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On boundary value problem with singular inhomogeneity concentrated on the boundary ☆

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Abstract

We study the asymptotic behavior of solutions to a boundary value problem for the Poisson equation with a singular right-hand side, singular potential and with alternating type of the boundary condition. Assuming that the boundary microstructure is periodic, we construct the limit problem and prove the homogenization theorem by means of the unfolding method. The proof requires that the dimension be larger than two.

Résumé

Le but de cet article est d'étudier le comportement asymptotique des solutions d'une équation de Poisson avec un potentiel et un membre de droite singuliers et des conditions aux limites oscillantes. Le problème aux limites est posé dans un domaine de $\mathbb{R}^n$, $n \geqslant 3$. Sous l'hypothèse que la microstructure de la frontière est périodique, on démontre un théorème d'homogénéisation en utilisant la méthode d'éclatement périodique.

Keywords: Homogenization; Periodic unfolding; Mass concentration; Strange term

1. Introduction

In this work we consider the homogenization of a boundary value problem for the Poisson equation with singular (asymptotically high contrast) zero order term and right-hand side, the support of which is concentrated near a fixed subset of the domain boundary and with a periodic microstructure. The boundary condition alternates rapidly between Dirichlet and Neumann on this subset.

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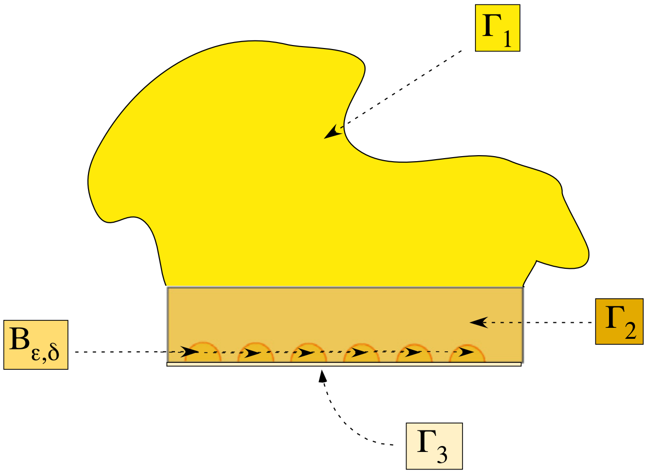

Fig. 1. Side view.

Problems in domains with singularly perturbed density ("concentrated masses") have been widely discussed previously (see [16] already back in 1913). The behavior of solutions of a wave equation with one concentrated mass and the vibration of a body with a concentrated mass were studied in [29] and [30], respectively. The behavior of the spectrum of the elasticity system with volume distributed concentrated masses was described in [25,28,24]. The eigenvalue problem for an elastic membrane with a concentrated mass was treated in [26,17], and the case of concentrated masses located along the boundary of a domain was investigated in [19,6,9,7].

The spectral problem with mass concentration on periodic rod structures was considered in [21–23].

Other spectral and boundary value problems in domains with high contrast and singularly perturbed densities can be found in [15,14,1,2].

Problems with rapidly alternating boundary conditions have also been intensively studied (see [12,20,4,13,3,10]).

In this paper we consider a homogenization problem with two small parameters (going to zero), the first one, ε, characterizes the boundary microstructure period, while the second, δ, characterizes the volume fraction of the set where the source term is large, as well as the portion of the boundary where the Dirichlet condition is imposed. It should be noted that, depending on the ratio between ε and δ, one can obtain different boundary conditions in the limit problem (see, for instance, [18,4]).

In this paper, the periodic unfolding method is used for the first time for such a type of problems. It allows to characterize the oscillation of solutions, build the boundary layer term, show the convergence in H^1 norm and improve on the estimates for the rate of convergence. We use the version of the unfolding procedure adapted to the boundary homogenization. The boundary unfolding method was originally introduced in [27] and [11]. For technical reasons, the dimension has to be larger or equal to three.

The main results are presented in Theorem 5.4 where the unfolded limit of solutions is constructed, and in Theorem 6.2 where the macroscopic effective model is derived. In particular, the singular inhomogeneity concentrated near the boundary can give rise to a nontrivial term (a kind of "strange term") in the boundary operator of the limit equation.

A problem similar to that studied in the present work, was previously considered in [5].

2. Settings

For a given fixed $h \geqslant 0$, consider a domain Ω in $\mathbb{R}^n$, $n \geqslant 3$, which lies in the upper half-space, with a piecewise smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ consisting of three parts (see Fig. 1).

The part Γ_3 is the $(n-1)$-dimensional unit cube

$$\Gamma_3 \doteq \left\{x\colon -\frac{1}{2} < x_i < \frac{1}{2} \text{ for } i = 1, \ldots, n-1,\ x_n = 0\right\}.$$

The part Γ_2 is the union of Γ_2^i for $i = 1, \ldots, n-1$, where

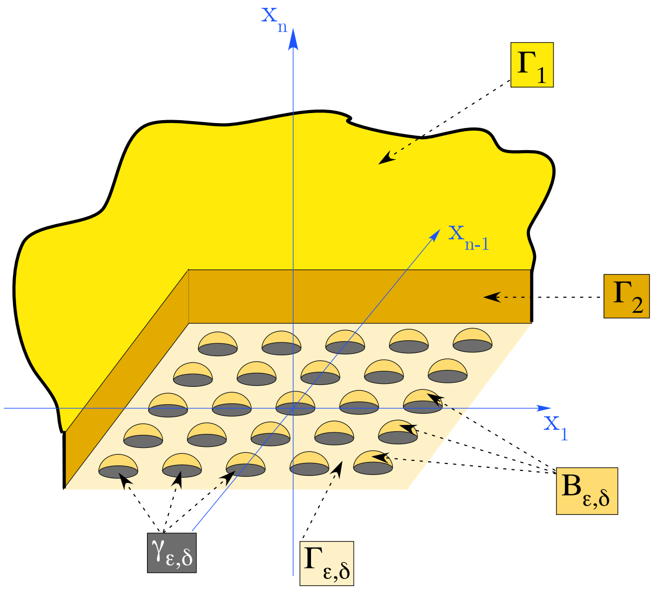

Fig. 2. Perspective.

$$\Gamma_2^i \doteq \left\{x\colon\ x_i = \pm\frac{1}{2},\ -\frac{1}{2} < x_j < \frac{1}{2} \text{ for } j \neq i,\ 1 \leqslant j \leqslant n-1,\ 0 \leqslant x_n \leqslant h\right\}.$$

The remainder Γ_1 is the part of $\partial\Omega$ located in the half-space $x_n \geqslant h$. Moreover, Γ_3 (see Fig. 2) has a periodic microstructure associated to the small parameters $\delta < 1$ and $\varepsilon = \frac{1}{(2\mathcal{N}+1)}$ where $\mathcal{N}$ is a natural number, $\mathcal{N} \gg 1$ and $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$,

$$\Gamma_3 = \Gamma_{\varepsilon,\delta} \cup \gamma_{\varepsilon,\delta}.$$

To describe the microstructure, let D be the hyperdisc

$$D = \left\{z \in \mathbb{R}^n\colon\ |z| < 1,\ z_n = 0\right\}. \tag{1}$$

Then

$$\gamma_{\varepsilon,\delta} = \Gamma_3 \cap \left(\bigcup_{\xi \in \mathbb{Z}^{n-1}} \varepsilon(\xi + \delta D)\right), \qquad \Gamma_{\varepsilon,\delta} \doteq \Gamma_3 \setminus \gamma_{\varepsilon,\delta}.$$

In the figures, we use the following notation. Let $\alpha_0 > 0$ and B be the half-ball

$$B = \left\{z\colon\ |z| < \alpha_0,\ z_n > 0\right\}.$$

Then $B_{\varepsilon,\delta}$ is the set

$$B_{\varepsilon,\delta} = \Omega \cap \left(\bigcup_{\xi \in \mathbb{Z}^{n-1}} B_{\varepsilon,\delta}^{\xi}\right), \quad \text{where } B_{\varepsilon,\delta}^{\xi} = \varepsilon(\xi + \delta B).$$

Remark 2.1. The set B defined above as a half-ball, can be replaced by any bounded connected open subset of the upper half-space with Lipschitz boundary. All the results of this paper remain valid for such a choice of B.

Our aim is to study the asymptotic behavior as ε and δ go to 0, of the solutions to the following boundary value problem:

$$\begin{cases} -\Delta u_{\varepsilon,\delta} + (\varepsilon\delta)^{-\varkappa}\rho_{\varepsilon,\delta}u_{\varepsilon,\delta} = f_{\varepsilon,\delta} & \text{in } \Omega, \\ u_{\varepsilon,\delta} = 0 & \text{on } \gamma_{\varepsilon,\delta} \cup \Gamma_1, \\ \dfrac{\partial u_{\varepsilon,\delta}}{\partial \nu} = 0 & \text{on } \Gamma_{\varepsilon,\delta} \cup \Gamma_2, \end{cases} \tag{2}$$

where ν is the outward unit normal on $\partial\Omega$, and $(\varepsilon\delta)^{-\varkappa}\rho_{\varepsilon,\delta}$ is a nonnegative density supported in $B_{\varepsilon,\delta}$, and such that $\rho_{\varepsilon,\delta}$ is bounded in $L^\infty(B_{\varepsilon,\delta})$. We also suppose $\varkappa \geqslant 0$. The right-hand side $f_{\varepsilon,\delta} \in L^2(\Omega)$ is of the form

$$f_{\varepsilon,\delta}(x) = \begin{cases} f & \text{in } \Omega\backslash\overline{B}_{\varepsilon,\delta}, \\ \tilde{f}_{\varepsilon,\delta} & \text{in } B_{\varepsilon,\delta}, \end{cases} \tag{3}$$

where $f_{\varepsilon,\delta}$ belongs to $L^2(\Omega)$, and, for simplicity, f is fixed in $L^2(\Omega)$. The function $\tilde{f}_{\varepsilon,\delta}$ is not necessarily bounded with respect to ε and δ, the exact scaling being specified in Proposition 3.1 below.

Let $\mathcal{V}_{\varepsilon,\delta}$ be the following space:

$$\mathcal{V}_{\varepsilon,\delta} = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_1 \cup \gamma_{\varepsilon,\delta}\}.$$

The variational formulation of problem (2) is now

$$\begin{cases} \text{Find } u_{\varepsilon,\delta} \in \mathcal{V}_{\varepsilon,\delta} \text{ satisfying} \\ \displaystyle\int_\Omega \nabla u_{\varepsilon,\delta} \nabla\phi\, dx + \int_\Omega (\varepsilon\delta)^{-\varkappa}\rho_{\varepsilon,\delta}\, u_\varepsilon \phi\, dx = \int_\Omega f_{\varepsilon,\delta}\phi\, dx, \\ \forall\phi \in \mathcal{V}_{\varepsilon,\delta}. \end{cases} \tag{$\mathcal{P}_{\varepsilon,\delta}$}$$

In the sequel we will make use of the following spaces:

$$\mathcal{V}_0 = \{w \in H^1(\Omega) \mid w = 0 \text{ on } \Gamma_1\},$$
$$\mathcal{V}_0^c = \{w \in C^\infty(\overline{\Omega}) \mid w = 0 \text{ in some neighborhood of } \Gamma_1\}.$$

Notice that $\mathcal{V}_0^c$ is a dense subspace of $\mathcal{V}_0$.

Remark 2.2. The spaces $\mathcal{V}_{\varepsilon,\delta}$ are all closed subspaces of $H^1(\Omega)$ and more precisely, of $\mathcal{V}_0$. The Poincaré–Friedrichs inequality holds for $\mathcal{V}_0$, hence uniformly for the spaces $\mathcal{V}_{\varepsilon,\delta}$.

Remark 2.3. The presence of the density $(\varepsilon\delta)^{-\varkappa}\rho_{\varepsilon,\delta}$ means that Eq. (2) is asymptotically singular near the boundary Γ_3 within the set $B_{\varepsilon,\delta}$.

2.0.1. The case of periodic data

Consider now the particular case where the functions $\rho_{\varepsilon,\delta}$ and $f_{\varepsilon,\delta}$ are periodic or locally periodic in the variable $x' \doteq (x_1, \ldots, x_{n-1})$ (see Example 5.3 below). In this case the functions $\rho_{\varepsilon,\delta}$ and $f_{\varepsilon,\delta}$ are of the form

$$\rho_{\varepsilon,\delta}(x) = \bar{\rho}\left(\frac{1}{\delta}\left\{\frac{x'}{\varepsilon}\right\}, \frac{x_n}{\varepsilon\delta}\right), \qquad \tilde{f}_{\varepsilon,\delta}(x) = \frac{1}{\varepsilon\delta^n}\bar{f}\left(\frac{1}{\delta}\left\{\frac{x'}{\varepsilon}\right\}, \frac{x_n}{\varepsilon\delta}\right)$$

with $\bar{\rho}(z)$ and $\bar{f}(z)$ defined in $\mathbb{R}^n_+$ and supported in B, here $\{\cdot\}$ stands for the fractional part. Note that this definition implies the fact that $\rho_{\varepsilon,\delta}$ and $\tilde{f}_{\varepsilon,\delta}$ are ε-periodic with respect to x'.

We suppose that

$$\bar{\rho} \in L^\infty(\mathbb{R}^n_+), \qquad \bar{f} \in L^2(\mathbb{R}^n_+).$$

As shown below, the effective (homogenized) boundary condition in problem (2) depends crucially on the ratio between ε and δ^{n-2}. We assume that δ is a function of ε such that there exists $k \in [0, +\infty]$ satisfying

$$k = \lim_{\varepsilon\to 0}\frac{\delta^{n-2}}{\varepsilon}. \tag{4}$$

To formulate the convergence results, we will need the following auxiliary problems stated in the half-space $\mathbb{R}^n_+ = \{z \in \mathbb{R}^n\colon z_n > 0\}$ (recall that D is defined by (1)):

$$\begin{cases} -\Delta_z\overline{U} + \bar{\rho}(z)\overline{U} = 0, \\ \overline{U}|_D = 1, \quad \left.\dfrac{\partial\overline{U}}{\partial\nu_z}\right|_{\mathbb{R}^{n-1}\backslash D} = 0, \\ \nabla\overline{U} \in L^2(\mathbb{R}^n_+), \end{cases} \tag{5}$$

and

$$\begin{cases} -\Delta_z \widetilde{U} + \bar{\rho}(z)\widetilde{U} = \bar{f}(z), \\ \widetilde{U}|_D = 0, \quad \left.\dfrac{\partial \widetilde{U}}{\partial \nu_z}\right|_{(\mathbb{R}^{n-1}\setminus D)} = 0, \\ \nabla \widetilde{U} \in L^2(\mathbb{R}^n_+). \end{cases}$$

Define

$$\Theta \doteq \int\limits_B \bar{\rho}(z)\overline{U}(z)\,dz - \int\limits_D \frac{\partial \overline{U}}{\partial \nu_z}(z')\,dz' \tag{6}$$

and

$$F \doteq \int\limits_B \bar{f}(z)\,dz - \int\limits_B \bar{\rho}(z)\widetilde{U}(z)\,dz + \int\limits_D \frac{\partial \widetilde{U}}{\partial \nu_z}(z')\,dz'. \tag{7}$$

Theorem 2.1. *Assume that $\varkappa = 2$ and that (4) holds. Let $u_{\varepsilon,\delta}$ be the solution of (2). Then, there exists a unique u_0 such that*

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad \textit{weakly in } \mathcal{V}_0.$$

This u_0 is the unique solution of a limit problem which depends on the value of k.

- *If $k \in (0, +\infty)$, the limit problem is*

$$\begin{cases} -\Delta u_0 = f \quad \text{in } \Omega, \\ u_0|_{\Gamma_1} = 0, \quad \left.\dfrac{\partial u_0}{\partial \nu}\right|_{\Gamma_2} = 0, \\ \left.\left(\dfrac{\partial u_0}{\partial \nu} + \Theta k u_0\right)\right|_{\Gamma_3} = F, \end{cases} \tag{8}$$

  with Θ and F defined in (6) and (7), respectively.
- *If $k = 0$, then the limit problem is*

$$\begin{cases} -\Delta u_0 = f \quad \text{in } \Omega, \\ u_0|_{\Gamma_1} = 0, \quad \left.\dfrac{\partial u_0}{\partial \nu}\right|_{\Gamma_2} = 0, \\ \left.\dfrac{\partial u_0}{\partial \nu}\right|_{\Gamma_3} = F \end{cases} \tag{9}$$

  (*note the Neumann boundary condition on Γ_3*).
- *If $k = +\infty$, then the limit problem is*

$$\begin{cases} -\Delta u_0 = f \quad \text{in } \Omega, \\ u_0|_{\Gamma_1 \cup \Gamma_3} = 0, \quad \left.\dfrac{\partial u_0}{\partial \nu}\right|_{\Gamma_2} = 0 \end{cases} \tag{10}$$

  (*note the Dirichlet boundary condition on Γ_3*).

In order to formulate the convergence result in the case $\varkappa \neq 2$, we introduce the usual half-space harmonic capacities of the sets D and $D \cup \overline{B}$. We denote these capacities by Θ_D and $\Theta_{D\cup\bar{B}}$ (see Definition 6.1 below).

Theorem 2.2. *Assume that $\varkappa < 2$ and that (4) holds with k finite. Then, there exists a unique u_0 such that*

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad \textit{weakly in } \mathcal{V}_0,$$

where u_0 is the unique solution of the following problem

$$\begin{cases} -\Delta u_0 = f \quad in\ \Omega, \\ u_0|_{\Gamma_1} = 0, \quad \left.\dfrac{\partial u_0}{\partial \nu}\right|_{\Gamma_2} = 0, \\ \left.\left(\dfrac{\partial u_0}{\partial \nu} + \Theta_\gamma k u_0\right)\right|_{\Gamma_3} = F_D, \end{cases} \tag{11}$$

with F_D defined by

$$F_D \doteq \int\limits_B \bar{f}(z)\,dz + \int\limits_D \frac{\partial \widetilde{U}_D}{\partial \nu_z}(z')\,dz',$$

where $\widetilde{U}_D$ is a solution in $\mathbb{R}^n_+$ of the problem

$$\begin{cases} -\Delta_z \widetilde{U}_D = \bar{f}, \\ \widetilde{U}_D|_D = 0, \quad \left.\dfrac{\partial \widetilde{U}_D}{\partial \nu_z}\right|_{(\mathbb{R}^{n-1}\setminus D)} = 0, \\ \nabla \widetilde{U}_D \in L^2(\mathbb{R}^n_+). \end{cases}$$

For $\varkappa > 2$ the following result holds true.

Theorem 2.3. *Assume that $\varkappa > 2$ and that (4) holds with k finite, and suppose that $\bar{\rho}(z) > 0$ everywhere in B. Then a solution $u_{\varepsilon,\delta}$ of problem (2) converges in $L^2(\Omega)$, as $\varepsilon \to 0$, toward a unique solution of the problem*

$$\begin{cases} -\Delta u_0 = f \quad in\ \Omega, \\ u_0|_{\Gamma_1} = 0, \quad \left.\dfrac{\partial u_0}{\partial \nu}\right|_{\Gamma_2} = 0, \\ \left.\left(\dfrac{\partial u_0}{\partial \nu} + \Theta_{D\cup\bar{B}} k u_0\right)\right|_{\Gamma_3} = 0. \end{cases} \tag{12}$$

Finally, in the case of $k = +\infty$, the homogenized problem takes the form (10) whatever the value of $\varkappa$.

2.0.2. The case of locally periodic data

In the locally periodic case, the functions $\rho_{\varepsilon,\delta}$ and $\tilde{f}_{\varepsilon,\delta}$ are of the form

$$\rho_{\varepsilon,\delta}(x) = \bar{\rho}\left(x', \frac{1}{\delta}\left\{\frac{x'}{\varepsilon}\right\}, \frac{x_n}{\varepsilon\delta}\right), \qquad \tilde{f}_{\varepsilon,\delta}(x) = \frac{1}{\varepsilon\delta^n}\bar{f}\left(x', \frac{1}{\delta}\left\{\frac{x'}{\varepsilon}\right\}, \frac{x_n}{\varepsilon\delta}\right)$$

with $\bar{\rho}(x', z)$ and $\bar{f}(x', z)$ defined in $\Gamma_3 \times \mathbb{R}^n_+$ and supported in $\Gamma_3 \times B$. We suppose that

$$\bar{\rho} \in C\left(\Gamma_3; L^\infty(\mathbb{R}^n_+)\right), \qquad \bar{f} \in C\left(\Gamma_3; L^2(\mathbb{R}^n_+)\right).$$

All the auxiliary functions $\overline{U}$, $\widetilde{U}$, etc. and the quantities Θ, F, etc. will depend on x' as a parameter. In particular, problem (5) reads

$$\begin{cases} -\Delta_z \overline{U}(x', z) + \bar{\rho}(x', z)\overline{U}(x', z) = 0, \\ \overline{U}(x', \cdot)\big|_D = 1, \quad \left.\dfrac{\partial \overline{U}}{\partial \nu_z}(x', \cdot)\right|_{\mathbb{R}^{n-1}\setminus D} = 0, \\ \nabla \overline{U}(x', \cdot) \in L^2(\mathbb{R}^n_+), \end{cases}$$

and $\Theta = \Theta(x')$ is defined by

$$\Theta(x') \doteq \int\limits_B \bar{\rho}(x', z)\overline{U}(x', z)\,dz - \int\limits_D \frac{\partial \overline{U}}{\partial \nu_z}(x', z')\,dz'.$$

The definitions of $\widetilde{U}$ and F should be modified accordingly.

The statement of Theorem 2.1 in the locally periodic case reads:

Theorem 2.4. *Assume that $\varkappa = 2$, and that (4) holds. Then, there exists a unique u_0 such that*

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad \text{weakly in } \mathcal{V}_0.$$

This u_0 is the unique solution of a limit problem which depends on the value of k.

- *If $k \in (0, +\infty)$, then the limit problem is*

$$\begin{cases} -\Delta u_0 = f \quad \text{in } \Omega, \\ u_0|_{\Gamma_1} = 0, \quad \left.\dfrac{\partial u_0}{\partial \nu}\right|_{\Gamma_2} = 0, \\ \left.\left(\dfrac{\partial u_0}{\partial \nu} + \Theta\big(x'\big)ku_0\right)\right|_{\Gamma_3} = F\big(x'\big). \end{cases} \tag{13}$$

- *If $k = 0$ then the limit problem is*

$$\begin{cases} -\Delta u_0 = f \quad \text{in } \Omega, \\ u_0|_{\Gamma_1} = 0, \quad \left.\dfrac{\partial u_0}{\partial \nu}\right|_{\Gamma_2} = 0, \\ \left.\dfrac{\partial u_0}{\partial \nu}\right|_{\Gamma_3} = F\big(x'\big). \end{cases} \tag{14}$$

- *If $k = +\infty$, then the limit problem coincides with problem (10).*

Theorems 2.1–2.4 are corollaries of Theorems 5.4, 5.5 and 5.7 where the assumptions on $\rho_{\varepsilon,\delta}$ and $f_{\varepsilon,\delta}$ are more general.

The plan of the paper is as follows. In Section 2 uniform estimates are established. Section 3 introduces the boundary layer operator periodic, which is the main tool in the proof presented in Section 5 (the unfolded limit problems). Section 6 gives the macroscopic form of these limit problems. Section 7 is devoted to the convergence of the energy in these problems and improves on the convergence of the solutions.

3. Estimates

In this section we establish uniform estimates for the solution of problem $(\mathcal{P}_{\varepsilon,\delta})$. Here we assume that ε and δ are two independent small parameters.

Proposition 3.1. *There is a constant C independent of ε and δ such that*

$$\begin{aligned} &\|u_{\varepsilon,\delta}\|_{H^1(\Omega)} + (\varepsilon\delta)^{-\varkappa/2}\big\|\rho_{\varepsilon,\delta}^{1/2} u_{\varepsilon,\delta}\big\|_{L^2(B_{\varepsilon,\delta})} \\ &\quad \leqslant C\big(\|f\|_{L^2(\Omega)} + \min\big\{(\varepsilon\delta)^{\varkappa/2}\big\|\rho_{\varepsilon,\delta}^{-1/2}\tilde{f}_{\varepsilon,\delta}\big\|_{L^2(B_{\varepsilon,\delta})}, \varepsilon\delta\|\tilde{f}_{\varepsilon,\delta}\|_{L^2(B_{\varepsilon,\delta})}\big\}\big). \end{aligned}$$

Proof. We denote by c a generic constant which does not depend on ε and δ.

Using $u_{\varepsilon,\delta}$ as a test function in $(\mathcal{P}_{\varepsilon,\delta})$, we obtain

$$\int_\Omega |\nabla u_{\varepsilon,\delta}|^2\, dx + (\varepsilon\delta)^{-\varkappa}\int_\Omega \rho_{\varepsilon,\delta} u_{\varepsilon,\delta}^2\, dx = \int_{B_{\varepsilon,\delta}} \tilde{f}_{\varepsilon,\delta} u_{\varepsilon,\delta}\, dx + \int_{\Omega\setminus B_{\varepsilon,\delta}} f u_{\varepsilon,\delta}\, dx.$$

By the Poincaré–Friedrichs inequality for $\mathcal{V}_0$ and the standard use of the Young inequality,

$$\frac{1}{2}\int_\Omega |\nabla u_{\varepsilon,\delta}|^2\, dx + (\varepsilon\delta)^{-\varkappa}\int_\Omega \rho_{\varepsilon,\delta} u_{\varepsilon,\delta}^2\, dx \leqslant \int_{B_{\varepsilon,\delta}} |\tilde{f}_{\varepsilon,\delta} u_{\varepsilon,\delta}|\, dx + c\|f\|_{L^2(\Omega)}^2. \tag{15}$$

The Cauchy–Schwarz inequality and the Young inequality, give

$$\int_{B_{\varepsilon,\delta}} |\tilde{f}_{\varepsilon,\delta}||u_{\varepsilon,\delta}|\,dx \leqslant \left\|\rho_{\varepsilon,\delta}^{-1/2}\tilde{f}_{\varepsilon,\delta}\right\|_{L^2(B_{\varepsilon,\delta})}\left\|\tilde{\rho}_{\varepsilon,\delta}^{1/2}u_{\varepsilon,\delta}\right\|_{L^2(B_{\varepsilon,\delta})}$$

and

$$\int_\Omega |\nabla u_{\varepsilon,\delta}|^2\,dx + (\varepsilon\delta)^{-\varkappa}\int_\Omega \rho_{\varepsilon,\delta}u_{\varepsilon,\delta}^2\,dx \leqslant c\big((\varepsilon\delta)^{\varkappa}\left\|\rho_{\varepsilon,\delta}^{-1/2}\tilde{f}_{\varepsilon,\delta}\right\|^2_{L^2(B_{\varepsilon,\delta})} + \|f\|^2_{L^2(\Omega)}\big). \tag{16}$$

On the other hand, by [8], there is a Poincaré–Friedrichs inequality in the set B for functions vanishing on $D\cap\overline{B}$. By scaling, it follows that

$$\|u_{\varepsilon,\delta}\|_{L^2(B_{\varepsilon,\delta})} \leqslant c\varepsilon\delta\|\nabla u_{\varepsilon,\delta}\|_{L^2(B_{\varepsilon,\delta})}.$$

Using this estimate and the Young inequality (again) in (15), gives

$$\frac{1}{4}\int_\Omega |\nabla u_{\varepsilon,\delta}|^2\,dx + (\varepsilon\delta)^{-\varkappa}\int_\Omega \rho_{\varepsilon,\delta}u_{\varepsilon,\delta}^2\,dx \leqslant c\big(\varepsilon\delta\|\tilde{f}_{\varepsilon,\delta}\|^2_{L^2(B_{\varepsilon,\delta})} + \|f\|^2_{L^2(\Omega)}\big). \tag{17}$$

The conclusion is obtained by combining (16) and (17). □

Corollary 3.2. *If*

$$\min\big\{(\varepsilon\delta)^{\varkappa/2}\left\|\rho_{\varepsilon,\delta}^{-1/2}\tilde{f}_{\varepsilon,\delta}\right\|_{L^2(B_{\varepsilon,\delta})}, \varepsilon\delta\|\tilde{f}_{\varepsilon,\delta}\|_{L^2(B_{\varepsilon,\delta})}\big\} \leqslant C$$

as ε and δ tend to zero, then $u_{\varepsilon,\delta}$ is bounded in $H^1(\Omega)$. Furthermore,

$$\left\|\rho_{\varepsilon,\delta}^{1/2}u_{\varepsilon,\delta}\right\|_{L^2(B_{\varepsilon,\delta})} \leqslant C(\varepsilon\delta)^{\varkappa/2}.$$

4. The boundary-layer unfolding operator $\mathcal{T}^{bl}_{\varepsilon,\delta}$

Recall the notation $x' \doteq (x_1,\ldots,x_{n-1})$. We use the periodicity cells

$$Y' \doteq (-1/2, 1/2)^{n-1}, \qquad Y \doteq Y'\times(0,1), \tag{18}$$

and define the layer ω_ε as

$$\omega_\varepsilon = \Omega\cap\{x\colon\ 0<x_n<\varepsilon\}. \tag{19}$$

For y' in $\mathbb{R}^{n-1}$, $[y']_{Y'}$ denotes the point $\xi\in\mathbb{Z}^{n-1}$ such that $y'-\xi$ belongs to Y'. This is defined uniquely (except on a set of measure zero). Similarly, $\{y'\}_{Y'}$ denotes $y'-\xi$ which belongs to Y'. From now on, when referring to a point $(x',0)$ in Γ_3, we often drop the last coordinate and just write x'.

Definition 4.1. For $\phi\in L^p(\omega_\varepsilon)$, $p\in[1,+\infty)$, the unfolding operator $\mathcal{T}^{bl}_{\varepsilon,\delta}: L^p(\omega_\varepsilon)\to L^p(\Gamma_3\times\mathbb{R}^n_+)$ is defined by

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(\phi)\big(x',z\big) = \begin{cases} \phi\left(\varepsilon\left[\dfrac{x'}{\varepsilon}\right]_{Y'} + \varepsilon\delta z\right) & \text{if } (x',z)\in\Gamma_3\times\dfrac{1}{\delta}Y, \\ 0 & \text{otherwise.}\end{cases} \tag{20}$$

This operation, designed to capture the contribution of the barriers in the limit process, was originally used in [27]. We also introduce the notion of local average in the neighborhood of the hyperplane Γ_3.

Definition 4.2. The local average $M^{\varepsilon,bl}_Y: L^p(\omega_\varepsilon)\mapsto L^p(\Gamma_3)$, is defined for every ϕ in $L^p(\omega_\varepsilon)$, $1\leqslant p<+\infty$, by

$$M^{\varepsilon,bl}_Y(\phi)\big(x'\big) = \frac{\delta^n}{|Y|}\int_{\frac{1}{\delta}Y}\mathcal{T}^{bl}_{\varepsilon,\delta}(\phi)\big(x',z\big)\,dz = \frac{1}{\varepsilon^n}\int_{\varepsilon[\frac{x'}{\varepsilon}]+\varepsilon Y}\phi(\zeta)\,d\zeta, \quad \text{for } x'\in\Gamma_3$$

(note that the measure of Y is equal to 1).

Remark 4.3. Since elements of $L^p(\Gamma_3)$ can be considered as functions of $L^p(\omega_\varepsilon)$, $M_Y^{\varepsilon,bl}$ can be applied to them. With this convention,

$$\mathcal{T}^{bl}_{\varepsilon,\delta}\big(M_Y^{\varepsilon,bl}(\phi)\big) = M_Y^{\varepsilon,bl}(\phi) \quad \text{on } \Gamma_3.$$

The following statements are straightforward modifications of the corresponding results of [11].

Proposition 4.4. *Let w_ε be a sequence such that $w_\varepsilon \rightharpoonup w$ weakly in $H^1(\Omega)$. Then*

$$M_Y^{\varepsilon,bl}(w_\varepsilon) \to w_{|\Gamma_3} \quad \text{strongly in } L^2(\Gamma_3).$$

Theorem 4.5 *(Properties of the operator $\mathcal{T}^{bl}_{\varepsilon,\delta}$).*

1. *For any $v, w \in L^p(\omega_\varepsilon)$,*

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(vw) = \mathcal{T}^{bl}_{\varepsilon,\delta}(v)\mathcal{T}^{bl}_{\varepsilon,\delta}(w).$$

2. *For any $u \in L^1(\omega_\varepsilon)$,*

$$\varepsilon\delta^n \int\limits_{\Gamma_3\times\mathbb{R}^n_+} \mathcal{T}^{bl}_{\varepsilon,\delta}(u)\,dx'\,dz = \int\limits_{\omega_\varepsilon} u\,dx,$$

and

$$\varepsilon\delta^n \int\limits_{\Gamma_3\times\mathbb{R}^n_+} \big|\mathcal{T}^{bl}_{\varepsilon,\delta}(u)\big|\,dx'\,dz = \int\limits_{\omega_\varepsilon} |u|\,dx.$$

3. *For any $u \in L^2(\omega_\varepsilon)$,*

$$\big\|\mathcal{T}^{bl}_{\varepsilon,\delta}(u)\big\|^2_{L^2(\Gamma_3\times\mathbb{R}^n_+)} = \frac{1}{\varepsilon\delta^n}\|u\|^2_{L^2(\omega_\varepsilon)}.$$

4. *Let u be in $H^1(\omega_\varepsilon)$. Then,*

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla_x u) = \frac{1}{\varepsilon\delta}\nabla_z\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u)\big) \quad \text{in } \Gamma_3 \times \frac{1}{\delta}Y.$$

5. *Suppose $n \geqslant 3$ and let Q be an open and bounded set in $\mathbb{R}^n_+$. Then the following estimates hold*:

$$\big\|\nabla_z\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u)\big)\big\|^2_{L^2(\Gamma_3\times\frac{1}{\delta}Y)} \leqslant \frac{\varepsilon}{\delta^{n-2}}\|\nabla u\|^2_{L^2(\omega_\varepsilon)},$$
$$\big\|\mathcal{T}^{bl}_{\varepsilon,\delta}\big(u - M_Y^{\varepsilon,bl}(u)\big)\big\|^2_{L^2(\Gamma_3;L^{2^*}(\mathbb{R}^n_+))} \leqslant \frac{C\varepsilon}{\delta^{n-2}}\|\nabla u\|^2_{L^2(\omega_\varepsilon)},$$

and

$$\big\|\mathcal{T}^{bl}_{\varepsilon,\delta}(u)\big\|^2_{L^2(\Gamma_3,L^{2^*}(Q))} \leqslant 2\frac{C\varepsilon}{\delta^{n-2}}\|\nabla u\|^2_{L^2(\omega_\varepsilon)} + 2|Q|^{2/2^*}\|u\|^2_{L^2(\omega_\varepsilon)},$$

where C denotes the Sobolev–Poincaré–Wirtinger constant for $H^1(Y)$ and 2^* is the Sobolev exponent defined by $\frac{1}{2^*} \doteq \frac{1}{2} - \frac{1}{n}$.

6. *Assume $n \geqslant 3$ and $\frac{\varepsilon}{\delta^{n-2}}$ is bounded. Let $w_{\varepsilon,\delta}$ be in $H^1(\omega_\varepsilon)$ such that*

$$\|\nabla w_{\varepsilon,\delta}\|_{L^2(\omega_\varepsilon)} \leqslant C.$$

Then, up to a subsequence, there exist two functions $W \in L^2(\Gamma_3; L^{2^*}(\mathbb{R}^n_+))$ and U in $L^2(\Gamma_3; H^1_{\mathrm{loc}}(\mathbb{R}^n_+))$ with $\nabla_z W$ and $\nabla_z U$ in $L^2(\Gamma_3 \times \mathbb{R}^n_+)$, such that

$$\big(M_Y^\varepsilon(w_{\varepsilon,\delta})1_{\frac{1}{\delta}Y} - \mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})\big) \rightharpoonup W \quad \textit{weakly in } L^2\big(\Gamma_3; L^{2^*}\big(\mathbb{R}^n_+\big)\big),$$
$$\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) \rightharpoonup U \quad \textit{weakly in } L^2\big(\Gamma_3; L^{2^*}_{\mathrm{loc}}\big(\mathbb{R}^n_+\big)\big),$$
$$\nabla_z\big(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})\big)1_{\frac{1}{\delta}Y} \rightharpoonup \nabla_z U \quad \textit{weakly in } L^2\big(\Gamma_3 \times \mathbb{R}^n_+\big).$$

Furthermore, $\nabla_z W = -\nabla_z U$*,* $W + U$ *is independent of* z *and*

$$w_{\varepsilon,\delta}|_{\Gamma_3} \to U + W \quad \textit{strongly in } L^2(\Gamma_3).$$

Remark 4.6. In the present work, for simplicity we assume that Γ_3 is the exact union of $\varepsilon Y'$-cells. The general case of Γ_3 with Lipschitz boundary can actually be handled as in [11].

5. Unfolding procedure

5.1. Functional setting

In the study of the limit behavior of problem $(\mathcal{P}_{\varepsilon,\delta})$ as $\varepsilon, \delta \to 0$, the following functional space, well-known in potential theory, plays an essential role ($n \geqslant 3$ is required so that 2^* is finite):

$$K_D \doteq \big\{\Phi \in L^{2^*}\big(\mathbb{R}^n_+\big);\ \nabla\Phi \in L^2\big(\mathbb{R}^n_+\big),\ \Phi|_D \text{ is a constant}\big\}. \tag{21}$$

It is known that $\|\Phi\|_{K_D} \doteq \|\nabla\Phi\|_{L^2(\Omega)}$ is a Hilbert norm on K_D and the space

$$K_D^c \doteq \big\{\Phi \in K_D \cap C^\infty\big(\overline{\mathbb{R}^n_+}\big),\ \text{support of } \Phi \text{ is bounded}\big\}, \tag{22}$$

is dense in K_D. Moreover, the map $\Phi \to \Phi(D) \doteq \Phi|_D$ is a continuous linear form on K_D and its kernel is

$$K_D^0 = \big\{\Phi \in L^{2^*}\big(\mathbb{R}^n_+\big);\ \nabla\Phi \in L^2\big(\mathbb{R}^n_+\big),\ \Phi|_D = 0\big\}.$$

Associated with K_D, is the space

$$\widetilde{K}_D \doteq \big\{\Psi = \Phi(D) - \Phi,\ \Phi \in K_D\big\},$$

which is a Hilbert space isometric to K_D when endowed with the norm $\|\Psi\|_{\widetilde{K}_D} = \|\nabla\Psi\|_{L^2(\Omega)}$. The elements of $\widetilde{K}_D$ vanish on D and the map

$$\ell(\Psi) \doteq \Phi(D)$$

is a continuous linear form on $\widetilde{K}_D$.

Analogously, let $\widetilde{K}^c_D$ be defined by

$$\widetilde{K}^c_D \doteq \big\{\Psi = \Phi(D) - \Phi,\ \Phi \in K^c_D\big\}.$$

This subspace is constituted of smooth functions which are constant outside a bounded subset in $\mathbb{R}^n_+$ and is dense in $\widetilde{K}_D$. One should remark that ℓ is just the limit at $+\infty$ for the elements of $\widetilde{K}^c_D$, so it is a generalization of this limit for the full space $\widetilde{K}_D$. Note also that $\widetilde{K}^0_D = K^0_D = \operatorname{Ker}\ell$.

Associated with these spaces,consider the auxiliary boundary layer problem, for F in $L^{2^{*\prime}}(\mathbb{R}^n_+)$ ($2^{*\prime} = \frac{2n}{n+2}$), $G \in L^{n/2}(\mathbb{R}^n_+)$ and nonnegative, and C a real number,

$$\begin{cases} \text{Find } w \in K_D \text{ satisfying } w(D) = C \text{ and} \\ \displaystyle\int_{\mathbb{R}^n_+} (\nabla w \nabla\varphi + Gw\varphi)\,dz = \int_{\mathbb{R}^n_+} F\varphi\,dz, \\ \forall\varphi \in K^0_D. \end{cases} \tag{23}$$

Proposition 5.1. *Problem* (23) *has a unique solution and the following Green formula holds for every* ϕ *in* K_D:

$$\int_{\mathbb{R}^n_+} (\nabla w \nabla \varphi + G w \varphi)\, dz = \int_{\mathbb{R}^n_+} F \phi\, dz - \phi(D) \int_D \frac{\partial w}{\partial z_n}\, dz'. \tag{24}$$

Proof. Let Φ_1 be an arbitrary element of K_D with $\Phi_1|_D = 1$ and look for $\tilde{w}$ in K_D^0 solution of

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n_+} (\nabla \tilde{w} \nabla \varphi + G \tilde{w} \varphi)\, dz \\ \displaystyle\quad = \int_{\mathbb{R}^n_+} F \varphi\, dz - C \int_{\mathbb{R}^n_+} (\nabla \Phi_1 \nabla \varphi + G \Phi_1 \varphi)\, dz, \\ \forall \phi \in K_D^0. \end{cases} \tag{25}$$

The second integral makes sense since, by the Hölder inequality, Gw belongs to $L^{2^{*\prime}}(\mathbb{R}^n_+)$. Therefore, by the Lax–Milgram theorem, $\tilde{w}$ exists and is unique, hence $w = C\Phi_1 + \tilde{w}$ is the unique solution of (23).

To obtain the Green formula, for $\phi \in K_D$ use $\varphi \doteq \phi - \phi(D)\Phi_1 \in K_D^0$ as a test function in (23) to get

$$\int_{\mathbb{R}^n_+} (\nabla w\, \nabla \phi + G w \phi)\, dz = \int_{\mathbb{R}^n_+} F \phi\, dz + \phi(D) \int_{\mathbb{R}^n_+} (\nabla w \nabla \Phi_1 + G w \Phi_1 - F \Phi_1)\, dz. \tag{26}$$

The last integral does not depend upon the choice of Φ_1 (use $\varphi \doteq \Phi_1 - \widehat{\Phi}_1$ in (24)) and can be interpreted as (a generalization of) $-\int_D \frac{\partial w}{\partial z_n}\, dz'$. □

Corollary 5.2. *For* F' *in* $L^{2^{*\prime}}(\mathbb{R}^n_+)$, $G \in L^{2^{*\prime}}(\mathbb{R}^n_+) \cap L^{n/2}(\mathbb{R}^n_+)$ *and nonnegative, and* C' *a real number, there is a unique solution* u *for the auxiliary boundary layer problem*

$$\begin{cases} \text{Find}\, u \in \widetilde{K}_D \text{ satisfying } \ell(u) = C' \text{ and} \\ \displaystyle\int_{\mathbb{R}^n_+} (\nabla u \nabla \varphi + G u \varphi)\, dz = \int_{\mathbb{R}^n_+} F' \varphi\, dz, \\ \forall \varphi \in K_D^0, \end{cases} \tag{27}$$

and for every ϕ *in* K_D:

$$\int_{\mathbb{R}^n_+} (\nabla u \nabla \phi + G u \varphi)\, dz = \int_{\mathbb{R}^n_+} F \phi\, dz - \phi(D) \int_D \frac{\partial u}{\partial z_n}\, dz'. \tag{28}$$

Proof. Note that for every $u \in \widetilde{K}_D$ and $\varphi \in K_D^0$, the product $Gu\varphi$ is integrable since it equals $G(u - \ell(u))\varphi + \ell(u) G \varphi$ and each term is integrable by the Hölder inequality due to the two conditions on G.

Let $F \doteq F' - C'G$, which belongs to $L^{2^{*\prime}}(\mathbb{R}^n_+)$, and $C \doteq -C'$. Then, the solution w of (23) exists and is unique. It is straightforward to check that $u \doteq w - w(D)$ is the unique solution of (27) and that formula (28) follows from formula (24). □

5.2. The unfolded limit for $0 < k < +\infty$

In this subsection, we assume that (4) holds with $0 < k < +\infty$. Note that this implies the relation $\varepsilon\delta^n \sim k(\varepsilon\delta)^2$. Also we suppose that the following conditions on the functions $\rho_{\varepsilon,\delta}$ and $\tilde{f}_{\varepsilon,\delta}$ are satisfied:

H1. The functions $\rho_{\varepsilon,\delta}$ satisfy the estimate $\|\rho_{\varepsilon,\delta}\|_{L^\infty} \leqslant C$ uniformly in ε, δ, and $\mathcal{T}^{bl}_{\varepsilon,\delta}(\rho_{\varepsilon,\delta})$ converges in measure (or almost everywhere) in $\Gamma_3 \times B$ to a function $\bar\rho$.
H2. The functions $\varepsilon\delta^n \mathcal{T}^{bl}_{\varepsilon,\delta}(\tilde f_{\varepsilon,\delta})$ converge weakly to some $\bar f$ in $L^2(\Gamma_3 \times B)$.

Hypothesis **H2** implies that $\varepsilon\delta\|\tilde f\|_{L^2(B_{\varepsilon,\delta})}$ is uniformly bounded, so that Corollary 3.2 applies.

Example 5.3. A typical example of $\rho_{\varepsilon,\delta}$ and $\tilde f_{\varepsilon,\delta}$ satisfying Hypotheses **H1** and **H2** is the case of Section 2.0.1, where

$$\rho_{\varepsilon,\delta}(x) = \bar\rho\left(\frac{1}{\delta}\left\{\frac{x'}{\varepsilon}\right\}, \frac{x_n}{\varepsilon\delta}\right), \qquad \tilde f_{\varepsilon,\delta}(x) = \frac{1}{\varepsilon\delta^n}\bar f\left(\frac{1}{\delta}\left\{\frac{x'}{\varepsilon}\right\}, \frac{x_n}{\varepsilon\delta}\right),$$

with $\bar\rho$ and $\bar f$ defined in $\mathbb{R}^n_+$ and supported in B.

Our first statement deals with the case $\varkappa = 2$.

Theorem 5.4. *Let $u_{\varepsilon,\delta}$ be a solution of problem $(\mathcal{P}_{\varepsilon,\delta})$. Assume that $\varkappa = 2$, and that conditions* **H1** *and* **H2** *are fulfilled. Then*

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad \text{weakly in } \mathcal{V}_0, \tag{29}$$

and there exists $U = U(x', z)$ in $L^2(\Gamma_3; \widetilde K_D)$ with $\ell(U) = u_0|_{\Gamma_3}$, such that the pair (u_0, U) solves the equations

$$\int_{\mathbb{R}^n_+} \nabla_z U(x', z)\nabla_z v\,dz + \int_B \bar\rho(x', z)U(x', z)v(z)\,dz = \int_B \bar f(x', z)v(z)\,dz \tag{30}$$

for a.e. x' in Γ_3 and all $v \in K^0_D$;

$$\int_\Omega \nabla u_0 \nabla\psi\,dx + k\int_{\Gamma_3}\left(\int_B \bar\rho(x', z)U(x', z)\,dz - \int_D \frac{\partial U}{\partial \nu_z}\,dz'\right)\psi(x')\,dx'$$
$$= \int_\Omega f\psi\,dx + \int_{\Gamma_3}\left(\int_B \bar f(x', z)\,dz\right)\psi(x')\,dx' \tag{31}$$

for all $\psi \in \mathcal{V}_0$. Furthermore, the solution (u_0, U) of (30)–(31) is unique.

The next statement treats the case $\varkappa < 2$.

Theorem 5.5. *Let $u_{\varepsilon,\delta}$ be a solution of problem $(\mathcal{P}_{\varepsilon,\delta})$. Assume that $\varkappa < 2$, and that conditions* **H1** *and* **H2** *are fulfilled. Then*

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad \text{weakly in } \mathcal{V}_0,$$

and there exists $U = U(x', z)$ with $U - u_0$ in $L^2(\Gamma_3; K_D)$, $U(x', z) = 0$ for $z \in D$, such that the pair (u_0, U) solves the equations

$$\int_{\mathbb{R}^n_+} \nabla_z U(x', z)\,\nabla_z v\,dz = \int_B \bar f(x', z)v(z)\,dz \tag{32}$$

for a.e. x' in Γ_3 and all $v \in K^0_D$;

$$\int_\Omega \nabla u_0 \nabla\psi\,dx - k\int_{\Gamma_3}\left(\int_D \frac{\partial U}{\partial \nu_z}\,dz'\right)\psi(x')\,dx'$$
$$= \int_\Omega f\psi\,dx + \int_{\Gamma_3}\left(\int_B \bar f(x', z)\,dz\right)\psi(x')\,dx' \tag{33}$$

for all $\psi \in \mathcal{V}_0$.

Remark 5.6. The statements of Theorem 5.5 is actually the same as that of Theorem 5.4 if we set $\bar{\rho} \equiv 0$.

We now consider the case $\varkappa > 2$. For simplicity we assume:

H1′. There is a subset B' of B with Lipschitz boundary such that $\mathcal{T}^{bl}_{\varepsilon,\delta}(\rho_{\varepsilon,\delta})$ vanishes on $\Gamma_3 \times (B \setminus B')$, and

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(\rho_{\varepsilon,\delta}) \xrightarrow{\text{a.e.}} \bar{\rho},$$

where $\bar{\rho} > 0$ a.e. on $\Gamma_3 \times B'$.

We introduce the following notations:

$$B'' = B' \cup D, \qquad D'' = \big(\partial B' \cup D\big) \setminus \big(\partial B' \cap D\big),$$

and define in the same way as K_D (see (21)) the following space:

$$K_{B''} \doteq \big\{\Phi \in L^{2^*}\big(\mathbb{R}^n_+\big);\ \nabla\Phi \in L^2\big(\mathbb{R}^n_+\big),\ \Phi|_{B''} \text{ is a constant}\big\}.$$

The spaces $K^c_{B''}$ and $K^0_{B''}$ are defined similarly.

Theorem 5.7. *Let $u_{\varepsilon,\delta}$ be a solution of problem $(\mathcal{P}_{\varepsilon,\delta})$. Assume that $\varkappa > 2$, and that conditions* **H1′** *and* **H2** *are satisfied. Then*

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad \text{weakly in } \mathcal{V}_0,$$

and there exists $U = U(x', z)$ with $U - u_0$ in $L^2(\Gamma_3; K_{B''})$, $U(x', z) = 0$ for $z \in B''$, such that the pair (u_0, U) solves the equations

$$\int_{\mathbb{R}^n_+ \setminus B'} \nabla_z U\big(x', z\big)\, \nabla_z v\, dz = \int_{B \setminus B'} \bar{f}\big(x', z\big) v(z)\, dz \tag{34}$$

for a.e. x' in Γ_3 and all $v \in K^0_{B''}$;

$$\int_{\Omega} \nabla u_0 \nabla\psi\, dx - k \int_{\Gamma_3} \bigg(\int_{D''} \frac{\partial U}{\partial \nu_z}\, d\sigma(z) \bigg) \psi\big(x'\big)\, dx' = \int_{\Omega} f\psi\, dx + \int_{\Gamma_3} \bigg(\int_{B \setminus B'} \bar{f}\big(x', z\big)\, dz \bigg) \psi\big(x'\big)\, dx' \tag{35}$$

for all $\psi \in \mathcal{V}_0$, where ν_z is the outward normal to D''.

Remark 5.8. In Theorem 5.7 assumption **H1′** can be relaxed to the case where the subset B' depends on $x' \in \Gamma_3$ in a regular enough way, in which case D'' depends on x' too.

For the proofs of Theorems 5.4, 5.5 and 5.7, we use the following lemma:

Lemma 5.9. *For v in K^c_D and δ small enough, set*

$$w^{bl}_{\varepsilon,\delta}(x) = v(D) - v\bigg(\frac{1}{\delta} \bigg\{ \frac{x'}{\varepsilon} \bigg\}_Y, \frac{x_n}{\varepsilon\delta} \bigg) \quad \text{for } x \in \mathbb{R}^n_+. \tag{36}$$

Then, for $0 < k < \infty$,

$$w^{bl}_{\varepsilon,\delta} \rightharpoonup v(D) \quad \text{weakly in } H^1(\Omega),$$

and for $k = 0$,

$$w^{bl}_{\varepsilon,\delta} \to v(D) \quad \text{strongly in } H^1(\Omega).$$

Furthermore, $\nabla w^{bl}_{\varepsilon,\delta}$ vanishes outside the layer ω_ε (defined by (19)).

The proof of the lemma is similar to the proof of Lemma 4.2 from [11].

Proof of Theorem 5.4. By Proposition 3.1, the solutions $u_{\varepsilon,\delta}$ are bounded in $\mathcal{V}_0$, so that, up to a subsequence, we can assume that

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad \text{weakly in } \mathcal{V}_0.$$

By item 6 of Theorem 4.5, there exists a U in $L^2(\Gamma_3; L^2_{\mathrm{loc}}(\mathbb{R}^n_+))$ such that, up to a subsequence,

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup U \quad \text{weakly in } L^2\big(\Gamma_3; L^2_{\mathrm{loc}}\big(\mathbb{R}^n_+\big)\big). \tag{37}$$

Since $\mathcal{T}^{bl}_{\varepsilon,\delta}(M^{\varepsilon,bl}_Y(u_{\varepsilon,\delta})) = M^{\varepsilon,bl}_Y(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}$, Proposition 4.4 implies

$$M^{\varepsilon,bl}_Y(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y} \to u_{0|_{\Gamma_3}} \quad \text{strongly in } L^2\big(\Gamma_3; L^2_{\mathrm{loc}}\big(\mathbb{R}^n_+\big)\big). \tag{38}$$

On the other hand, item 6 of Theorem 4.5 gives a W in $L^2(\Gamma_3; L^{2^*}(\mathbb{R}^n_+))$ with $\nabla_z W$ in $L^2(\Gamma_3 \times \mathbb{R}^n_+)$, such that

$$M^{\varepsilon,bl}_Y(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y} - \mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup W \quad \text{weakly in } L^2\big(\Gamma_3; L^{2^*}\big(\mathbb{R}^n_+\big)\big). \tag{39}$$

From (37), (38) and (39) it follows

$$U + W = u_{0|_{\Gamma_3}} \quad \text{and} \quad \nabla_z U + \nabla_z W = 0,$$

and

$$\varepsilon\delta\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta}) = \nabla_z\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\big)1_{\frac{1}{\delta}Y} \rightharpoonup \nabla_z U \quad \text{weakly in } L^2\big(\Gamma_3 \times \mathbb{R}^n_+\big). \tag{40}$$

This, combined with (37), implies the weak convergence of $\mathcal{T}^{bl}_{\varepsilon,\delta}(u_\varepsilon, \delta)$ toward U in $L^2(\Gamma_3; H^1_{\mathrm{loc}}(\mathbb{R}^n_+))$.

By Definition 4.1, $\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0$ in $\Gamma_3 \times D$, so (37) implies the relation

$$U = 0 \quad \text{on } \Gamma_3 \times D. \tag{41}$$

Therefore, U belongs to $L^2(\Gamma_3; \widetilde{K}_D)$ and $W = u_{0|_{\Gamma_3}} - U$ belongs to $L^2(\Gamma_3; K_D)$. Recall that by Corollary 3.2

$$\big\|\rho^{1/2}_{\varepsilon,\delta} u_{\varepsilon,\delta}\big\|_{L^2(B_{\varepsilon,\delta})} \leqslant C(\varepsilon\delta)^{\varkappa/2}.$$

Under unfolding, this yields

$$\big\|\mathcal{T}^{bl}_{\varepsilon,\delta}\big(\rho^{1/2}_{\varepsilon,\delta}\big)\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\big|_{L^2(\Gamma_3\times B)} \leqslant C(\varepsilon\delta)^{(\varkappa/2-1)}. \tag{42}$$

Summarizing the above estimates, we obtain

$$\big\|\mathcal{T}^{bl}_{\varepsilon,\delta}\big(\rho^{1/2}_{\varepsilon,\delta}\big)\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\big\|_{L^2(\Gamma_3\times B)} \leqslant C\min\big\{1, (\varepsilon\delta)^{(\varkappa/2-1)}\big\}. \tag{43}$$

In order to capture the contribution of the singular terms in the limit problem, we adapt the proof of Theorem 3.1 from [11] and use Lemma 5.9.

For $\psi \in \mathcal{V}^c_0$ and $v \in K^c_D$, we set

$$w^{bl}_{\varepsilon,\delta}(x) = v(D) - v\left(\frac{1}{\delta}\left\{\frac{x'}{\varepsilon}\right\}_Y, \frac{x_n}{\varepsilon\delta}\right) \quad \text{for } x \in \mathbb{R}^n_+,$$

and let $\Phi \doteq \psi w^{bl}_{\varepsilon,\delta}$. Since $w^{bl}_{\varepsilon,\delta}$ vanishes on $\gamma_{\varepsilon,\delta}$, Φ is a test function for problem $(\mathcal{P}_{\varepsilon,\delta})$. Thus,

$$\int_{\omega_\varepsilon} \nabla u_{\varepsilon,\delta}\nabla w^{bl}_{\varepsilon,\delta}\psi\,dx + \int_{\Omega} \nabla u_{\varepsilon,\delta}\nabla\psi w^{bl}_{\varepsilon,\delta}\,dx + (\varepsilon\delta)^{-\varkappa}\int_{B_{\varepsilon,\delta}} \rho_{\varepsilon,\delta}u_{\varepsilon,\delta}\psi w^{bl}_{\varepsilon,\delta}\,dx$$
$$= \int_{\Omega\setminus B_{\varepsilon,\delta}} f w^{bl}_{\varepsilon,\delta}\psi\,dx + \int_{B_{\varepsilon,\delta}} \tilde{f}_{\varepsilon,\delta}w^{bl}_{\varepsilon,\delta}\psi\,dx. \tag{44}$$

We now determine the limits for each of the terms in (44).

By item 2 of Theorem 4.5,

$$\int_{\omega_\varepsilon} \nabla u_{\varepsilon,\delta} \nabla w^{bl}_{\varepsilon,\delta} \psi \, dx = \varepsilon \delta^n \int_{\Gamma_3 \times \mathbb{R}^n_+} \mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta}) \mathcal{T}^{bl}_{\varepsilon,\delta}\big(\nabla w^{bl}_{\varepsilon,\delta}\big) \mathcal{T}^{bl}_{\varepsilon,\delta}(\psi) \, dx' \, dz. \tag{45}$$

By item 4 of the same theorem,

$$\mathcal{T}^{bl}_{\varepsilon,\delta}\big(\nabla w^{bl}_{\varepsilon,\delta}\big) = -\frac{1}{\varepsilon\delta} \nabla_z v \quad \text{and} \quad \mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta}) = \frac{1}{\varepsilon\delta} \nabla_z \big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\big) \quad \text{in } \Gamma_3 \times \frac{1}{\delta} Y,$$

so that (45) yields

$$\int_{\omega_\varepsilon} \nabla u_{\varepsilon,\delta} \nabla w^{bl}_{\varepsilon,\delta} \psi \, dx = \frac{\delta^{n-2}}{\varepsilon} \int_{\Gamma_3 \times \mathbb{R}^n_+} \nabla_z \big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\big)(-\nabla_z v) \mathcal{T}^{bl}_{\varepsilon,\delta}(\psi) \, dx' \, dz. \tag{46}$$

Since v is with compact support, the obvious inequality

$$\big\| \mathcal{T}^{bl}_{\varepsilon,\delta}(\psi) - \psi \big\|_{L^\infty(\Gamma_3 \times \frac{1}{\delta} Y)} \leqslant c\varepsilon \|\nabla_x \psi\|_{L^\infty(\Omega)^n},$$

implies

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(\psi) \nabla_z v \to \psi \nabla_z v \quad \text{strongly in } L^2\big(\Gamma_3 \times \mathbb{R}^n_+\big). \tag{47}$$

This, together with (40), allows to pass to the limit in (46) to get

$$\lim_{\varepsilon\to 0} \int_{\omega_\varepsilon} \nabla u_{\varepsilon,\delta} \nabla w^{bl}_{\varepsilon,\delta} \psi \, dx = -k \int_{\Gamma_3 \times \mathbb{R}^n_+} \nabla_z U\big(x', z\big) \nabla_z v(z) \psi(x) \, dx' \, dz. \tag{48}$$

The second term in (44) converges as follows:

$$\lim_{\varepsilon\to 0} \int_{\Omega} \nabla u_{\varepsilon,\delta} \nabla \psi w^{bl}_{\varepsilon,\delta} \, dx = v(D) \int_{\Omega \times Y} \nabla u_0 \nabla \psi \, dx \, dy.$$

By item 2 of Theorem 4.5 again, the last term on the left-hand side of (44) now reads

$$\begin{aligned}
&(\varepsilon\delta)^{-\varkappa} \int_{B_{\varepsilon,\delta}} \rho_{\varepsilon,\delta} u_{\varepsilon,\delta} \psi w^{bl}_{\varepsilon,\delta} \, dx \\
&\quad = \varepsilon^{1-\varkappa} \delta^{n-\varkappa} \int_{\Gamma_3 \times B} \mathcal{T}^{bl}_{\varepsilon,\delta}(\rho_{\varepsilon,\delta}) \mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \mathcal{T}^{bl}_{\varepsilon,\delta}(\psi) \mathcal{T}^{bl}_{\varepsilon,\delta}\big(w^{bl}_{\varepsilon,\delta}\big) \, dx' \, dz \\
&\quad = \frac{\delta^{n-2}}{\varepsilon} (\varepsilon\delta)^{2-\varkappa} \int_{\Gamma_3 \times B} \mathcal{T}^{bl}_{\varepsilon,\delta}(\rho_{\varepsilon,\delta}) \mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \mathcal{T}^{bl}_{\varepsilon,\delta}(\psi) \, \mathcal{T}^{bl}_{\varepsilon,\delta}\big(w^{bl}_{\varepsilon,\delta}\big) \, dx' \, dz.
\end{aligned} \tag{49}$$

Since $\varkappa = 2$, this is simply

$$(\varepsilon\delta)^{-2} \int_{B_{\varepsilon,\delta}} \rho_{\varepsilon,\delta} u_{\varepsilon,\delta} \psi w^{bl}_{\varepsilon,\delta} \, dx = \frac{\delta^{n-2}}{\varepsilon} \int_{\Gamma_3 \times B} \mathcal{T}^{bl}_{\varepsilon,\delta}(\rho_{\varepsilon,\delta}) \mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \mathcal{T}^{bl}_{\varepsilon,\delta}(\psi) \mathcal{T}^{bl}_{\varepsilon,\delta}\big(w^{bl}_{\varepsilon,\delta}\big) \, dx' \, dz.$$

Note that $\mathcal{T}^{bl}_{\varepsilon,\delta}(w^{bl}_{\varepsilon,\delta})$ is just $-v(z)$, and that by assumption **H1**, $\mathcal{T}^{bl}_{\varepsilon,\delta}(\rho_{\varepsilon,\delta})$ converges a.e. to $\bar\rho(x', z)$. At the limit,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} (\varepsilon\delta)^{-2} \int_{B_{\varepsilon,\delta}} \rho_{\varepsilon,\delta} u_{\varepsilon,\delta} \psi w^{bl}_{\varepsilon,\delta} \, dx \\
&\quad = k \int_{\Gamma_3 \times B} \bar\rho\big(x', z\big) U\big(x', z\big) \psi\big(x', z\big) \big(v(D) - v(z)\big) \, dx' \, dz.
\end{aligned} \tag{50}$$

Regarding the right-hand side of (44), using again the unfolding, it is easily seen that

$$\lim_{\varepsilon\to 0}\int_{\Omega\setminus B_{\varepsilon,\delta}} f w^{bl}_{\varepsilon,\delta}\psi\,dx = v(D)\int_{\Omega} f(x)\psi(x)\,dx,$$

$$\lim_{\varepsilon\to 0}\int_{B_{\varepsilon,\delta}} \tilde f_{\varepsilon,\delta} w^{bl}_{\varepsilon,\delta}\psi\,dx = k\int_{\Gamma_3\times B} \bar f(x',z)\bigl(v(D)-v(z)\bigr)\Psi(x')\,dx'\,dz.$$

Summarizing, the limit of Eq. (44) reads

$$\begin{aligned} v(D)\int_{\Omega}\nabla u_0\nabla\psi\,dx &- k\int_{\Gamma_3}\biggl(\int_{\mathbb{R}^n_+}\nabla_z U(x',z)\,\nabla_z v\,dz\biggr)\psi(x')\,dx' \\ &+ k\int_{\Gamma_3}\biggl(\int_B \bar\rho(x',z)U(x',z)\psi(x',z)\bigl(v(D)-v(z)\bigr)\,dz\biggr)\psi(x')\,dx' \\ &= v(D)\int_{\Omega} f\psi\,dx + k\int_{\Gamma_3}\biggl(\int_B \bar f(x',z)\bigl(v(D)-v(z)\bigr)\,dz\biggr)\psi(x')\,dx'. \end{aligned} \tag{51}$$

By density arguments, the last relation holds true for every $\psi\in\mathcal{V}_0$ and $v\in K_D$.

With $v\in K^0_D$, i.e. $v(D)=0$, and since ψ is arbitrary, Eq. (51) gives (30) for a.e. x' in Γ_3.

In view of Corollary 5.2, for $v\in K_D$, it follows that,

$$\begin{aligned} &\int_{\mathbb{R}^n_+}\nabla_z U(x',z)\nabla_z v\,dz + \int_B \bar\rho(x',z)U(x',z)v(z)\,dz \\ &\quad = \int_B \bar f(x',z)v(z)\,dz - v(D)\int_D \frac{\partial U}{\partial z_n}\,dz'. \end{aligned}$$

Multiplying this equality by $\psi=\psi(x')$ and integrating over Γ_3 and subtracting the result from (51) gives (31).

The uniqueness of the solution of (30)–(31) will be proved in Section 6. □

Proof of Theorem 5.5. The proof is similar. In this case, in view of (49) and $\kappa<2$, the third term converges to 0 in (44) and the limit problem is the same as before but with ρ replaced by 0. □

Proof of Theorem 5.7. The proof proceeds essentially along the same lines. Under the extra hypotheses, however, estimate (43) implies that U vanishes a.e. on $\Gamma_3\times B''$. Then, by choosing $v\in K_{B''}$, the third term in (44) is already zero and we obtain the weak formulation (34)–(35). □

5.3. *Unfolded limit for $k=0$*

In the case $k=0$ the contribution of $\gamma_{\varepsilon,\delta}$ and $B_{\varepsilon,\delta}$ is asymptotically negligible, and the limit problem includes the Neumann boundary condition on Γ_3.

We introduce the following hypothesis:

H3. The number $\varkappa$ and the functions $\rho_{\varepsilon,\delta}$ and $\tilde f_{\varepsilon,\delta}$ are such that, as ε and δ tend to zero,

$$\min\bigl\{(\varepsilon\delta)^{\varkappa/2}\bigl\|\rho^{-1/2}_{\varepsilon,\delta}\tilde f_{\varepsilon,\delta}\bigr\|_{L^2(B_{\varepsilon,\delta})},\ \varepsilon\delta\|\tilde f_{\varepsilon,\delta}\|_{L^2(B_{\varepsilon,\delta})}\bigr\}$$

is bounded (so that Corollary 3.2 applies).

Theorem 5.10. *Let $u_{\varepsilon,\delta}$ be a solution of problem $(\mathcal{P}_{\varepsilon,\delta})$. Assume that Hypothesis* **H3** *is fulfilled and $k=0$. Then*

$$u_{\varepsilon,\delta}\rightharpoonup u_0 \quad \textit{weakly in } \mathcal{V}_0,$$

and u_0 is the solution of the following variational problem

$$\int_\Omega \nabla u_0 \nabla \psi \, dx = \int_\Omega f \psi \, dx \tag{52}$$

for all $\psi \in \mathcal{V}_0$.

Remark 5.11. Note that for $k = 0$ the value of $\varkappa$ has no influence on the structure of the limit problem. Its formulation (52) only involves the function u_0 and thus, represents the macroscopic limit problem.

Proof of Theorem 5.10. We take an arbitrary $v \in K_B$ and construct $w^{bl}_{\varepsilon,\delta}$ as in Lemma 5.9. Since $k = 0$, actually $w^{bl}_{\varepsilon,\delta}$ converges strongly in $H^1(\Omega)$ to $v(B)$. For $\psi \in C^\infty(\overline{\Omega})$ and vanishing in a neighborhood of Γ_1, let $\Phi \doteq \psi w^{bl}_{\varepsilon,\delta}$. Since $w^{bl}_{\varepsilon,\delta}$ vanishes on $B_{\varepsilon,\delta}$, Φ is a test function in problem $(\mathcal{P}_{\varepsilon,\delta})$. One has

$$\int_{\omega_\varepsilon} \nabla u_{\varepsilon,\delta} \nabla w^{bl}_{\varepsilon,\delta} \psi \, dx + \int_\Omega \nabla u_{\varepsilon,\delta} \nabla \psi w^{bl}_{\varepsilon,\delta} \, dx = \int_{\Omega \setminus B_{\varepsilon,\delta}} f w^{bl}_{\varepsilon,\delta} \psi \, dx.$$

Passing to the limit $\varepsilon \to 0$ and using a density argument (of the ψ's in $\mathcal{V}_0$) completes the proof. □

5.4. Unfolded limit for $k = \infty$

In this case the "spots" $\gamma_\varepsilon, \delta$ are large enough to ensure the Dirichlet boundary condition on Γ_3 in the limit problem.

Theorem 5.12. *Let $u_{\varepsilon,\delta}$ be the solution of problem $(\mathcal{P}_{\varepsilon,\delta})$. Assume that Hypothesis* **H3** *is fulfilled, and $k = \infty$. Then*

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad \text{weakly in } \mathcal{V}_0,$$

and u_0 is the solution of the following variational problem

$$\int_\Omega \nabla u_0 \nabla \psi \, dx = \int_\Omega f \psi \, dx, \qquad u_0 = 0 \quad \text{on } \Gamma_3,$$

for all $\psi \in \mathcal{V}_0$ with $\psi = 0$ on Γ_3.

Proof. By items 5 and 6 of Theorem 4.5 it follows

$$\begin{aligned}
\big(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) - M^\varepsilon_Y(w_{\varepsilon,\delta}) 1_{\frac{1}{\delta}Y}\big) \to W \equiv 0 \quad &\text{weakly in } L^2\big(\Gamma_3; L^{2^*}\big(\mathbb{R}^n_+\big)\big),\\
\nabla_z\big(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})\big) 1_{\frac{1}{\delta}Y} \to 0 \quad &\text{strongly in } L^2\big(\Gamma_3 \times \mathbb{R}^n_+\big),\\
\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) \rightharpoonup U \quad &\text{weakly in } L^2\big(\Gamma_3; L^2_{\mathrm{loc}}\big(\mathbb{R}^n_+\big)\big),
\end{aligned}$$

and $W(x', z) = U(x', z) + u_0(x', 0)$. Since $U(x', z) = 0$ for $z \in B$ and $W \equiv 0$, it follows that $u_0 = 0$ on Γ_3. The desired statement follows by taking the test ψ vanishing on a neighborhood of Γ_3, and by using density arguments. □

Remark 5.13. Note that for $k = +\infty$ the value of $\varkappa$ has no influence on the structure of the limit problem. Its formulation in the statement of Theorem 5.12 only involves the function u_0 and thus, represents the macroscopic limit problem.

6. Macroscopic description of the limit problem for $0 < k < \infty$

6.1. The case $\varkappa = 2$

One can rewrite system (30), (31) in a form where only u_0 appears.

Let $\overline{U}$ be the unique solution of the following problem, where $x' \in \Gamma_3$ appears as a parameter:

$$\int_{\mathbb{R}^n_+} \nabla_z \overline{U}(x', z) \nabla v \, dz + \int_B \bar{\rho}(x', z) \overline{U}(x', z) v(z) \, dz = 0 \tag{53}$$

for all $v \in K^0_D$, where $\overline{U}$ belongs to $L^2(\Gamma_3; \widetilde{K}_D)$ and $\ell(\overline{U}) = 1$.

Because $\bar{\rho}$ is nonnegative, essentially bounded and with compact support in $z \in \mathbb{R}^n_+$, this problem admits a unique solution given by Corollary 5.2 for $\overline{U}$ in $L^2(\Gamma_3; \widetilde{K}_D)$.

Definition 6.1. For a.e. $x' \in \Gamma_3$, the generalized capacity in $\mathbb{R}^n_+$ associated with the weight function $\bar{\rho}(x', z)$ for the set D is

$$\Theta(x') \doteq \int_B \bar{\rho}(x', z) \overline{U}(x', z) \, dz - \int_D \frac{\partial \overline{U}}{\partial \nu_z}(x', z') \, dz'. \tag{54}$$

Note that by Hypothesis **H1**, Θ belongs to $L^\infty(\Gamma_3)$.

Define also $\widetilde{U} \in K^0_D$ to be the unique solution of the following problem:

$$\int_{\mathbb{R}^n_+} \nabla_z \widetilde{U}(x', z) \nabla v \, dz + \int_B \bar{\rho}(x', z) \widetilde{U}(x', z) v(z) \, dz = \int_B \bar{f}(x', z) v(z) \, dz \tag{55}$$

for all $v \in K^0_D$. Here again, the Lax–Milgram theorem applies directly in K^0_D. We then set

$$F(x') \doteq \int_B \bar{f}(x', z) \, dz - \int_B \bar{\rho}(x', z) \widetilde{U}(x', z) \, dz + \int_D \frac{\partial \widetilde{U}}{\partial \nu_z}(x', z') \, dz'.$$

By Hypothesis **H2**, the function F belongs to $L^2(\Gamma_3)$.

The macroscopic formulation can now be expressed in terms of the functions Θ and F.

Theorem 6.2. *The limit function u_0 given by Theorem 5.4 is the unique solution of the homogenized equation*

$$\begin{cases} u_0 \in \mathcal{V}_0, \\ \displaystyle\int_\Omega \nabla u_0 \nabla \psi \, dx + k \int_{\Gamma_3} \Theta u_0 \psi \, dx' = \int_\Omega f \psi \, dx + \int_{\Gamma_3} F \psi \, dx', \\ \forall \psi \in \mathcal{V}_0. \end{cases} \tag{56}$$

Proof. It is straightforward that

$$U(x', z) = u_0(x') \overline{U}(x', z) + \widetilde{U}(x', z).$$

Combining this with (31) gives (56). Uniqueness for the solution of (56) is standard, and also implies uniqueness in Theorem 5.4. Consequently, the whole sequence $\{u_{\varepsilon,\delta}\}$ converges weakly to u_0 in the space $\mathcal{V}_0$. □

Remark 6.3. The strong formulation for (56) is:

$$\begin{cases} -\Delta u_0 = f & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial \nu} + k\Theta u_0 = F & \text{on } \Gamma_3, \\ \dfrac{\partial u_0}{\partial \nu} = 0 & \text{on } \Gamma_2, \\ u_0 = 0 & \text{on } \Gamma_1. \end{cases}$$

6.2. The case $\varkappa \neq 2$

For $\varkappa < 2$, the formulation (32)–(33), being the same as (30)–(31) for $\varkappa = 2$ with $\bar\rho \equiv 0$, the macroscopic formulation is the same as above. In this case, Θ is a constant which is the usual capacity of the set D in $\mathbb{R}^n_+$ (half of its capacity in $\mathbb{R}^n$).

For $\varkappa > 2$, the system (34)–(35) is again of the same form as (30)–(31) with D replaced by D'' and B replaced by $B \setminus B'$. The macroscopic formulation is therefore as above with these modifications.

7. Convergence of the energy and improved convergence results

In Section 6, the sequence $u_{\varepsilon,\delta}$ was shown to converge weakly to u_0 in the space $\mathcal{V}_0$. Can strong convergence hold? The following theorem gives a positive answer (we give the details only for the case $0 < k < \infty$, $\varkappa = 2$, for which the proof is the most elaborate). It improves on converges (29), (37) and (40).

Theorem 7.1. *Under the hypotheses of Theorem 5.4, the following strong convergences hold*:

$$u_{\varepsilon,\delta} \to u_0 \quad \text{strongly in } \mathcal{V}_0,$$
$$\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \to U \quad \text{strongly in } L^2\big(\Gamma_3; L^2_{\text{loc}}(\mathbb{R}^n_+)\big),$$
$$\big(\nabla_z \mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\big)1_{\frac{1}{\delta}Y} \to \nabla_z U \quad \text{strongly in } L^2\big(\Gamma_3 \times \mathbb{R}^n_+\big).$$

The limit U of the boundary layer term is in $L^2(\Gamma_3; \widetilde{K}_D)$. Due to the discontinuity of the boundary layer term $\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})$ at $\partial(\frac{1}{\delta}Y)$, one cannot expect its convergence in this space. However, the last two convergences above imply that one can extend $\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})$ into $(\Gamma_3 \times \mathbb{R}^n_+)$, so that this extension converges strongly to U in $L^2(\Gamma_3; \widetilde{K}_D)$.

The complete information at the limit is encapsulated in the pair (u_0, U). It belongs to the Hilbert space $\mathcal{G}(\mathcal{V}_0, D)$, defined as

$$\mathcal{G}(\mathcal{V}_0, D) \doteq \big\{(\phi,\psi) \in \mathcal{V}_0 \times L^2(\Gamma_3; \widetilde{K}_D);\ \ell\big(\psi\big(x',\cdot\big)\big) = \phi_{|\Gamma_3}\big(x'\big) \text{ for a.e. } x' \in \Gamma_3\big\}.$$

We first show a density result in $\mathcal{G}(\mathcal{V}_0, D)$.

Lemma 7.2. *The subspace $\mathcal{G}^c_0 \doteq \{(\ell(v)\varphi, \varphi_{|\Gamma_3} v), \varphi \in \mathcal{V}^c_0$ and $v \in \widetilde{K}^c_D\}$ is total in $\mathcal{G}(\mathcal{V}_0, D)$.*

Proof. Let (p,q) be an element of the product Hilbert space $\mathcal{V}_0 \times L^2(\Gamma_3; \widetilde{K}_D)$. We show that if it is orthogonal to $\mathcal{G}^c_0$ in $\mathcal{V}_0 \times L^2(\Gamma_3; \widetilde{K}_D)$, then it is also orthogonal to $\mathcal{G}(\mathcal{V}_0, D)$.

Now, (p,q) orthogonal to $\mathcal{G}^c_0$ reads

$$\ell(v)\int_\Omega \nabla p \nabla\varphi\, dx + \int_{\Gamma_3\times\mathbb{R}^n_+} \varphi \nabla_z q \nabla v\, dx'\, dz = 0 \quad \text{for all } \varphi \in \mathcal{V}^c_0 \text{ and } v \in \widetilde{K}^c_D.$$

Choosing v with $\ell(v) = 0$ implies $\int_{\Gamma_3\times\mathbb{R}^n_+} \varphi\nabla_z q \nabla v\, dx'\, dz = 0$, which in turn implies that, for a.e. $x' \in \Gamma_3$, q satisfies

$$\begin{cases} -\Delta_z q\big(x',\cdot\big) = 0 & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial q(x',\cdot)}{\partial \nu_z} = 0 & \text{on } \mathbb{R}^{n-1} \setminus D. \end{cases}$$

Using Green's formula (28) with $\varphi \doteq v - \ell(v)$, for v in $\widetilde{K}_D^c$, gives

$$\int_{\mathbb{R}_+^n} \nabla_z q(x', z) \nabla v \, dz = \ell(v) \int_D \frac{\partial q(x', z)}{\partial z_n} \, dz' \tag{57}$$

for a.e. $x' \in \Gamma_3$ and all $v \in \widetilde{K}_D$. From the above formulae we deduce the relation

$$\ell(v)\left(\int_\Omega \nabla p \nabla \varphi \, dx + \int_{\Gamma_3 \times D} \varphi \frac{\partial q(x', z)}{\partial z_n} \, dx' \, dz'\right) = 0.$$

Consequently, for all $\varphi \in \mathcal{V}_0^c$ one has

$$\int_\Omega \nabla p \nabla \varphi \, dx + \int_{\Gamma_3 \times D} \varphi \frac{\partial q(x', z)}{\partial z_n} \, dx' \, dz' = 0. \tag{58}$$

This holds also for every φ in $\mathcal{V}_0$ by density, and can be interpreted as p satisfying

$$\begin{cases} -\Delta p = 0 & \text{in } \Omega, \\ p = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial p}{\partial \nu} = 0 & \text{on } \Gamma_2, \\ \dfrac{\partial p}{\partial x_3}(x') = \displaystyle\int_D \frac{\partial q(x', z)}{\partial z_3} \, dz' & \text{for almost all } x' \text{ in } \Gamma_3. \end{cases}$$

Now let (Φ, Ψ) belong to $\mathcal{G}(\mathcal{V}_0, D)$. Applying (57) for a.e. $x' \in \Gamma_3$ with v replaced by $\Psi(x', \cdot)$ and integrating over Γ_3 gives

$$\int_{\Gamma_3 \times \mathbb{R}_+^n} \nabla_z q \nabla_z \Psi \, dx' \, dz = \int_{\Gamma_3 \times D} \ell(\Psi) \frac{\partial q}{\partial z_n} \, dx' \, dz'.$$

But by definition, $\ell(\Psi) = \Phi_{|\Gamma_3}$, so by (58) it follows that (p, q) is orthogonal to (Φ, Ψ), which concludes the proof. $\square$

Proposition 7.3. *The following convergence holds*:

$$\int_\Omega |\nabla u_{\varepsilon,\delta}|^2 \, dx + (\varepsilon\delta)^{-2} \int_\Omega \rho_{\varepsilon,\delta} u_{\varepsilon,\delta}^2 \, dx \to \int_\Omega |\nabla u_0|^2 \, dx$$
$$+ k \int_{\Gamma_3 \times \mathbb{R}_+^n} \left|\nabla_z U(x', z)\right|^2 dx' \, dz + k \int_{\Gamma_3 \times B} \bar{\rho}(x', z) \left|U(x', z)\right|^2 dx' \, dz. \tag{59}$$

Proof. By Lemma 7.2, equality (51) implies for every (Φ, Ψ) in $\mathcal{G}(\mathcal{V}_0, D)$

$$\int_\Omega \nabla u_0 \nabla \Phi \, dx + k \int_{\Gamma_3 \times \mathbb{R}_+^n} \nabla_z U(x', z) \nabla_z \Psi(x', z) \, dx' \, dz$$
$$+ k \int_{\Gamma_3 \times B} \bar{\rho}(x', z) U(x', z) \Psi(x', z) \, dx' \, dz$$
$$= \int_\Omega f \Phi \, dx + \int_{\Gamma_3 \times B} \bar{f}(x', z) \Psi(x', z) \, dx' \, dz. \tag{60}$$

This is true in particular for $(\Phi, \Psi) = (u_0, U)$. Hence

$$\int_\Omega |\nabla u_0|^2\,dx + k\int_{\Gamma_3\times\mathbb{R}^n_+} \left|\nabla_z U(x',z)\right|^2 dx'\,dz + k\int_{\Gamma_3\times B} \bar\rho(x',z)\left|U(x',z)\right|^2 dx'\,dz$$
$$= \int_\Omega f u_0\,dx + \int_{\Gamma_3\times B} \bar f(x',z)U(x',z)\,dx'\,dz. \tag{61}$$

The variational formulation $\mathcal{P}_{\varepsilon\delta}$ (rewritten here with $\varkappa = 2$) with $u_{\varepsilon\delta}$ as a test function, and together with (3) implies

$$\int_\Omega |\nabla u_{\varepsilon,\delta}|^2\,dx + (\varepsilon\delta)^{-2}\int_\Omega \rho_{\varepsilon,\delta}u^2_{\varepsilon,\delta}\,dx = \int_{B_{\varepsilon,\delta}} \tilde f_\varepsilon, \delta u_{\varepsilon,\delta}\,dx + \int_{\Omega\setminus B_{\varepsilon,\delta}} f u_{\varepsilon,\delta}\,dx.$$

By unfolding, it is easy to see that under Hypothesis **H2**,

$$\lim_{\varepsilon\to 0}\int_{B_{\varepsilon,\delta}} \tilde f_{\varepsilon,\delta}u_{\varepsilon,\delta}\,dx = \int_{\Gamma_3\times B} \bar f U\,dx',$$
$$\lim_{\varepsilon\to 0}\int_{\Omega\setminus B_{\varepsilon,\delta}} f u_{\varepsilon,\delta}\,dx = \int_\Omega f u_0\,dx,$$

so that, confronting with (61) completes the proof. □

Now we claim that from the above convergence, Theorem 7.1 follows. But this proof is not straightforward. Indeed, if we unfold $(\varepsilon\delta)^{-\varkappa}\int_\Omega \rho_{\varepsilon,\delta}u^2_{\varepsilon,\delta}\,dx$, it is not too hard to see that

$$(\varepsilon\delta)^{-2}\int_\Omega \rho_{\varepsilon,\delta}u^2_{\varepsilon,\delta}\,dx = \frac{\delta^{n-2}}{\varepsilon}\int_{\Gamma_3\times\mathbb{R}^n_+} \mathcal{T}^{bl}_{\varepsilon,\delta}(\rho_{\varepsilon,\delta})\left|\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\right|^2 dx'\,dz$$

so, by the weak lower semi-continuity of norms,

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla u_{\varepsilon,\delta}|^2\,dx + (\varepsilon\delta)^{-2}\int_\Omega \rho_{\varepsilon,\delta}u^2_{\varepsilon,\delta}\,dx$$
$$\geqslant \int_\Omega |\nabla u_0|^2\,dx + k\int_{\Gamma_3\times B} \bar\rho(x',z)\left|U(x',z)\right|^2 dx'\,dz.$$

Here we have used the following simple integration result:

Lemma 7.4. *Let O be a measure space with measure μ, $\{\alpha_m\}$ a sequence in $L^2(O)$ which weakly converges to some α, $\{p_m\}$ a sequence of nonnegative functions which is bounded in $L^\infty(O)$ and converges to some p almost everywhere. Then,*

$$\liminf_{m\to\infty}\int_O |\alpha_m|^2 p_m\,d\mu \geqslant \int_O |\alpha|^2 p\,d\mu.$$

Furthermore, if equality holds, then

$$\sqrt{p_m}\alpha_m \to \sqrt{p}\alpha \quad \text{strongly in } L^2(O),$$

and if p_m/p is bounded, then α_m also converges to α. □

Therefore, (59) is more precise and indicates a gap, we will with it in the proof of Theorem 7.1.

Proof of Theorem 7.1. We introduce a sequence of functions v_δ in K^c_D such that

$$v_\delta(D)=1, \qquad 0\leqslant v_\delta\leqslant 1, \qquad v_\delta(z)\nearrow 1, \quad \forall z \text{ as } \delta\to 0,$$
$$\operatorname{supp}(v_\delta)\subset \frac{1}{\delta^{1/2}}Y.$$

The important part is that the support of v_δ grows slower that $1/\delta$. From this function, using formula (36) from Lemma 5.9, we construct the sequence $\tilde{w}_{\varepsilon,\delta}$ as follows:

$$\tilde{w}_{\varepsilon,\delta}(x)=v_\delta(D)-v_\delta\left(\frac{1}{\delta}\left\{\frac{x'}{\varepsilon}\right\}_Y,\frac{x_n}{\varepsilon\delta}\right) \quad \text{for } x\in\mathbb{R}^n_+$$

and introduce the sequence $\tilde{v}_{\varepsilon,\delta}$,

$$\tilde{v}_{\varepsilon,\delta}=1-\tilde{w}_{\varepsilon,\delta}.$$

So

$$\tilde{w}_{\varepsilon,\delta}+\tilde{v}_{\varepsilon,\delta}=1, \qquad 0\leqslant \tilde{v}_{\varepsilon,\delta},\tilde{w}_{\varepsilon,\delta}\leqslant 1 \quad \text{and} \quad \tilde{w}_{\varepsilon,\delta}\nearrow 1 \quad \text{a.e. in } \Omega. \tag{62}$$

Now we rewrite the left-hand side of (59) as

$$\int_\Omega |\nabla u_{\varepsilon,\delta}|^2\tilde{w}_{\varepsilon,\delta}\,dx+\int_\Omega |\nabla u_{\varepsilon,\delta}|^2\tilde{v}_{\varepsilon,\delta}\,dx+(\varepsilon\delta)^{-2}\int_\Omega \rho_{\varepsilon,\delta}u_{\varepsilon,\delta}^2\,dx.$$

By Lemma 7.4, the first term satisfies

$$\liminf_{\varepsilon,\delta\to 0}\int_\Omega |\nabla u_{\varepsilon,\delta}|^2\tilde{w}_{\varepsilon,\delta}\,dx\geqslant\int_\Omega|\nabla u_0|^2\,dx. \tag{63}$$

By unfolding the second term

$$\int_\Omega |\nabla u_{\varepsilon,\delta}|^2\tilde{v}_{\varepsilon,\delta}\,dx=\frac{\delta^{n-2}}{\varepsilon}\int_{\Gamma_3\times\mathbb{R}^n_+}\left|\nabla_z\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\big)\right|^2 v_\delta(z)\,dx'\,dz,$$

and Lemma 7.4 again gives

$$\liminf_{\varepsilon,\delta\to 0}\int_\Omega |\nabla u_{\varepsilon,\delta}|^2\tilde{v}_{\varepsilon,\delta}\,dx\geqslant k\int_{\Gamma_3\times\mathbb{R}^n_+}|\nabla_z U|^2\,dx'\,dz. \tag{64}$$

In an analogous way,

$$\liminf_{\varepsilon,\delta\to 0}(\varepsilon\delta)^{-2}\int_\Omega \rho_{\varepsilon,\delta}u_{\varepsilon,\delta}^2\,dx\geqslant k\int_{\Gamma_3\times B}\bar{\rho}\big(x',z\big)\big|U\big(x',z\big)\big|^2\,dx'\,dz.$$

Combining this with (63), (64) and (59), one finally obtains the term by term convergence:

$$\int_\Omega |\nabla u_{\varepsilon,\delta}|^2\tilde{w}_{\varepsilon,\delta}\,dx\to\int_\Omega|\nabla u_0|^2\,dx,$$
$$\frac{\delta^{n-2}}{\varepsilon}\int_{\Gamma_3\times\mathbb{R}^n_+}\left|\nabla_z\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\big)\right|^2 v_\delta(z)\,dx'\,dz\to k\int_{\Gamma_3\times\mathbb{R}^n_+}|\nabla_z U|^2\,dx'\,dz,$$
$$\frac{\delta^{n-2}}{\varepsilon}\int_{\Gamma_3\times\mathbb{R}^n_+}\mathcal{T}^{bl}_{\varepsilon,\delta}(\rho_{\varepsilon,\delta})\left|\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\right|^2\,dx'\,dz\to k\int_{\Gamma_3\times B}\bar{\rho}\big(x',z\big)\big|U\big(x',z\big)\big|^2\,dx'\,dz.$$

Now applying Lemma 7.4 repeatedly completes the proof of Theorem 7.1. □

8. Some generalizations

All the above results can be extended for the case of a second order elliptic operator with a possibly oscillating matrix $A_{\varepsilon,\delta}(x)$. The original problem is changed to

$$\begin{cases} \text{Find } u_{\varepsilon,\delta} \in \mathcal{V}_{\varepsilon,\delta} \text{ satisfying} \\ \int_\Omega A_{\varepsilon,\delta}(x)\nabla u_{\varepsilon,\delta}\nabla\phi\,dx + \int_\Omega (\varepsilon\delta)^{-\varkappa}\rho_{\varepsilon,\delta}u_{\varepsilon,\delta}\phi\,dx = \int_\Omega f_{\varepsilon,\delta}\phi\,dx, \\ \forall\phi \in \mathcal{V}_{\varepsilon,\delta}. \end{cases} \tag{$\mathcal{P}_{\varepsilon,\delta}$}$$

The extra hypotheses for $A_{\varepsilon,\delta}$ are

A1. $A_{\varepsilon,\delta}$ is positive definite and bounded uniformly in ε and δ and for a.e. in $x \in \Omega$.
A2. $A_{\varepsilon,\delta}$ H-converges to some limit matrix A^{hom}.
A3. $\mathcal{T}^{bl}_{\varepsilon,\delta}(A_{\varepsilon,\delta})$ converges to some $\mathcal{A}^0$ a.e. on $\Gamma_3 \times \mathbb{R}^n_+$.

Let us briefly describe the limit problem in the case $0 < k < +\infty$, and $\varkappa = 2$. Eqs. (30) and (31) become respectively,

$$\int_{\mathbb{R}^n_+} \mathcal{A}^0(x',z)\nabla_z U(x',z)\,\nabla_z v\,dz + \int_B \bar\rho(x',z)U(x',z)v(z)\,dz = \int_B \bar f(x',z)v(z)\,dz$$

for a.e. x' in Γ_3 and all $v \in K^0_D$;

$$\int_\Omega A^{\mathrm{hom}}(x)\nabla u_0\nabla\psi\,dx + k\int_{\Gamma_3}\biggl(\int_B \bar\rho(x',z)U(x',z)\,dz + \int_D \frac{\partial U}{\partial\nu_{\mathcal{A}^0}}\,dz'\biggr)\psi(x')\,dx'$$
$$= \int_\Omega f\psi\,dx + \int_{\Gamma_3}\biggl(\int_B \bar f(x',z)\,dz\biggr)\psi(x')\,dx'.$$

For the corresponding macroscopic formulation, the auxiliary problem (53) reads

$$\int_{\mathbb{R}^n_+} \mathcal{A}^0\nabla_z\overline{U}(x',z)\nabla_z v\,dz + \int_B \bar\rho(x',z)\overline{U}(x',z)v(z)\,dz = 0,$$

and the corresponding generalized capacity becomes

$$\Theta(x') \doteq \int_B \bar\rho(x',z)\overline{U}(x',z)\,dz + \int_D \frac{\partial\overline{U}}{\partial\nu_{\mathcal{A}^0}}(x',z')\,dz'.$$

The first term of (55) is modified in a similar way. The proof goes along the same lines making also use of the definition of H-convergence.

The convergence of the energy still holds (with obvious modifications) and implies the strong convergence for the boundary layer term. The strong convergence of $u_{\varepsilon,\delta} \to u_0$ in $\mathcal{V}_0$ is replaced by the standard corrector result associated with the H-convergence of the operators $A_{\varepsilon,\delta}(x)$.

The other cases for k and $\varkappa$ are modified accordingly.

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