The right-hand side of (3.28) contains the β-derivative of the quadratic polynomial from the left-hand side of (3.22) which is computed on the eigenpair $\{\beta, u^0\}$ and is taken with the opposite sign. Assume that a solution $u^1 \in \mathring{H}^1(\Omega)^k$ to the problem (3.28) exists. We set $v = u^0$ in both integral identities (3.22) and (3.28). The left-hand side of (3.28) vanishes in view of formula (3.22) with $v = u^1$. Multiplying (3.28) by β and subtracting the equality (3.22), we find

$$\begin{aligned} 0 &= \beta\mathbf{b}(u^0, u^0; \Omega) + 2\beta^2(\mathbf{m}u^0, u^0)_\Omega + (\mathbf{a}(u^0, u^0; \Omega) - \beta\mathbf{b}(u^0, u^0; \Omega) - \beta^2(\mathbf{m}u^0, u^0)_\Omega) \\ &= \mathbf{a}(u^0, u^0; \Omega) + \beta^2(\mathbf{m}u^0, u^0)_\Omega > 0. \end{aligned} \tag{3.29}$$

We arrive at a contradiction, which means that there are no adjoint vectors.

Since the same Hermitian positive definite forms appear in (3.27) and (3.29), it becomes possible to satisfy the normalization condition (3.26). The same relations appear for the orthogonality conditions as follows: in the integral identity for the eigenpair $\{\beta_j, u^0_{(j)}\}$ (and the pair $\{\beta_l, u^0_{(l)}\}$), we set $v = \beta_j^{-1}u^0_{(l)}$ (and $v = \beta_l^{-1}u^0_{(j)}$). After the complex conjugation, we subtract the second equality from the equality and find that

$$(\beta_j^{-1} - \beta_l^{-1})\mathbf{a}(u^0_{(j)}, u^0_{(l)}; \Omega) - (\beta_j - \beta_l)(\mathbf{m}u^0_{(j)}, u^0_{(l)})_\Omega = 0. \tag{3.30}$$

In the case $\beta_j \neq \beta_l$, formula (3.30) implies the required relation (3.26).

It remains to show that there exist exactly two infinitely large sequences (3.24) and (3.25) of eigenvalues[4)]. Let $\mathbf{m}^{1/2}$ be the positive square root of the matrix $\mathbf{m}$. We write the system of differential equations (3.21) as follows:

[4)] In the case $\mathbf{s} = 0$, this fact is obvious since $\beta_{\pm k} = \pm B_k^{1/2}$, where $0 < B_1 \leqslant B_2 \leqslant \ldots \leqslant B_j \leqslant\to +\infty$ is the sequence of eigenvalues of the problem $\mathbf{a}(w, v; \Omega) = B(\mathbf{m}w, v)_\Omega$, $v \in \mathring{H}^1(\Omega)^k$ (cf. the system (3.21) with removed two terms in the middle of the left-hand side).

$$
\begin{aligned}
&\mathbf{L}(\nabla_x)u^0(x) + \frac{1}{4}\mathbf{B}(\nabla_x)\mathbf{m}^{-1}\mathbf{B}(\nabla_x)u^0(x) + \frac{1}{2}\mathbf{B}(\nabla_x)\mathbf{m}^{-1/2}U^0(x) = \beta\mathbf{m}^{1/2}U^0(x),\\
&\frac{1}{2}\mathbf{m}^{-1/2}\mathbf{B}(\nabla_x)u^0(x) + U^0(x) = \beta\mathbf{m}^{1/2}u^0(x), \quad x \in \Omega.
\end{aligned}
\tag{3.31}
$$

Here, the second row serves as the definition of the new unknown $U^0 = (U_1^0, \dots, U_k^0)^\top$ and $\mathbf{B}(\nabla_x)$ is a formally selfadjoint differential $(k \times k)$–matrix-valued operator of the first order that appears on the left-hand side of (3.21); namely,

$$
\mathbf{B}(\nabla_x) = -\mathbf{s}\mathscr{D}(\nabla_x) + \overline{\mathscr{D}(\nabla_x)}^\top \mathbf{s}^\top. \tag{3.32}
$$

The first row in (3.31) appears because of the new notation in (3.21). It is easy to check that the system (3.31) for $\{u^0, U^0, \beta\}$ yields the system (3.21) for $\{u^0, \beta\}$.

The variational statement of the problem (3.31), (3.11) for the extended vector-valued function $\mathfrak{u} = (u^0, U^0)$ can be written as follows:

$$
\begin{aligned}
\mathfrak{q}(\mathfrak{u}, \mathfrak{v}; \Omega) &:= (\mathbf{A}\mathscr{D}(\nabla_x)u^0, \mathscr{D}(\nabla_x)v^0)_\Omega + \frac{1}{4}(\mathbf{m}^{-1}\mathbf{B}(\nabla_x)u^0, \mathbf{B}(\nabla_x)v^0)_\Omega\\
&\quad + \frac{1}{2}(\mathbf{U}^0, \mathbf{m}^{1/2}\mathbf{B}(\nabla_x z)v^0)_\Omega + \frac{1}{2}(\mathbf{m}^{1/2}\mathbf{B}(\nabla_x)u^0, V^0)_\Omega + (U^0, V^0)_\Omega\\
&\quad = \beta\mathfrak{t}(\mathfrak{u}, \mathfrak{v}; \Omega) := \beta((\mathbf{m}^{1/2}U^0, v^0)_\Omega + (u^0, \mathbf{m}^{1/2}V^0)_\Omega),\\
&\quad \mathfrak{v} = (v^0, V^0) \in \mathfrak{H} := \mathring{H}^1(\Omega)^k \times L_2(\Omega)^k.
\end{aligned}
\tag{3.33}
$$

Since

$$
\mathfrak{q}(\mathfrak{u}, \mathfrak{u}; \Omega) = (\mathbf{A}\mathscr{D}(\nabla_x)u^0, \mathscr{D}(\nabla_x)u^0)_\Omega + \left\| U^0 + \frac{1}{2}\mathbf{m}^{1/2}\mathbf{B}(\nabla_x)u^0; L_2(\Omega) \right\|^2,
$$

the Hermitian sesquilinear form $\mathfrak{q}$ is positive definite in the Hilbert space $\mathfrak{H}$ and possesses the polynomial property [**2, 3**]; moreover, the lineal of polynomials $\mathfrak{P}$ in an assertion similar to (1.6) takes the form $\mathfrak{P} = \mathscr{P} \times \mathbb{C}^k$ (cf. the definition (1.8)). Note that the relation

$$
\mathfrak{q}(\mathfrak{u}, \mathfrak{u}; \Omega) \geqslant C(\|u^0; \mathring{H}^1(\Omega)\|^2 + \|U^0; L_2(\Omega)\|^2), \quad C > 0,
$$

follows from the inequality (2.1) and positive definiteness of the matrix $\mathbf{A}$. With $\mathfrak{q}$ we can associate a continuous selfadjoint positive definite operator $\mathfrak{Q} : \mathfrak{H} \to \mathfrak{H}^*$. Denote by $\mathfrak{Q}^{1/2}$ its positive square root (cf., for example, [**10**, Section 10.4]).

The polynomial property guarantees (cf. [**2, 3**]) the ellipticity of the system (3.31) in the sense of Douglis–Nirenberg with framing

$$
\{t_1, \dots, t_{2k}\} = \{s_1, \dots, s_{2k}\} = \{1, \dots, 1, 0, \dots, 0\}
$$

(k units and k zeros) and also yields its covering of the Dirichlet boundary condition (3.11) at each point $x \in \partial\Omega$. In other words, the boundary value problem (3.21), (3.11) is elliptic in Ω.

The Hermitian form $\mathfrak{t}$ on the right-hand side of (3.33) is not sign-definite and takes positive and negative values on the infinite-dimensional linear sets

$$
\mathfrak{H}_\pm = \left\{ \mathfrak{u} = \{u^0, \pm u^0\} : u^0 \in \mathring{H}^1(\Omega)^k \right\}
$$

respectively. Since the mapping

$$
\mathfrak{H} \ni \mathfrak{u} = (u^0, U^0) \mapsto \mathfrak{T}\mathfrak{u} = \{\mathbf{m}^{1/2}U^0, \mathbf{m}^{1/2}u^0\} \in \mathfrak{H}^* = H^{-1}(\Omega)^k \times L_2(\Omega)^k
$$

is compact, the above-listed observations yield the existence of sequences (3.24) and (3.25) of eigenvalues of the problem (3.33) or (3.31), (3.11), and, consequently, the problem (3.22) or the problem (3.21), (3.11). To obtain this assertion directly, it suffices to apply the max-min principle (cf., for example, [**10**, Theorem 10.2.4]) to the compact operators $\mathfrak{S} = \mathfrak{Q}^{1/2}\mathfrak{T}\mathfrak{Q}^{1/2}$ and $-\mathfrak{S}$; the first operator appears in the abstract spectral equation

$$\mathfrak{S}\mathfrak{w} = \sigma\mathfrak{q}\mathfrak{w},$$

obtained from the problem (3.33) by the replacements

$$\mathfrak{u} \mapsto \mathfrak{w} = \mathfrak{Q}^{1/2}\mathfrak{u}, \quad \beta \mapsto \sigma = \beta^{-1}$$

and are equivalent to this problem, Theorem 3.4 is proved. □

Remark 3.5. As in Remark 3.3, the vector-valued functions $u^0_{(j)}$ belong to the space $C^{2,\delta_*}(\Omega)^k \subset C^2(\Omega)^k$ for small δ_*, and are infinitely differentiable inside the domain Ω. □

3. Attempt of homogenization in high-frequency range. We consider the asymptotic ansätze

$$u^\varepsilon(x) = w(x)U(\varepsilon^{-1}x) + \varepsilon w^{(1)}(\varepsilon^{-1}x, x) + \varepsilon^2 w^{(2)}(\varepsilon^{-1}x, x) + \dots, \tag{3.34}$$

$$\lambda^\varepsilon = \varepsilon^{-2}\tau + \varepsilon^{-1}\tau^{(1)} + \varepsilon^0\tau^{(2)} + \dots, \tag{3.35}$$

where $\{\tau, U\}$ is the spectral pair of the problem on the cell

$$\overline{\mathscr{D}(-\nabla_y)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_y)U(y) = \tau\rho(y)U(y), \quad y \in S, \tag{3.36}$$

with the periodicity condition on the opposite faces of the cube $S = (0,1)^n$. Numbers $\tau^{(q)}$ and vector-valued functions w, $w^{(q)}$ are unknown and should be found (in particular, and height of the column w).

Modifying the arguments of Subsection 1.3, we obtain the following assertion.

Proposition 3.6. *The problem* (3.36) *in the variational form*

$$a(U, V; S) = \tau(\rho U, V)_S, \quad V \in H^1_{\mathrm{per}}(S)^k, \tag{3.37}$$

has a discrete spectrum. With the eigenvalue $\tau_0 = 0$ we associate the eigensubspace $\mathbb{C}^k$ of constant vector-valued functions. The remaining eigenvalues form two infinitely large sequences, positive and negative:

$$0 < \tau_{+1} \leqslant \tau_{+2} \leqslant \dots \tau_{+j} \leqslant \dots \to +\infty,$$

$$0 > \tau_{-1} \geqslant \tau_{-2} \geqslant \dots \tau_{-j} \geqslant \dots \to -\infty,$$

and the corresponding vector-valued eigenfunctions $U_{\pm j} \in H^1_{\mathrm{per}}(S)^k$ satisfy the orthogonality and normalization conditions

$$a(U_{(j)}, U_{(l)}; S) = \delta_{j,l}, \quad j, l = \pm 1, \pm 2, \dots \tag{3.38}$$

We introduce several simplifying assumptions. First, $\tau \neq 0$ is a simple eigenvalue and U is the corresponding eigenfunction which, by (3.38) and (3.37), satisfies the relation

$$(\rho U, U)_S = \tau^{-1}. \tag{3.39}$$

By this assumption, the unknown w on the right-hand side of (3.34) becomes scalar. We substitute the ansätze (3.34) and (3.35) into the system of differential equations (1.1) and, using the chain rule (3.3), collect coefficients at the same powers of the small parameter. The coefficient at ε^{-2} vanishes since $\{\tau, U\}$ is an eigenpair of the problem (3.36). The coefficients at ε^{-1} form the following system of differential equations with parameter $x \in \Omega$:

$$\begin{aligned}
&\mathscr{L}(y, \nabla_y) w^{(1)}(y,x) - \tau\rho(y) w^{(1)}(y,x) \\
&= \overline{\mathscr{D}(\nabla_x)}^{\top} \mathscr{A}(y)\mathscr{D}(\nabla_y)U(y)w(x) + \overline{\mathscr{D}(\nabla_y)}^{\top} \mathscr{A}(y)\mathscr{D}(\nabla_x)U(y)w(x) + \tau^{(1)}\rho(y)U(y)w(x) \\
&=: F^{(1)}(y,x), \quad y \in S. \qquad (3.40)
\end{aligned}$$

Since τ is a simple eigenvalue, the solvability condition for (3.40) in the class of periodic functions is the orthogonality condition in the space $L_2(S)^k$ of vectors $F^{(1)}$ and U. Thus, the following relation must hold:

$$\tau^{(1)}(\rho U, U)_S = \sum_{p=1}^{n} \Big((\mathscr{A}\mathscr{D}(\nabla_y)U, \mathscr{D}(e_p)U)_S - (\mathscr{A}\mathscr{D}(e_p), \mathscr{D}(\nabla_y)U)_S \Big) \frac{\partial w}{\partial x_p}(x), \quad x \in \Omega. \quad (3.41)$$

It is clear that the factor $i\mathbf{t}_p$ at the derivative $\partial w/\partial x_p$ on the right-hand side of (3.41) is purely imaginary. In particular, it vanishes for the real matrices $\mathscr{A}$ and $\mathscr{D}$. We formulate the second simplified assumption: for a complex matrix the equality $\mathbf{t}_p = 0$ holds. Then $\tau^{(1)} = 0$ in view of (3.41) and (3.39), whereas the second term in the ansatz (3.34) takes the form

$$w^{(1)}(y,x) = \sum_{p=1}^{n} N^{\tau}_{(p)}(y) \frac{\partial w}{\partial x_p}(x), \qquad (3.42)$$

where $N^{\tau}_{(p)} \in H^1_{\mathrm{per}}(S)^k$ is an analog of the above asymptotic corrector, i.e., a solution to the problem

$$\begin{aligned}
\mathscr{L}(y, \nabla_y) N^{\tau}_{(p)}(y) - \tau\rho(y) N^{\tau}_{(p)}(y) &= \overline{\mathscr{D}(e_p)}^{\top} \mathscr{A}(y)\mathscr{D}(\nabla_y)U(y) \\
&\quad + \overline{\mathscr{D}(\nabla_y)}^{\top} \mathscr{A}(y)\mathscr{D}(e_p)U(y), \quad y \in S. \qquad (3.43)
\end{aligned}$$

By the above assumption, the problem has a solution determined up to a summand cU and becomes unique if the orthogonality condition is satisfied:

$$(\rho N^{\tau}_{(p)}, U)_S = 0. \qquad (3.44)$$

We compose a $(k \times n)$–matrix N^{τ} of columns $N^{\tau}_{(1)}, \dots, N^{\tau}_{(n)}$ and write (3.42) in the form

$$w^{(1)}(y,x) = N^{\tau}(y)\nabla_x w(x).$$

The superscript τ indicates that the matrix N^{τ} depends on the eigenvalue and vector-valued eigenfunction.

We continue calculations. The terms $w^{(2)}$ and $\tau^{(2)}$ of the asymptotic ansätze (3.34) and (3.35) satisfy the system of equations

$$\begin{aligned}
&\mathscr{L}(y,\nabla_y)w^{(2)}(y,x)-\tau\rho(y)w^{(2)}(y,x)\\
&=\overline{\mathscr{D}(\nabla_x)}^\top\mathscr{A}(y)\mathscr{D}(\nabla_x)U(y)w(x)+\overline{\mathscr{D}(\nabla_x)}^\top\mathscr{A}(y)\mathscr{D}(\nabla_y)N^\tau(y)\nabla_x w(x)\\
&\quad+\overline{\mathscr{D}(\nabla_y)}^\top\mathscr{A}(y)\mathscr{D}(\nabla_x)N^\tau(y)\nabla_x w(x)+\tau^{(2)}\rho(y)U(y)w(x),\quad y\in S.
\end{aligned}\tag{3.45}$$

By (3.39), the above-mentioned solvability condition for this problem on a cell in the class of periodic functions is written as the following scalar differential equation:

$$-\nabla_x^\top T^\tau\nabla_x w(x)=\tau^{(2)}\tau^{-1}w(x),\quad x\in\Omega,\tag{3.46}$$

where T^τ is an $(n\times n)$–matrix with entries

$$T^\tau_{qp}=(\mathscr{A}\mathscr{D}(\nabla_y)N^\tau_{(p)},\mathscr{D}(e_q)U)_S-(\mathscr{A}\mathscr{D}(e_q)N^\tau_{(p)},\mathscr{D}(\nabla_x)U)_S.\tag{3.47}$$

For Equation (3.46) we consider the Dirichlet condition following from the original boundary condition (1.2):

$$w(x)=0,\quad x\in\partial\Omega.\tag{3.48}$$

Substituting the test function $V=N^\tau_q$ into the integral identity (3.43), we find

$$T^\tau_{qp}=(\mathscr{A}\mathscr{D}(\nabla_y)N^\tau_{(p)},\mathscr{D}(\nabla_y)N^\tau_{(q)})_S-\tau(\rho N^\tau_{(p)},N^\tau_{(q)})_S.$$

Thus, T^τ is the difference of two (symmetric) Gram matrices. The first matrix is nonnegative definite, but the sign of the second matrix is unknown because, at least, ρ is a changing sign density (cf. the assumption (1.18)).

The authors see no way of proving the positive definiteness of the matrix T^τ in the general case (the matrix is a priori nonnegative, but can be singular) so that, at least an elliptic operator appears on the left-hand side of (3.46), which provides the discreteness of the spectrum of the problem (3.46), (3.48). If the matrix T^τ is singular, the spectrum is not necessarily discrete. In all the cases, it is impossible to make an informative conclusion about eigenvalues of the problem (1.1), (1.2). Furthermore, there are situations (cf., for example, comments in [**19**] to the paper [**20**])), where the formal asymptotic analysis leads to wrong conclusions.

The following remark shows how to get rid of the above-introduced simplified assumptions. However, such generalizations are little informative, as in the case of the above calculations.

Remark 3.7. If τ is a multiple eigenvalue and $\{U^1,\dots,U^\kappa\}$ is a basis for the corresponding eigenspace, then the ansatz (3.34) is replaced with the following:

$$u^\varepsilon(x)=\sum_{p=1}^{\kappa}w_p(x)U^p(\varepsilon^{-1}x)+\varepsilon w^{(1)}(\varepsilon^{-1}x,x)+\varepsilon^2w^{(2)}(\varepsilon^{-1}x,x)+\dots$$

and the vector $w=(w_1,\dots,w_\kappa)^\top$ becomes unknown. In this sense, the eigenvalue $\tau_0=0$ and the corresponding vector-valued eigenfunctions were also used in the asymptotic analysis of the previous subsections, where $U^p=e_p$, $p=1,\dots,n$. □

Remark 3.8. Let τ be a simple eigenvalue. If not all coefficients $i\mathbf{t}_1,\dots,i\mathbf{t}_n$ on the right-hand side of (3.41) vanish, then the solvability condition (3.40) can be written as the differential equation

$$i\tau\mathbf{t}^\top\nabla_x w(x)=\tau^{(1)}w(x),\quad x\in\Omega,\tag{3.49}$$

where $\mathbf{t} = (\mathbf{t}_1, \dots, \mathbf{t}_n)^\top$ is a number real column. Respectively, the problems (3.43) for the columns N^τ_p in the asymptotic corrector (3.42) are also modified:

$$\begin{aligned}&(\mathscr{A}\mathscr{D}(\nabla_y)N^\tau_{(p)}, \mathscr{D}(\nabla_y)V)_S - \tau(\rho N^\tau_{(p)}, V)_S\\&= (\mathscr{A}\mathscr{D}(\nabla_y)U, \mathscr{D}(e_p)V)_S - (\mathscr{A}\mathscr{D}(e_p)U, \mathscr{D}(\nabla_y)V)_S - i\mathbf{t}_p(\rho U, V)_S, \quad V \in H^1_{\mathrm{per}}(S)^k.\end{aligned}$$

Furthermore, we need a general solution to the system (3.40), i.e.,

$$w^{(1)}(y, x) = N^\tau(y)\nabla_x w(x) + w^{(10)}(x)U(y)$$

and $w^{(10)}$ is a new unknown function. As above, the columns $N^\tau_{(p)}$ satisfy the orthogonality condition (3.44).

The system (3.45) with respect to the terms $w^{(2)}$ and $\tau^{(2)}$ of the asymptotic ansätze (3.34) and (3.35) takes the form

$$\begin{aligned}&\mathscr{L}(y, \nabla_y)w^{(2)}(y, x) - \tau\rho(y)w^{(2)}(y, x)\\&= \overline{\mathscr{D}(\nabla_x)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_x)U(y)w(x) + \overline{\mathscr{D}(\nabla_x)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_y)(N^\tau(y)\nabla_x w(x) + U(y)w^{(10)}(x))\\&\quad + \overline{\mathscr{D}(\nabla_y)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_x)(N^\tau(y)\nabla_x w(x) + U(y)w^{(10)}(x))\\&\quad + \tau^{(1)}\rho(y)(N^\tau(y)\nabla_x w(x) + U(y)w^{10}(x)) + \tau^{(2)}\rho(y)U(y)w(x), \quad y \in S.\end{aligned}$$

Owing to the orthogonality condition (3.44), the modification of the corrector N^τ does not affect the final formulas (3.47) for the entries of the matrix T^τ, but the homogenized equation (3.46) takes the form

$$-\nabla_x^\top T^\tau \nabla_x w(x) = \tau^{(2)}\tau^{-1}w(x) + (\tau^{(1)}\tau^{-1}w^{(10)}(x) - i\tau\mathbf{t}^\top\nabla_x w^{(10)}(x)), \quad x \in \Omega, \tag{3.50}$$

Now, the right-hand side of (3.50) involves the differential operator $\tau^{(1)}\tau^{-1} - i\tau\mathbf{t}^\top\nabla_x$ which already appeared in (3.49). Therefore, it is reasonable to project this equation onto the subspace

$$\Big\{f \in L_2(\Omega) : (f, \tau^{(1)}\tau^{-1}v - i\tau\mathbf{t}^\top\nabla_x v)_\Omega = 0, \ v \in \mathring{H}^1(\Omega)\Big\},$$

and complete the obtained relation with the equalities (3.50) and (3.48). The obtained problem contains two spectral parameters, $\tau^{(1)}$ and $\tau^{(2)}$; moreover, $\tau^{(1)}$ is involved in the projection onto the subspace (3.50). Since the justification of asymptotics in the high-frequency range $|\lambda^\varepsilon| = O(\varepsilon^{-2})$ goes wrong, the authors do not consider this nonstandard spectral problem. □

4. Justification of Asymptotics

1. Case $\overline{\rho} = 0$; treatment of residuals. According to the asymptotic ansätze (3.16) and (3.15), the pair $\{\mathscr{M}, \mathscr{U}\}$ in Lemma 2.7 includes the following number and one of the vector-valued functions

$$\mathscr{M} = \varepsilon\beta^{-1}_{\pm j}, \quad \mathscr{U}_{(p)} = \|U_{(p)}; \mathscr{H}\|^{-1}U_{(p)}, \tag{4.1}$$

where $j \in \mathbb{N}$, $p = j, \dots, j + \varkappa_{\pm j} - 1$, and

$$U_{(p)}(x) = u^0_{(\pm p)}(x) + \varepsilon X_\varepsilon(x)\big(N(\varepsilon^{-1}x)\mathscr{D}(\nabla_x)u^0_{(\pm p)}(x) + \beta_{\pm j}N^0(\varepsilon^{-1}x)u^0_{(\pm p)}(x)\big); \tag{4.2}$$

$\beta_{\pm j}$ is an eigenvalue of the pencil (3.22) with multiplicity $\varkappa_{\pm j}$, i.e.,

$$\pm\beta_{\pm j\mp 1} < \pm\beta_{\pm j} = \cdots = \pm\beta_{\pm j\pm\varkappa_{\pm j}\mp 1} < \pm\beta_{\pm j\pm\varkappa_{\pm j}}, \tag{4.3}$$

and β_q are elements of the sequences (3.24) and (3.25). Finally, $u^0_{\pm j}, \dots, u^0_{\pm j\pm\varkappa_{\pm j}\mp 1}$ are the corresponding vector-valued eigenfunctions satisfying the orthogonality and normalization conditions (3.26), N and N^0 are the asymptotic correctors, i.e., a $(k\times K)$–matrix and a $(k\times k)$–matrix solving the problems (3.7) and (3.17) respectively, and X_ε is a smooth cut-off function introduced before Proposition 2.6.

We begin by computing the inner products $\langle U_{(p)}, U_{(q)}\rangle$ in accordance with the definitions (4.2) and (1.15). Differentiating and making the change of variables $y \mapsto x = \varepsilon y$, we find

$$\begin{aligned}
\mathscr{D}(\nabla_x)U_{(p)}(x) = {} & X_\varepsilon(x)\Big((\mathbb{I}_k + \mathscr{D}(\nabla_y)N(y))\mathscr{D}(\nabla_x)u^0_{(\pm p)}(x) \\
& + \beta_{\pm p}\mathscr{D}(\nabla_y)N^0(y)u^0_{(\pm p)}(x)\Big) \\
& + \varepsilon X_\varepsilon(x)\mathscr{D}(\nabla_x)\Big(N(y)\mathscr{D}(\nabla_x)u^0_{(\pm p)}(x) + \beta_{\pm p}N^0(y)u^0_{(\pm p)}(x)\Big) \\
& + \Big((1 - X_\varepsilon(x))\mathscr{D}(\nabla_x)u^0_{(\pm p)}(x) \\
& + \varepsilon[\mathscr{D}(\nabla_x), X_\varepsilon]\big(N(y)\mathscr{D}(\nabla_x)u^0_{(\pm p)}(x) + \beta_{\pm p}N^0(y)u^0_{(\pm p)}(x)\big)\Big) \\
& =: X_\varepsilon(x)J^{\varepsilon 1}_{(p)}(x) + \varepsilon X_\varepsilon(x)J^{\varepsilon 2}_{(p)}(x) + J^{\varepsilon 3}_{(p)}(x),
\end{aligned} \tag{4.4}$$

where $[\mathscr{D}(\nabla_x), X_\varepsilon]$ is the operator of multiplication by a $(K\times k)$–matrix-valued function $x \mapsto \mathscr{D}(\nabla_x)X_\varepsilon(x)$ with entries of order ε^{-1} in accordance with formula (2.23). The supports of these entries, as well as the support of the difference $1-X_\varepsilon$, are concentrated in the $(2h\varepsilon)$-neighborhood of $\partial\Omega$. Denote by $\Sigma^\varepsilon(\partial\Omega)$ the union of those periodicity cells that intersect this neighborhood; moreover,

$$\operatorname{mes}_n \Sigma^\varepsilon(\partial\Omega) = O(\varepsilon)$$

and

$$|u^0_{(\pm p)}(x)| \leqslant c\varepsilon$$

for $x \in \Omega \cap \Sigma^\varepsilon(\partial\Omega)$ by the boundary condition (3.11). Consequently, taking into account the inclusion $u^0_{(\pm p)} \in C^{2,\delta}(\Omega)^k$ (cf. Remark 3.5), which yields the boundedness of the vector-valued eigenfunction and its derivatives, we find

$$\begin{aligned}
\|J^{\varepsilon 3}_{(p)}; L_2(\Omega)\|^2 & \leqslant c \int\limits_{\Sigma^\varepsilon(\partial\Omega)} \Big(\sup_{x\in\Omega}|\nabla_x u^0_{(\pm p)}(x)|^2(1 + |N(\varepsilon^{-1}x)|^2) + \sup_{x\in\Omega}|u^0_{(\pm p)}(x)|^2||N^0(\varepsilon^{-1})|^2\Big)\, dx \\
& \leqslant c\,(1 + \|N; L_2(S)\|^2 + \varepsilon^2\|N^0; L_2(S)\|^2)\ \operatorname{mes}_n \Sigma^\varepsilon(\partial\Omega) \leqslant c\varepsilon.
\end{aligned}$$

Let $\Sigma^\varepsilon(\Omega)$ be the union of the cells intersecting Ω. Then

$$\begin{aligned}
& \varepsilon^2\|X_\varepsilon J^{\varepsilon 2}_{(p)}; L_2(\Omega)\|^2 \\
& \leqslant c\varepsilon^2 \int\limits_{\Sigma^\varepsilon(\Omega)} \Big(\sup_{x\in\Omega}|\nabla^2_x u^0_{(\pm p)}(x)|^2|N(\varepsilon^{-1}x)|^2 + \sup_{x\in\Omega}|\nabla_x u^0_{(\pm p)}(x)|^2|N^0(\varepsilon^{-1}x)|^2\Big)\, dx \leqslant c\varepsilon^2.
\end{aligned}$$

Thus, we have established that

$$\Big|\Big(\mathscr{A}^\varepsilon \mathscr{D}(\nabla_x)U_{(p)}, \mathscr{D}(\nabla_x)U_{(q)})_\Omega - (\mathscr{A}^\varepsilon X_\varepsilon J^{\varepsilon 1}_{(p)}, X_\varepsilon J^{\varepsilon 1}_{(q)}\Big)_\Omega\Big| \leqslant c\varepsilon. \tag{4.5}$$

To compute the subtrahend on the left-hand side of (4.5), we use Proposition 2.6 (the version (2.26)) and obtain the formula

$$\Big|\Big(\mathscr{A}^\varepsilon X_\varepsilon J^{\varepsilon 1}_{(p)}, X_\varepsilon J^{\varepsilon 1}_{(q)})_\Omega - \mathbf{a}(u^0_{(\pm p)}, u^0_{(\pm q)}; \Omega) - \beta_{\pm p}\beta_{\pm q}(\mathbf{m}u^0_{(\pm p)}, u^0_{(\pm q)}\Big)_\Omega\Big| \leqslant c\varepsilon. \tag{4.6}$$

Explanation. We replaced $J^{\varepsilon 1}_{(p)}$ and $J^{\varepsilon 1}_{(q)}$ by the sums (cf. (4.4)), took into account that the product of entries of $\mathscr{D}(\nabla_x)N$ and $\mathscr{D}(\nabla_x)N^0$ belongs to the space $L_1(S)$, whereas the product of components of the vector-valued functions $u^0_{(\pm p)}$ and $u^0_{(\pm q)}$, as well as their derivatives, belongs to the space $C^1(\Omega)$, and, finally, we computed the mean of rapidly oscillating factors over the cell S. We also used the definitions (3.10) and (3.19) of the matrices $\mathbf{A}$ and $\mathbf{m}$ and took into account the equality

$$\int_S \overline{(\mathscr{D}(\nabla_y)N^0(y))}^\top \mathscr{A}(y)(\mathbb{I}_K + \mathscr{D}(\nabla_y)N(y))\,dy$$

$$= \int_S \overline{N^0(y)}^\top \overline{(\mathscr{D}(-\nabla_y)}^\top \mathscr{A}(y)(\mathbb{I}_K + \mathscr{D}(\nabla_y)N(y))\,dy = 0 \in \mathbb{C}^{k\times K}$$

which annihilates crossing terms, i.e., terms containing both correctors N and N^0, and can be obtained by using the periodicity of the matrix-valued functions N, N^0, and $\mathscr{A}$, the integration–by–part formula, and the relations (3.6), (1.5).

Since the sum of two subtrahends under the sign of modulus in (4.6) forms the left-hand side of the orthogonality and normalization condition (3.26), from (4.4)–(4.6) it follows that

$$\big|(\mathscr{A}^\varepsilon \mathscr{D}(\nabla_x)U_{(p)}, \mathscr{D}(\nabla_x)U_{(q)})_\Omega - \delta_{p,q}\big| \leqslant c\varepsilon. \tag{4.7}$$

Therefore, for small $\varepsilon > 0$

$$\|U_{(p)}; \mathscr{H}\| \geqslant 1/2. \tag{4.8}$$

In order to apply Lemma 2.7, we estimate δ in (2.30). Taking into account (4.1), (1.15), and (1.16), we find

$$\begin{aligned}
\delta &= \big\|\mathscr{K}^\varepsilon \mathscr{U}_{(p)} - \mathscr{M}\mathscr{U}_{(p)}; \mathscr{H}\big\| \\
&= \sup_{\langle w,w\rangle=1} \big|\langle \mathscr{K}^\varepsilon \mathscr{U}_{(p)} - \mathscr{M}\mathscr{U}_{(p)}, w\rangle\big| \\
&= \mathscr{M}\|U_{(p)}; \mathscr{H}\|^{-1} \sup_{\langle w,w\rangle=1} \big|\mathscr{M}^{-1}\langle \mathscr{K}^\varepsilon U_{(p)}, w\rangle - \langle U_{(p)}, w\rangle\big| \\
&= \varepsilon\beta^{-1}_{\pm j} \sup_{\langle w,w\rangle=1} \Big|\Big(\mathscr{A}^\varepsilon \mathscr{D}(\nabla_x)U_{(p)}, \mathscr{D}(\nabla_x)w)_\Omega - \varepsilon^{-1}\beta_{\pm j}(U_{(p)}, w\Big)_\Omega\Big| \\
&\leqslant c\varepsilon \sup_{\langle w,w\rangle=1} \Big|\Big(\mathscr{L}U_{(p)} - \varepsilon^{-1}\beta_{\pm j}U_{(p)}, w\Big)_\Omega\Big|,
\end{aligned} \tag{4.9}$$

where the supremum is taken over $w \in \mathscr{H} = \mathring{H}^1(\Omega)^k$ such that $\|w; \mathscr{H}\| = 1$. We emphasize that, in the last transformation in the chain (4.9), we took into account the estimate (4.8) and used the Green formula. It is important that the support of the cut-off function X_ε is located inside the domain Ω and the vector-valued eigenfunctions are infinitely differentiable there (cf. Remark 3.5), i.e., the expression $\mathscr{L}U_{(p)}$ is well defined. However, for obtaining estimates, uniform with respect to ε, we should get rid of the third order derivatives of $u^0_{(\pm p)j}$.

We remove the cut-off function X_ε from the differential operator $\mathscr{L}^\varepsilon$ and then apply the inequalities (2.23) and (2.21):

$$\begin{aligned}&\varepsilon\big|([\mathscr{L}^\varepsilon, X_\varepsilon](N\mathscr{D}u^0_{(\pm p)} + \beta_{\pm j}N^0u^0_{(\pm p)}), w)_\Omega\big|\\&\leqslant c\varepsilon\varepsilon^{-2}(\|N; H^1(S)\| + \|N^0; H^1(S)\|)\|u^0_{(\pm p)}; C^2(\Omega)\|\,\varepsilon\int\limits_{\operatorname{supp}|\nabla_x X_\varepsilon|} r^{-1}|w(x)|\,dx\\&\leqslant c\varepsilon^{1/2}\|w; \mathscr{H}\|.\end{aligned}$$

Here, we took into account that $r < c\varepsilon$ on the set $\operatorname{supp}|\nabla_x X_\varepsilon|$ of volume $O(\varepsilon)$. Similarly,

$$\begin{aligned}&\big|((1 - X_\varepsilon)(\mathscr{L}^\varepsilon u^0_{(\pm p)} - \varepsilon^{-1}\beta_{\pm j}\rho u^0_{(\pm p)}), w)_\Omega\big|\\&\qquad\leqslant c\varepsilon^{-1}\left(\operatorname{mes}_n \operatorname{supp}(1 - X_\varepsilon)\right)^{1/2}\varepsilon\|r^{-1}w; \mathscr{H}\|\\&\qquad\leqslant c\varepsilon^{1/2}\|w; \mathscr{H}\|.\end{aligned}$$

In other words, with an (admissible) error $O(\varepsilon^{1/2})$ the last inner product in formula (4.9) is equal to the expression

$$(\varepsilon^{-1}I_1 + \varepsilon^{-1}I_2 + \varepsilon^0 I_3 + I_4, W^\varepsilon)_\Omega, \tag{4.10}$$

where $W^\varepsilon = X_\varepsilon w$ and

$$I_1(y, x) = \overline{\mathscr{D}(-\nabla_y)}^\top \mathscr{A}(y)(\mathbb{I}_K + \mathscr{D}(\nabla_y)N(y))\mathscr{D}(\nabla_x)u^0_{(\pm p)}(x),$$

$$I_2(y, x) = \beta_{\pm j}\left(\overline{\mathscr{D}(-\nabla_y)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_y)N^0(y) + \rho(y)\right)u^0_{(\pm p)}(x),$$

$$\begin{aligned}I_3(y, x) &= \overline{\mathscr{D}(-\nabla_x)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_x)u^0_{(\pm p)}(x)\\&+ \overline{\mathscr{D}(-\nabla_x)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_y)(N(y)\mathscr{D}(\nabla_x)u^0_{(\pm p)}(x) + \beta_{\pm j}N^0(y)u^0_{(\pm p)}(x))\\&+ \beta_{\pm j}\rho(y)(N(y)\mathscr{D}(\nabla_x)u^0_{(\pm p)}(x) + \beta_{\pm j}N^0(y)u^0_{(\pm p)}(x)),\end{aligned} \tag{4.11}$$

$$\begin{aligned}I_4(y, x) &= -\varepsilon\overline{\{\varepsilon^{-1}\mathscr{D}(\nabla_y) + \mathscr{D}(\nabla_x)\}}^\top \mathscr{A}(y)\mathscr{D}(\nabla_x)(N(y)\mathscr{D}(\nabla_x)u^0_{(\pm p)}(x)\\&+ \beta_{\pm j}N^0(y)u^0_{(\pm p)}(x)).\end{aligned}$$

As in (4.4), on the right-hand sides of (4.11) after the differentiaton, we should return from the fast variables (1.3) to the slow variable x. It is remarkable that the expression in the curly brackets in the last formula of (4.11) contains the operator $\mathscr{D}(\nabla_x)$ of "full" differentiation (cf. the rule (3.3)). Hence, integrating by parts, we obtain the equality

$$(I_4, W^\varepsilon)_\Omega = \varepsilon\int\limits_\Omega (\overline{\mathscr{D}(\nabla_x)W^\varepsilon(x)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_y)(N(y)\mathscr{D}(\nabla_x)u^0_{(\pm p)}(x) + \beta_{\pm j}N^0(y)u^0_{(\pm p)}(x))\,dx. \tag{4.12}$$

On the right-hand side of (4.12), there are only the first order and second order derivatives of vector-valued function $u^0_{(\pm p)}$ (cf. comment to (4.9)).

Again, taking into account the property of supports of the derivatives of X_ε, we find

$$\begin{aligned}\|\nabla_x W^\varepsilon; L_2(\Omega)\|^2 &\leqslant c\,(\|\nabla_x w; L_2(\Omega)\|^2 + \varepsilon^{-2}\|w; L_2(\operatorname{supp}|\nabla_x X_\varepsilon|)\|^2 \\ &\leqslant c\,(\|\nabla_x w; L_2(\Omega)\|^2 + \|r^{-1} w; L_2(\Omega)\|^2 \\ &\leqslant c\|w; \mathscr{H}\|^2 = c. \end{aligned} \tag{4.13}$$

Since the matrices $\mathscr{A}$, N, and N^0 in the integrand of (4.12) depending on the fast variable y are not differentiated, we find

$$|(I_4, W^\varepsilon)_\Omega| \leqslant c\varepsilon\|\nabla_x W^\varepsilon; L_2(\Omega)\| \leqslant c\varepsilon.$$

The components I_1 and I_2 of the first factor in (4.10) vanish by the definition of correctors (cf. (3.7) and (3.17)). The relations (3.9) and (3.19)–(3.21) show that the mean of I_3 over the cell $y \in S$ vanishes since $\{\beta_{\pm j}, u^0_{\pm j}\}$ is the spectral pair of the problem (3.21), (3.11). Furthermore, it is equal to the sum of products $Z(\varepsilon^{-1}x)\mathscr{Y}(x)$, where $Z \in L_2(S)$ and $\mathscr{Y} \in C^1(\overline{\Omega})$. Consequently, the assumptions of Proposition 2.6 (the version (2.25)) are satisfied; moreover, $Y = \mathscr{Y} W^\varepsilon$. Thus, by (2.27) and (4.13),

$$|(I_3, W^\varepsilon)_\Omega| \leqslant c\varepsilon\|\nabla_x W^\varepsilon; L_2(\Omega)\| = c\varepsilon.$$

Combining the obtained estimates, we get

$$\|\mathscr{K}^\varepsilon \mathscr{U}_{(p)} - \mathscr{M}\mathscr{U}_{(p)}; \mathscr{H}\| \leqslant c\varepsilon^{3/2}.$$

By Lemma 2.7, there is an eigenvalue μ^ε_q of the operator $\mathscr{K}^\varepsilon$ such that

$$|\mu^\varepsilon_q - \varepsilon\beta^{-1}_{\pm j}| \leqslant c\varepsilon^{3/2}. \tag{4.14}$$

For the eigenvalue λ^ε_q of problem (1.1), (1.2) connected with μ^ε_q by formula (1.17), we have

$$|\lambda^\varepsilon_q - \varepsilon^{-1}\beta_{\pm j}| \leqslant c\varepsilon^{1/2}\beta_{\pm j}\lambda^\varepsilon_q \quad \Rightarrow \quad \lambda^\varepsilon_q \leqslant c\varepsilon^{-1}(1 + c\varepsilon^{3/2}\lambda^\varepsilon_q),$$

i.e., for $\varepsilon \in (0, \varepsilon_j]$ and small $\varepsilon_j > 0$

$$|\lambda^\varepsilon_q - \varepsilon^{-1}\beta_{\pm j}| \leqslant c_j\varepsilon^{-1/2}. \tag{4.15}$$

Now, we use the second part of Lemma 2.7 with $\mathbf{C}_j\varepsilon^{3/2}$ instead of δ_1, where $\mathbf{C}_j$ is a large constant. We find columns $a^\varepsilon_{(p)}$ of coefficients a^ε_{pl} such that

$$\Big\|\mathscr{U}_{(p)} - \sum a^\varepsilon_{pl} u^\varepsilon_{(l)}; \mathscr{H}\Big\| \leqslant 2\delta_1^{-1}\delta \leqslant \mathbf{C}_j^{-1} c_j, \quad \sum |a^\varepsilon_{pl}|^2 = 1, \tag{4.16}$$

moreover, $\pm p = \pm j, \dots, \pm j \pm \varkappa_{\pm j} \mp 1$, and the sum is taken over those eigenvalues of the operator $\mathscr{K}^\varepsilon$ that satisfy the inequality

$$|\mu^\varepsilon_l - \varepsilon\beta^{-1}_{\pm j}| \leqslant \mathbf{C}_j\varepsilon^{3/2}. \tag{4.17}$$

For small ε they have the same sign as $\beta^{-1}_{\pm j}$ (the assumption $\delta_1 \in (\delta, \varepsilon|\beta_{\pm j}|^{-1})$ of Lemma 2.7 is satisfied) and form the sequence $\{\mu^\varepsilon_{\pm J(\varepsilon)}, \dots, \mu^\varepsilon_{\pm J(\varepsilon) \pm K(\varepsilon) \mp 1}\}$. From (4.16), (2.30), (1.24) and (4.1), (4.7), (4.8) we find

$$\langle \mathscr{U}_{(p)}, \mathscr{U}_{(q)}\rangle = \Big\langle \mathscr{U}_{(q)} - \sum a^\varepsilon_{pl} u^\varepsilon_{(l)}, \mathscr{U}_{(q)}\Big\rangle + \Big\langle \mathscr{U}_{(p)}, \mathscr{U}_{(q)} - \sum a^\varepsilon_{ql} u^\varepsilon_{(l)}\Big\rangle + (\overline{a^\varepsilon_{(q)}})^\top a^\varepsilon_{(p)}$$

and

$$\left|\langle \mathscr{U}_{(q)}, \mathscr{U}_{(p)}\rangle - \delta_{p,q}\right| = \|\mathscr{U}_{(p)}; \mathscr{H}\|^{-1} \|\mathscr{U}_{(p)}; \mathscr{H}\|^{-1} \left|\langle U_{(p)}, U_{(q)}\rangle - \|U_{(p)}; \mathscr{H}\|^2\right| \leqslant c\varepsilon.$$

Then we obtain the estimate

$$\left|(\overline{a^{\varepsilon}_{(q)}})^{\top} a^{\varepsilon}_{(p)} - \delta_{p,q}\right| \leqslant c\,(\varepsilon + \mathbf{C}_j^{-1}) \tag{4.18}$$

which means that for small ε and $\mathbf{C}_j^{-1}$ the columns $a^{\varepsilon}_{(j)}, \ldots, a^{\varepsilon}_{(j+\varkappa_j-1)}$ of height $K(\varepsilon)$ are "almost orthonormal," which is possible only if

$$K(\varepsilon) \geqslant \varkappa_j. \tag{4.19}$$

Consequently, there are at least $\varkappa_j$ eigenvalues of the operator $\mathscr{K}^{\varepsilon}$ satisfying the inequality (4.17) which is not too different from the inequality (4.14) and, consequently, implies the estimate (4.15) for $\varkappa_j$ eigenvalues of the problem (1.1), (1.2). Our next goal is to verify the equalities $K(\varepsilon) = \varkappa_j$ and $J(\varepsilon) = j$.

2. Case $\overline{\rho} = 0$; convergence and approximation error. We fix $j \in \mathbb{N}$. By (1.24), the $H^1(\Omega)$-norms of vector-valued eigenfunctions $u^{\varepsilon}_{\pm j}$ are uniformly bounded with respect to the parameter $\varepsilon \in (0, \varepsilon_0]$:

$$\|u^{\varepsilon}_{(\pm j)}; H^1(\Omega)\| \leqslant c. \tag{4.20}$$

Consequently, there exists an infinitely small sequence $\{\varepsilon_m\}_{m=1}^{\infty}$, along which

$$u^{\varepsilon}_{(\pm j)} \to \widehat{u}_{(\pm j)} \quad \text{weakly in } H^1(\Omega)^k \text{ and strongly in } L_2(\Omega)^k. \tag{4.21}$$

Furthermore, taking into account (4.15), we find

$$\pm\varepsilon\lambda^{\varepsilon}_j \leqslant \pm\varepsilon\lambda^{\varepsilon}_q \leqslant \pm\beta_{\pm j} + C_j\varepsilon^{1/2} \leqslant c_j. \tag{4.22}$$

Explanation. Since for every $j \in \mathbb{N}$ there is a subscript $q = q(j) \in \mathbb{N}$ for which the inequality (4.15) holds, from (4.19) it follows that $\{\lambda^{\varepsilon}_{\pm q(1)}, \ldots, \lambda^{\varepsilon}_{\pm q(j)}\}$ are ordered, i.e., $q(j) \geqslant j$. By (4.22), there exists an infinitely small sequence $\{\varepsilon_m\}$ (we can assume that it coincides with the above sequence) such that

$$\varepsilon_m \lambda^{\varepsilon_m}_{\pm j} \to \widehat{\beta}_{\pm j} \quad \text{for } m \to +\infty. \tag{4.23}$$

For an arbitrary vector-valued function $V \in C^{\infty}_c(\Omega)^k$ we construct a test function in the integral identity (1.10) in accordance with the asymptotic ansatz (4.2):

$$v^{\varepsilon}(x) = V(x) + \varepsilon\left(N(\varepsilon^{-1}x)\mathscr{D}(\nabla_x)V(x) + \varepsilon\lambda^{\varepsilon}_{\pm j}N^0(\varepsilon^{-1}x)V(x)\right).$$

It is not necessary to use a cut-off function X_{ε} since $v^{\varepsilon} = 0$ in a neighborhood of the boundary $\partial\Omega$. Integrating by parts, we obtain the equality

$$0 = (u^{\varepsilon}_{(\pm j)}, \mathscr{L}v^{\varepsilon} - \lambda^{\varepsilon}_{\pm j}v^{\varepsilon})_{\Omega}. \tag{4.24}$$

Repeating the above calculations (which are even simpler since there is no cut-off function), we see that the difference in the second position of the inner product (4.24) is expressed as follows:

$$
\begin{aligned}
&\overline{\mathscr{D}(-\nabla_x)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_x)V(x) + \overline{\mathscr{D}(-\nabla_x)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_y)\big(N(y)\mathscr{D}(\nabla_x)V(x)\\
&+ \varepsilon\lambda^\varepsilon_{\pm j} N^0(y)V(x)\big) + \varepsilon\lambda^\varepsilon_{\pm j}\rho(y)\big(N(y)\mathscr{D}(\nabla_x)V(x) + \varepsilon\lambda^\varepsilon_{\pm j}N^0(y)V(x)\big)\\
&+ \overline{\mathscr{D}(-\nabla_y)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_x)\big(N(y)\mathscr{D}(\nabla_x)V(x) + \varepsilon\lambda^\varepsilon_{\pm j}N^0(y)V(x)\big)\\
&+ \varepsilon\overline{\mathscr{D}(-\nabla_x)}^\top \mathscr{A}(y)\mathscr{D}(\nabla_x)\big(N(y)\mathscr{D}(\nabla_x)V(x) + \varepsilon\lambda^\varepsilon_{\pm j}N^0(y)V(x)\big)\\
&=: I_1(V) + I_2(V) + I_3(V) + I_3(V) + \varepsilon I_5(V),
\end{aligned}
\tag{4.25}
$$

as usual, we mean that the replacement $y \mapsto x = \varepsilon y$ was made after the differentiation. It is clear that the estimates (4.22) and (4.20) imply

$$|(u^\varepsilon_{(\pm j)}, \varepsilon I_5(V))_\Omega| \leqslant c\varepsilon \to +0.$$

The mean of $I_4(V)$ over the cell $S \ni y$ vanishes, and $I_4(V)$ is represented as the sum of expressions $Z(\varepsilon^{-1}x)\mathscr{Y}(x)$, where $Z \in L_2(S)$ and $\mathscr{Y} \in C^{0,\gamma}(\Omega)$. Hence, by the estimate (4.20) and Proposition 2.6 with $Y = \mathscr{Y}u^\varepsilon_{(\pm j)}$, we have

$$|(u^\varepsilon_{(\pm j)}, I_4(V))_\Omega| \leqslant c\varepsilon^{1/2} \to +0.$$

From the sum of the first three terms in (4.25) we subtract the expression

$$\mathbf{I}(V, \varepsilon\lambda^\varepsilon_{\pm j}) = \mathbf{L}(\nabla_x)V(x) + \varepsilon\lambda^\varepsilon_{\pm j}\mathbf{B}(\nabla_x)V(x) - \varepsilon^2(\lambda^\varepsilon_{\pm j})^2\mathbf{m}V(x), \tag{4.26}$$

where the operators $\mathbf{L}$, $\mathbf{B}$ and matrices $\mathbf{A}$, $\mathbf{m}$ are defined by formulas (1.28), (3.32) and (3.9), (3.19) respectively. Then the resulting expression also has zero mean over the cell $S \ni y$. As above,

$$|(u^\varepsilon_{(\pm j)}, I_1(V) + I_2(V) + I_3(V) - \mathbf{I}(V, \varepsilon\lambda^\varepsilon_{\pm j}))_\Omega| \leqslant c\varepsilon^{1/2} \to +0.$$

It remains to note that (4.21) and (4.23) imply the convergence

$$(u^{\varepsilon_m}_{(\pm j)}, \mathbf{I}(V, \varepsilon_m\lambda^{\varepsilon_m}_{\pm j}))_\Omega \to (\widehat{u}_{(\pm j)}, \mathbf{I}(V, \widehat{\lambda}_{\pm j}))_\Omega \quad \text{as} \quad m \to +\infty.$$

Passing to the limit along the sequence $\{\varepsilon_m\}_{m=0}^\infty$ in (4.24), we obtain the integral identity

$$(\widehat{u}_{(\pm j)}, \mathbf{I}(V, \widehat{\beta}_{\pm j}))_\Omega = 0, \quad V \in C^\infty_c(\Omega)^k,$$

which can be written in the form

$$\mathbf{a}(\widehat{u}_{(\pm j)}, V; \Omega) - \widehat{\beta}_{\pm j}\mathbf{b}(\widehat{u}_{(\pm j)}, V; \Omega) - (\widehat{\beta}_{\pm j})^2(\mathbf{m}\widehat{u}_{(\pm j)}, V)_\Omega = 0, \quad V \in \mathring{H}^1(\Omega)^k,$$

in view of the definitions (3.12), (3.23), (4.26) and the closure in the $H^1(\Omega)^k$-norm. Thus, $\widehat{\beta}_{\pm j}$ is an eigenvalue of the problem (3.22) only if $\widehat{u}_{(\pm j)} \neq 0$.

We check the last relation. First, by (3.17), we have

$$
\begin{aligned}
\varepsilon^{-1}(\rho u^\varepsilon_{(\pm j)}, u^\varepsilon_{(\pm j)})_\Omega &= \varepsilon^{-1}\left(\left(\overline{\mathscr{D}(-\nabla_y)}^\top \mathscr{A}\mathscr{D}(\nabla_y)N^0\right)u^\varepsilon_{(\pm j)}, u^\varepsilon_{(\pm j)}\right)_\Omega\\
&= \left(\left(\mathscr{A}\mathscr{D}(\nabla_y)N^0\right)u^\varepsilon_{(\pm j)}, \mathscr{D}(\nabla_x)u^\varepsilon_{(\pm j)}\right)_\Omega\\
&+ \sum_{p=1}^n \left(\frac{\partial u^\varepsilon_{(\pm j)}}{\partial x_p}, \left(\overline{\mathscr{D}(\nabla_y)N^0}\right)^\top \mathscr{A}\mathscr{D}(e_p)u^\varepsilon_{(\pm j)}\right)_\Omega.
\end{aligned}
\tag{4.27}
$$

Second, $\mathscr{A}\mathscr{D}(\nabla_y)N^0 \in L_\infty(S)^{N\times k}$ in view of Proposition 2.3 (3). Consequently, by Proposition 2.6 (the variant (2.26)), from (4.21) it follows that

$$(\mathscr{A}\mathscr{D}(\nabla_y)N^0)u^\varepsilon_{(\pm j)} \quad \to \quad M\widehat{u}_{(\pm j)} \quad \text{strongly in } L_2(\Omega)^{N\times k}, \tag{4.28}$$

where M is the mean of the matrix-valued function $\mathscr{A}\mathscr{D}(\nabla_y)N^0$ over the cell S. We emphasize that the operator $\mathscr{D}(\nabla_y)$ in (4.27) and (4.28) acts on the corrector N^0, but not on $u^\varepsilon_{(\pm j)}$.

Third, owing to the normalization condition (1.24) and the identity (1.10), the left-hand side of (4.27) has the limit $(\widehat{\beta}_{\pm j})^{-1}$ (which can be infinite) as $\varepsilon \to 0$, whereas the right-hand side is the sum of inner products where one factor is a term of a weakly converging sequence in $L_2(\Omega)$ and another factor is a term of a strongly converging sequence. Hence the limit of the right-hand side vanishes because of the assumption $\widehat{u}_{(\pm j)} = 0$. The obtained contradiction means that $\widehat{u}_{(\pm j)} \neq 0$ is the vector-valued eigenfunction of the problem (3.22) corresponding to the eigenvalue $\widehat{\beta}_{\pm j}$.

Now, we are ready to check the first main assertion of the paper.

Theorem 4.1. *Suppose that the assumptions about the smoothness of the data of the problem* (1.1), (1.2) *are satisfied and the mean* (1.25) *of the density ρ vanishes. We also assume that $\beta_{\pm j}$ is an eigenvalue of the pencil* (3.21), (3.11) (*or* (3.22)) *with multiplicity $\varkappa_{\pm j}$, i.e., formula* (4.3) *is valid. Then there exists $\varepsilon_j > 0$ and c_j such that for $\varepsilon \in (0, \varepsilon_j]$ the eigenvalues $\lambda^\varepsilon_{\pm j}, \dots, \lambda^\varepsilon_{\pm j \pm \varkappa_{\pm j} \mp 1}$ of the problem* (1.10) *and only they satisfy the inequality* (4.15). *Furthermore, there are numbers C_j and columns $b^\varepsilon_{(p)} = (b^\varepsilon_{\pm j,p}, \dots, b^\varepsilon_{\pm j \pm \varkappa_{\pm j} \mp 1, p})^\top$, $p = \pm j, \dots, \pm j \pm \varkappa_{\pm j} \mp 1$, that form a Hermitian $(\varkappa_{\pm j} \times \varkappa_{\pm j})$–matrix b^ε and*

$$\Big\| u^\varepsilon_{(p)} - \sum_{q=\pm j}^{\pm j \pm \varkappa_{\pm j} \mp 1} b^\varepsilon_{qp} U^0_{(q)}; H^1(\Omega) \Big\| \leqslant C_j \varepsilon^{1/2}. \tag{4.29}$$

Here, $\varepsilon \in (0, \varepsilon_j]$, $p = \pm j, \dots, \pm j \pm \varkappa_{\pm j} \mp 1$,

$$U^0_{(q)}(\varepsilon, x) = u^0_{(q)}(x) - \varepsilon\big(N(\varepsilon^{-1}x)\mathscr{D}(\nabla_x)u^0_{(q)}(x) - \beta_{\pm j} N^0(\varepsilon^{-1}x)u^0_{(q)}(x)\big), \tag{4.30}$$

$u^\varepsilon_{(p)}$ and $u^0_{(p)}$ are vector-valued eigenfunctions of the problems (1.10) *and* (3.22) *satisfying the orthogonality and normalization conditions* (1.24) *and* (3.26) *respectively, and N and N^0 are asymptotic correctors i.e., periodic solutions to the problems* (3.7) *and* (3.17) *on the cell S.*

Remark 4.2. By Lemma 2.5, the vector-valued functions (4.2) and (4.30) (the cut-off function X_ε is absent in (4.30)) satisfy the inequality

$$\|U_{(p)} - U^0_{(p)}; H^1(\Omega)\| \leqslant c_p \varepsilon^{1/2}. \tag{4.31}$$

Consequently, the relation (4.7) yields the estimate

$$|\langle U^0_{(p)}, U^0_{(q)} \rangle - \delta_{p,q}| \leqslant c_p \varepsilon^{1/2}, \tag{4.32}$$

i.e., the asymptotic approximations (4.30) are “almost subject” to the orthogonality and normalization conditions (1.24). □

Proof of Theorem 4.1. As was already established, the subscript q in the inequality (4.15) takes the values $\pm J_{\pm j}(\varepsilon), \dots, \pm J_{\pm j}(\varepsilon) \pm K(\varepsilon) \mp 1$ and formula (4.19) holds. Thus, to verify the first assertion of the theorem, it remains to check that

$$J_{\pm j}(\varepsilon)=j,\quad K_{\pm j}(\varepsilon)=\varkappa_{\pm j}. \tag{4.33}$$

If $\varepsilon_j>0$ is sufficiently small, then formula (4.19) remains valid after the replacement $j\mapsto l$ for any $l=1,\dots,j$. Thus, if one of the equalities (4.33) fails, then for some l there are eigenvalues $\lambda^\varepsilon_{\pm h},\dots,\lambda^\varepsilon_{\pm h\pm\varkappa_{\pm l}}$ (the number of which is greater than the multiplicity $\varkappa_{\pm l}$ of the eigenvalue $\beta_{\pm l}$) such that

$$\varepsilon\lambda^\varepsilon_{\pm t}\to\beta_{\pm l},\quad t=h,\dots,h+\varkappa_{\pm l}.$$

By (1.10) and (1.24), any linear combination of the corresponding vector-valued eigenfunctions

$$\mathscr{U}^\varepsilon=c_{\pm h}u^\varepsilon_{(\pm h)}+\dots+c_{\pm h\pm\varkappa_{\pm l}}u^\varepsilon_{(\pm h)\pm\varkappa_{\pm l}},\qquad\sum_{t=h}^{t+\varkappa_{\pm l}}|c_{\pm t}|^2=1,$$

satisfies the relation

$$1=\langle\mathscr{U}^\varepsilon,\mathscr{U}^\varepsilon\rangle=\sum_{t=h}^{t+\varkappa_{\pm l}}|c_{\pm t}|^2\lambda^\varepsilon_{\pm t}(\rho u^\varepsilon_{(\pm t}, u^\varepsilon_{(\pm t)})_\Omega.$$

Consequently, as was shown above, the strong convergence $\mathscr{U}^\varepsilon\to\widehat{u}$ holds in $L_2(\Omega)^k$ and $\widehat{u}\neq0$ is a vector-valued eigenfunction of the pencil (3.22). The dimension of the space of such limits cannot exceed the dimension $\varkappa_{\pm l}$ of the eigensubspace of the pencil corresponding to $\beta_{\pm l}$. The obtained contradiction proves (4.33).

To verify the second assertion of the theorem, we again use Lemma 2.7. For $\mathbf{C}_j$ in the inequality (4.17) we set $\mathbf{C}^0_j\varepsilon^{-1/2}$ and choose a constant $\mathbf{C}^0_j$ so that the relation (4.17) is satisfied by only the eigenvalues $\mu^\varepsilon_{\pm j},\dots,\mu^\varepsilon_{\pm j\pm\varkappa_{\pm j}\mp1}$ of the operator $\mathscr{K}^\varepsilon$, which becomes possible owing to the first assertion of the theorem about eigenvalues $\lambda^\varepsilon_{\pm j},\dots,\lambda^\varepsilon_{\pm j\pm\varkappa_{\pm j}\mp1}$ of the problem (1.10) in terms of the inverses of (1.17). Now, formula (4.18) means that the columns $a^\varepsilon_{(\pm j)},\dots,a^\varepsilon_{(\pm j\pm\varkappa_{\pm j}\mp1)}$ form an "almost unitary" matrix, whereas the first relation in (4.16) takes the form

$$\Big\|u^\varepsilon_{(p)}-\sum_{q=\pm j}^{\pm j\pm\varkappa_{\pm j}\mp1}b^\varepsilon_{qp}U_{(q)};H^1(\Omega)\Big\|\leqslant C_j\varepsilon^{1/2}$$

in view of Lemma 2.8. Moreover, $b^\varepsilon=(b^\varepsilon_{(\pm j)},\dots,b^\varepsilon_{(\pm j\pm\varkappa_{\pm j}\mp1)})$ is a unitary matrix. To obtain the inequality (4.29), it remains to recall the definitions (4.1), (4.2), (4.30) and the estimates (4.7), (4.31). □

We weaken the results of Theorem 4.1 by stating them as assertions about convergence.

Corollary 4.3. 1) *Under the assumptions of Theorem* 4.1, *elements of the sequences* (1.22), (1.23) *and* (3.24), (3.25) *of eigenvalues of the problems* (1.10) *and* (3.22) *are related as follows*:

$$\varepsilon\lambda^\varepsilon_{\pm j}\quad\to\quad\beta_{\pm j}\quad\text{as}\quad\varepsilon\to+0.$$

2) *Suppose, in addition, that $\beta_{\pm j}$ is a simple eigenvalue. Then for small $\varepsilon>0$ the eigenvalue $\lambda^\varepsilon_{\pm j}$ is also simple and for the corresponding vector-valued eigenfunction $u^\varepsilon_{(\pm j)}$ of the problem* (1.10) *satisfying the normalization condition* (1.24) *the following assertions hold*:

$$\begin{aligned}&u^\varepsilon_{(\pm j)}\quad\to\quad u^0_{(\pm j)}\quad\text{strongly in } L_2(\Omega)^k,\\&\mathscr{A}^\varepsilon\mathscr{D}(\nabla_x)u^\varepsilon_{(\pm j)}\quad\to\quad\mathbf{A}\mathscr{D}(\nabla_x)u^0_{(\pm j)}+\beta_{\pm j}\mathbf{n}^0u^0_{(\pm j)}\quad\text{weakly in } L_2(\Omega)^k,\\&u^\varepsilon_{(\pm j)}-\varepsilon(N\mathscr{D}(\nabla_x)u^0_{(\pm j)}+\beta_{\pm j}N^0u^0_{(\pm j)})\quad\to\quad u^0_{(\pm j)}\quad\text{strongly in } H^1(\Omega)^k.\end{aligned}\tag{4.34}$$

Here, $u^0_{(\pm j)}$ *is a vector-valued eigenfunction of the problem* (3.22) *satisfying the normalization condition* (3.26) *and*

$$\mathbf{n}^0 = \int\limits_S \mathscr{A}(y)\mathscr{D}(\nabla_y)N^0(y)\,dy \in \mathbb{C}^{K\times k}. \tag{4.35}$$

Proof. It suffices to check only the second convergence in (4.34) and the equality (4.35) since the remaining assertions directly follow from Theorem 4.1. Differentiating (4.30), we find

$$\begin{aligned}\mathscr{A}(y)\mathscr{D}(\nabla_x)U^0(\varepsilon,x) = \Big(\mathscr{A}(y) + \mathscr{A}(y)\mathscr{D}(\nabla_y)N(y)\Big)\mathscr{D}(\nabla_x)u^0_{(\pm j)}(x)\\ + \beta_{\pm j}\mathscr{A}(y)\mathscr{D}(\nabla_y)N^0(y)u^0_{(\pm j)}(x)\\ + \varepsilon\Big(\mathscr{A}(y)\mathscr{D}(\nabla_x)N(y)\mathscr{D}(\nabla_x)u^0_{(\pm j)}(x) + \beta_{\pm j}\mathscr{A}(y)\mathscr{D}(\nabla_x)N^0(y)u^0_{(\pm j)}(x)\Big),\end{aligned}$$

where, as usual, we apply our agreement about the use of fast and slow variables. Now, the required assertion is obvious because the last term contains the small parameter ε. □

3. Approximation error and convergence in the case $\overline{\rho} > 0$. Partial analogs of Theorem 4.1 and Corollary 4.3 in the case $\overline{\rho} > 0$ are obtained by repeating the above arguments (which are considerably simpler) in the case $\overline{\rho} = 0$. Moreover, we can work within the framework of standard methods of the homogenization theory (cf. [**11, 12, 13**] etc.). Therefore, we restrict ourselves to formulations in the case of the positive part (1.22) of the spectrum of the problem (1.1), (1.2). Some (not complete) information about the negative part (1.23) of the spectrum will be given in the following subsection.

Theorem 4.4. *Suppose that the smoothness assumptions on the data of the problem* (1.1), (1.2) *listed in Subsection* 1.1 *are satisfied and the mean* (1.25) *of the density* ρ *is positive. Let* λ^0_j *be an eigenvalue of the problem* (3.8), (3.11) (*of* (3.12)) *with multiplicity* $\varkappa_j$, *i.e.,*

$$\lambda^0_{j-1} < \lambda^0_j = \dots = \lambda^0_{j+\varkappa_j-1} < \lambda^0_{j+\varkappa_j}.$$

Then there exist $\varepsilon_j > 0$ *and* c_j *such that for all* $\varepsilon \in (0, \varepsilon_j]$ *the eigenvalues* $\lambda^\varepsilon_j, \dots, \lambda^\varepsilon_{j+\varkappa_j-1}$ *of the problem* (1.10) *and only they satisfy the inequality*

$$|\lambda^\varepsilon_p - \lambda^0_j| \leqslant c_j\varepsilon^{1/2}. \tag{4.36}$$

Furthermore, there is a number C_j *and columns* $b^\varepsilon_{(p)} = (b^\varepsilon_{j,p}, \dots, b^\varepsilon_{j+\varkappa_j-1,p})^\top$, $p = j, \dots, j+\varkappa_j-1$, *that form a Hermitian* ($\varkappa_j \times \varkappa_j$)*–matrix* b^ε *and*

$$\Big\|u^\varepsilon_{(p)} - \sum_{q=j}^{j+\varkappa_j-1} b^\varepsilon_{qp}\left(u^0_{(q)} - \varepsilon N\mathscr{D}(\nabla_x)u^0_{(q)}\right); H^1(\Omega)\Big\| \leqslant C_j\varepsilon^{1/2}, \quad p = j, \dots, j+\varkappa_j-1.$$

Here, $\varepsilon \in (0, \varepsilon_j]$, $u^\varepsilon_{(p)}$ *and* $u^0_{(p)}$ *are vector-valued eigenfunctions of the problems* (1.10) *and* (3.12) *satisfying the orthogonality condition* (1.24) *and the normalization condition* (3.14) *respectively, and* N *an the asymptotic corrector, i.e., a periodic solution to the problem* (3.7) *on the cell* S.

Corollary 4.5. 1) *Under the assumptions of Theorem* 4.4, *elements of the sequences* (1.22) *and* (3.13) *of the eigenvalues of the problems* (1.10) *and* (3.22) *are related as follows*:

$$\lambda^\varepsilon_j \quad\to\quad \lambda^0_j \quad \text{as} \quad \varepsilon \to +0.$$

2) *Let, in addition,* λ_j^0 *be a simple eigenvalue. Then for small* $\varepsilon > 0$ *the eigenvalue* λ_j^ε *is also simple and the following assertions hold for the corresponding vector-valued eigenfunction* $u_{(j)}^\varepsilon$ *of the problem* (1.10) *satisfying the normalization condition* (1.24):

$$\begin{aligned} u_{(j)}^\varepsilon \quad &\to \quad u_{(j)}^0 \quad \text{strongly in } L_2(\Omega)^k, \\ \mathscr{A}^\varepsilon \mathscr{D}(\nabla_x) u_{(j)}^\varepsilon \quad &\to \quad \mathbf{A}\mathscr{D}(\nabla_x) u_{(j)}^0 \quad \text{weakly in } L_2(\Omega)^k, \\ u_{(j)}^\varepsilon - \varepsilon N \mathscr{D}(\nabla_x) u_{(j)}^0 \quad &\to \quad u_{(j)}^0 \quad \text{strongly in } H^1(\Omega)^k. \end{aligned}$$

Here, $u_{(\pm j)}^0$ *is a vector-valued eigenfunction of the problem* (3.12) *satisfying the normalization condition* (3.14).

4. Upper estimate for the negative part of the spectrum in the case $\overline{\rho} > 0$. The main goal of this subsection is to verify the last inequality in formula (1.27).

Theorem 4.6. *If* $\overline{\rho} > 0$, *then there is a positive number* θ *such that for* $\lambda \in (-\varepsilon^{-2}\theta, 0)$ *the variational problem*

$$(\mathscr{A}^\varepsilon \mathscr{D}(\nabla_x) u, \mathscr{D}(\nabla_x) v)_\Omega - \lambda(\rho u, v)_\Omega = f(v), \quad v \in \mathring{H}^1(\Omega)^k, \tag{4.37}$$

has a unique solution $u \in \mathring{H}^1(\Omega)^k$ *for any linear functional* $f \in \mathscr{H}^*$ *on the space* $\mathscr{H} = \mathring{H}^1(\Omega)^k$ *and the following estimate holds:*

$$\|u; \mathscr{H}\| \leqslant c\,(\lambda) \|f; \mathscr{H}^*\|,$$

where $c\,(\lambda)$ *is independent of* f.

Proof. Owing to the Riesz theorem on representation of linear functionals in a Hilbert space, it suffices to check that the left-hand side of (4.37) is the inner product in the space $\mathscr{H}$.

We extend $u \in \mathring{H}^1(\Omega)^k$ by zero outside the domain Ω and represent it on each cell (2.24) in the form

$$u(x) = \mathbf{p}(x - x^\alpha) b^\alpha + u_\perp^\alpha(x), \tag{4.38}$$

where x^α is the center of the cube S_ε^α, $\mathbf{p} = (p^1, \dots, p^d)$ is a polynomial $(k \times d)$–matrix, $d = \dim \mathscr{P}$, $\{p^1, \dots, p^d\}$ is a basis for the polynomial space (1.8) composed of columns $p^j(x)$ of homogeneous polynomials of degree σ^j in the variable x; moreover,

$$0 = \sigma^1 = \dots = \sigma^k < \sigma^{k+1} \leqslant \ldots \leqslant \sigma^d.$$

The number column $b^\alpha \in \mathbb{C}^d$ is defined by the formula

$$b^\alpha = \mathbf{P}(\varepsilon)^{-1} \int\limits_{S_\varepsilon^\alpha} \overline{\mathbf{p}(x - x^\alpha)}^\top u(x)\, dx, \tag{4.39}$$

$$\mathbf{P}(\varepsilon) = \int\limits_{S_\varepsilon^\alpha} \overline{\mathbf{p}(x - x^\alpha)}^\top \mathbf{p}(x - x^\alpha)\, dx.$$

It is clear that

$$\mathbf{P}(\varepsilon) = \mathscr{E}(\varepsilon) \mathbf{P}(1) \mathscr{E}(\varepsilon),$$

where

$$\mathscr{E}(\varepsilon)=\operatorname{diag}\{\varepsilon^{\sigma_1+n/2},\dots,\varepsilon^{\sigma_d+n/2}\}$$

and $\mathbf{P}(1)$ is a Hermitian positive definite Gram $(d\times d)$–matrix independent of the parameter ε. We construct the column $b^{\alpha\prime}$ from the first k elements b_j^α (they are associated with constant polynomials) and the column $b^{\alpha\prime\prime}\in\mathbb{C}^{d-k}$ from the remaining elements. Since $p^{k+1},\dots,p^d$ are polynomials of positive degree, we have

$$\|\nabla_x\mathbf{p}(x-x^\alpha)b^\alpha;L_2(S_\varepsilon^\alpha)\|^2\leqslant c\sum_{j=k+1}^d\varepsilon^{2(\sigma_j-1)+n}|b_j^\alpha|^2$$
$$\leqslant c\|\nabla_x u;L_2(S_\varepsilon^\alpha)\|^2. \tag{4.40}$$

By the definitions (4.39) and (4.38), the vector-valued function $u_\perp^\alpha$ satisfies the orthogonality condition

$$\int\limits_{S_\varepsilon^\alpha}\overline{\mathbf{p}(x-x^\alpha)}^\top u_\perp^\alpha(x)\,dx=0\in\mathbb{C}^d. \tag{4.41}$$

By Lemma 2.4 with $\mathscr{X}(u_\perp^\alpha)$ coinciding with the left-hand side of (4.41),

$$\varepsilon^{-2}\|u_\perp^\alpha;L_2(S_\varepsilon^\alpha)\|^2+\|\nabla_x u_\perp^\alpha;L_2(S_\varepsilon^\alpha)\|^2\leqslant c\|\mathscr{D}(\nabla_x)u_\perp^\alpha;L_2(S_\varepsilon^\alpha)\|^2$$
$$=c\|\mathscr{D}(\nabla_x)u;L_2(S_\varepsilon^\alpha)\|^2. \tag{4.42}$$

We emphasize that the factor ε^{-2} appears because of the coordinate stretching $x\mapsto y=\varepsilon^{-1}x$, made before applying the lemma; moreover, the last equality is valid since

$$\mathscr{D}(\nabla_x)\mathbf{p}=0\in\mathbb{C}^{K\times d}.$$

Taking into account (4.42) and (4.40), we get

$$\frac12(\mathscr{A}^\varepsilon\mathscr{D}(\nabla_x)u,\mathscr{D}(\nabla_x)u)_{S_\varepsilon^\alpha}-\lambda(\rho u,u)_{S_\varepsilon^\alpha}$$
$$\geqslant\frac12 c_{\mathscr{A}}\|\mathscr{D}(\nabla_x)u_\perp^\alpha;L_2(S_\varepsilon^\alpha)\|^2+\frac12|\lambda|\varepsilon^n\overline{\rho}|b^{\alpha\prime}|^2$$
$$-c|\lambda|\Bigg(\sum_{j=k+1}^d\varepsilon^{2\rho_j+n}|b_j^\alpha|^2+\|u_\perp^\alpha;L_2(S_\varepsilon^\alpha)\|^2\Bigg)$$
$$\geqslant\frac12|\lambda|\varepsilon^n\overline{\rho}|b^{\alpha\prime}|^2-c\varepsilon^2|\lambda|\Big(\|\nabla_x u;L_2(S_\varepsilon^\alpha)\|^2+\|\mathscr{D}(\nabla_x)u;L_2(S_\varepsilon^\alpha)\|^2\Big). \tag{4.43}$$

Note that the estimate from below becomes possible because $|\lambda|\overline{\rho}=-\lambda\overline{\rho}$ is positive.

Summing the inequality (4.43) over all the cells, we find

$$\Big(\mathscr{A}^\varepsilon\mathscr{D}(\nabla_x)u,\mathscr{D}(\nabla_x)u)_\Omega-\lambda(\rho u,u)\Big)_\Omega\geqslant\frac12 c_{\mathscr{A}}\|\mathscr{D}(\nabla_x)u;L_2(\Omega)\|^2$$
$$-c\varepsilon^2|\lambda|\Big(\|\nabla_x u;L_2(\Omega)\|^2+\|\mathscr{D}(\nabla_x)u;L_2(\Omega)\|^2\Big). \tag{4.44}$$

By the inequality (2.1), the existence of a required $\theta>0$ becomes obvious: in the case

$$c_{\mathscr{A}}-4c\varepsilon^2|\lambda|(1+c_{\mathscr{D}})\geqslant0$$

the left-hand side of (4.44) exceeds

$$\frac{1}{4}c_{\mathscr{A}}\|\mathscr{D}(\nabla_x)u; L_2(\Omega)\|.$$

The theorem is proved. □

The result of Theorem 4.6 shows that the half-interval $[\theta\varepsilon^{-2}, 0)$ is free from the spectrum of the problem (1.1), (1.2), i.e., $\lambda_{-j}^{\varepsilon} \leqslant -\theta\varepsilon^{-2}$, which agrees with the formal asymptotic of negative eigenvalues (cf. Subsection 3.4).

References

1. J. Nečas, *Les méthodes in théorie des équations elliptiques*, Masson-Academia, Paris-Prague (1967).
2. S. A. Nazarov, "Selfadjoint elliptic boundary value problems. The polynomial property and formally positive operators" [in Russian], *Probl. Mat. Anal.* **16**, 167–192 (1997); English transl.: *J. Math. Sci., New York* **92**, No. 6, 4338- 4353 (1998).
3. S. A. Nazarov, "The polynomial property of selfadjoint elliptic boundary value problems and the algebraic description of their attributes" [in Russian], *Uspekhi Mat. Nauk* **54**, No. 5, 77-142 (1999); English transl.: *Russ. Math. Surv.* **54**, No. 5, 947-1014 (1999).
4. O. A. Ladyzhenskaya, *Boundary-Value Problems in Mathematical Physics* [in Russian], Nauka, Moscow (1973).
5. S. G. Lekhnitskij, *Theory of Elasticity of an Anisotropic Body* [in Russian], Nauka, Moscow (1977); English transl.: Mir Publishers, Moscow (1981).
6. S. A. Nazarov, *Asymptotic Theory of Thin Plates and Rods. Vol. 1. Dimension Reduction and Integral Estimates*, Nauchanaya Kniga (IDMI), Novosibirck (2002).
7. V. T. Grinchenko, A. F. Ulitko, and N. A. Shul'ga, *Mechanics of Connected Fields in Elements of Constructions* [in Russian], Naukova Dumka, Kiev (1989).
8. V. Z. Parton and B. A. Kudryavtsev, *Electric/elasticity of piezoelectric and electrically conducting bodies* [in Russian], Nauka, Moscow (1988).
9. S. A. Nazarov, "Uniform estimates of remainders in asymptotic expansions of solutions to the problem on eigenoscillaitons of a piezoelectric plate" [in Russian], *Probl. Mat. Anal.* **25**, 99–188 (2003); English transl.: *J. Math. Sci., New York* **144**, No. 5, 1657- 1725 (2003).
10. M. Sh. Birman and M. Z. Solomyak, *Spectral Theory of Selfadjoint Operators in Hilbert Space* [in Russian], Leningrad State University, Leningrad (1980).
11. S. Kesavan, "Homogenization of Elliptic Eigenvalue Problems: Part 1," *Appl. Math. Optim.* **5**, 153–167 (1979); Part 2: ibid. **5**, 197–216 (1979).
12. V. V. Zhikov, S. M. Kozlov, and O. A. Oleinik, *Homogenization of Differential Operators and Integral Functions* [in Russian], Fizmatgiz, Moscow (1993); English transl.: Springer, Berlin etc. (1994).
13. M. Vanninatan, "Homogenization of eigenvalue problems in perforated domains," *Pro. Indian Acad. Sci. (Math. Sci.)* **90**, No. 3, 239–271 (1981).
14. O. A. Oleinik, A. S. Shamaev, and G. A. Yosifyan, *Mathematical Problems in the Theory of Strongly Inhomogeneous Elastic Media* [in Russian], Moscow State Univ., Moscow (1990).
15. G. Cardone, A. Corbo Esposito, and S. A. Nazarov, "Korn's inequality for periodic solids and convergence rate of homogenization," *Appl. Anal.* **88**, No. 6, 847–876 (2009).
16. I. M. Gelfand and G. E. Shilov, *Generalized Functions and Operations over Them* [in Russian], Fizmatgiz, Moscow (1958).

17. M. I. Vishik and L. A. Lusternik, "Regular degeneration and boundary layer for linear differential equations with small parameter" [in Russian] *Uspekhi Mat. Nauk* **12**, No. 5, 3–122 (1957).

18. I. C. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Nonself-Adjoint Operators in Hilbert Space* [in Russian], Nauka, Moscow (1965); English transl.: Am. Math. Soc., Providence, RI (1969).

19. S. A. Nazarov, "On the asymptotics of the spectrum of a thin plate problem of elasticity" [in Russian], *Sib. Mat. Zh.* **41**, No. 4, 895–912 (2000); English transl.: *Sib. Math. J.* **41**, No. 4, 744–750 (2000).

20. V. L. Berdicevskii, "High-frequency longwave oscillations of plates" [in Russian], *Dokl. Akad. Nauk SSSR* **236**, No. 6, 1319–1322 (1977).

Submitted on December 15, 2009