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Asymptotics of a spectral-sieve problem ☆

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ABSTRACT

In a bounded domain with a thin periodically punctured interface we study the limit behavior of the bottom of spectrum for a Steklov type spectral problem, the Steklov boundary condition being imposed on the perforation surface. For a certain range of parameters we construct the effective spectral problem and justify the convergence of eigenpairs.

1. Introduction

The paper deals with homogenization of elliptic Steklov type spectral problem in a domain consisting of two subdomains separated by a thin periodically punctured interface (sieve), Steklov spectral condition being imposed on the surface of thin cylindrical channels that form the interface perforation.

We consider a model spectral problem for the Laplacian that reads

$$\begin{cases} \Delta u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega \cap \partial\Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial n} = 0 \quad \text{on } \Gamma_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial n} = \lambda_\varepsilon u_\varepsilon \quad \text{on } \gamma_\varepsilon; \end{cases} \tag{1}$$

here Ω_ε is the union of two subdomains connected by the thin channels, the boundary of these channels is denoted by γ_ε, and Γ_ε is the lateral boundary of the perforated interface; ε is a small positive parameter characterizing the interface microstructure period. The domain Ω_ε is obtained by removing a thin perforated interface from a fixed domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$. The detailed description of the geometry is given in Section 2.

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Boundary-value problems in domains with perforated interfaces of infinitesimal or vanishing thickness were widely studied in the existing literature. The periodic spectral problem has been investigated in [2], where the higher order terms of the asymptotics were constructed. The boundary value problems in domains with perforation situated along an interior surface were homogenized in [5,9]. Theory of homogenization in perforated domains got started in the works [10,20,18].

Neumann sieve problem with the interface of zero thickness was considered in [19] and then in [1,4,6,12, 16]. The work [7] deals with the so called "thick Neumann's sieve" problem that reads

$$\begin{cases} -\Delta w_\varepsilon + w_\varepsilon = f \quad \text{in } \Omega_\varepsilon, \\ w_\varepsilon = 0 \quad \text{on } \partial\Omega \cap \partial\Omega_\varepsilon, \\ \frac{\partial w_\varepsilon}{\partial n} = 0 \quad \text{on } \Gamma_\varepsilon \cup \gamma_\varepsilon, \end{cases}$$

where $f \in L_2(\Omega)$. It was shown that w_ε converges to a function $w \in H^1(\Omega \setminus \Gamma)$ that solves the following boundary-value problem:

$$\begin{cases} -\Delta w + w = f \quad \text{in } \Omega \setminus \Gamma, \\ w = 0 \quad \text{on } \partial\Omega, \\ \frac{\partial w^-}{\partial n^-} - \frac{1}{2}\mu[w] = 0 \quad \text{on } \Gamma, \\ \frac{\partial w^+}{\partial n^+} - \frac{1}{2}\mu[w] = 0 \quad \text{on } \Gamma, \end{cases}$$

where Γ is the limit infinitesimally thin interface, $[w] = w^+ - w^-$ is the jump of w on Γ, where $w^\pm$ is the restriction of w on $\Omega^\pm$, $\Omega = \Omega^+ \cup \Gamma \cup \Omega^-$, $n^\pm$ are the respective outward unit normals on Γ, and μ is either a constant $0 \le \mu < \infty$, or $\mu = +\infty$, according to the ratio between the channels and the interface thickness. In the case when $\mu = +\infty$, the limit problem reads

$$\begin{cases} -\Delta w + w = f \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega. \end{cases}$$

See [7] for the details.

There is a vast literature devoted to homogenization of spectral problems including Steklov-type problems, see for instance [20,14,19]. Some results on homogenization of Steklov problems can be found in [3,11,13,17].

In the present paper we suppose that Steklov spectral condition is imposed on the surface of the interface channels. The limit behavior of eigenpairs, as $\varepsilon \to 0$, depends essentially on the ratio between the channels diameter and the period as well as the ratio between the interface thickness and the period. Here we assume that the channels diameter and the interface thickness are of the same order. Then for $N \ge 3$ three different cases are to be studied:

(i) the diameter is greater than $C\varepsilon^{\frac{N-1}{N-2}}$ (**subcritical case**),
(ii) the diameter is of order $\mathcal{O}(\varepsilon^{\frac{N-1}{N-2}})$ (**critical case**),
(iii) the diameter is less than $C\varepsilon^{\frac{N-1}{N-2}}$ (**supercritical case**).

This paper focuses on the subcritical case. Namely, we assume that the diameter of channels is of order ε^δ with $0 \le \delta < \frac{N-1}{N-2}$. In dimension 2 we assume that $0 \le \delta < \infty$. Under these conditions we construct the limit spectral problem and justify the convergence of eigenpairs. We show that in the subcritical case the principal eigenfunction, as well as other eigenfunctions corresponding to the bottom of the spectrum, exhibit a regular asymptotic behavior, in particular they have a non-trivial limit in $H^1(\Omega)$. On the contrary, in the supercritical case the principal eigenfunction localizes in the vicinity of the interface. The critical and supercritical cases will be considered in a separate paper.

Observe that the subset of the domain boundary where the Steklov condition is imposed asymptotically vanishes. Moreover, in the case $\delta > 0$ the surface volume of this subset also vanishes, as $\varepsilon \to 0$. Nevertheless, as long as the capacity of this subset remains uniformly positive, the eigenpairs related to the bottom of spectrum in (1) show a regular behavior, and the spectral condition of the original problem is inherited by the limit interface between two parts of the domain.

The paper is organized as follows. In Section 2 we provide the detailed description of the geometry and introduce the studied spectral problem. Section 3 focuses on constructing the limit spectral problem and the proof of convergence results.

2. Problem setup

Let Ω be a connected, open bounded set of $\mathbb{R}^N$ $(N \geq 2)$, with a piece-wise smooth Lipschitz continuous boundary $\partial\Omega$. Points in $\mathbb{R}^N$ are denoted by $x = (x', x_N)$ with $x' = (x_1, \cdots, x_{N-1}) \in \mathbb{R}^{N-1}$. We assume that the hyperplane $\{x \in \mathbb{R}^N : x_N = 0\}$ divides Ω into two non-empty subdomains Ω^- and Ω^+ with

$$\Omega^- = \{x \in \Omega : x_N < 0\}, \quad \Omega^+ = \{x \in \Omega : x_N > 0\},$$

and that, moreover, for some $m > 0$ we have $\Omega \cap \{x : -m < x_N < m\} = \Sigma \times (-m, m)$. Under our assumptions Σ is an open set in $\mathbb{R}^{N-1}$ with a Lipschitz boundary. In what follows we identify Σ with $\Omega \cap \{x \in \mathbb{R}^N : x_N = 0\}$. Then, $\Omega = \Omega^- \cup \Sigma \cup \Omega^+$. Denote $\Gamma_0 = \{x \in \partial\Omega : -m < x_N < m\}$.

Remark 2.1. The condition that $\Omega \cap \{x : -m < x_N < m\} = \Sigma \times (-m, m)$ for some $m > 0$ is imposed just for presentation simplicity. The results of the paper remain valid for domains of more general structure. In particular, the results hold for any bounded Lipschitz domain Ω that satisfies the following two conditions: (i) Ω^+ and Ω^- are non-empty; (ii) $\partial\Omega$ is smooth in the vicinity of the hyperplane $\{x_N = 0\}$, and for any $x \in \partial\Omega \cap \{x_N = 0\}$ the tangential hyperplane to $\partial\Omega$ does not coincide with $\{x_N = 0\}$.

Let Y be an open simply connected set in $\mathbb{R}^{N-1}$ with smooth boundary ∂Y; we assume that $\overline{Y} \subset (-\frac{1}{2}, \frac{1}{2})^{N-1}$. In the two dimensional case Y is a subinterval of $(-\frac{1}{2}, \frac{1}{2})$. For small real numbers $\varepsilon > 0$, $r_\varepsilon > 0$ and $h_\varepsilon > 0$ with $r_\varepsilon \leq \varepsilon$ we define

$$\Sigma_\varepsilon = \left\{x \in \Omega : -\frac{h_\varepsilon}{2} \leq x_N \leq \frac{h_\varepsilon}{2}\right\}, \qquad T_\varepsilon = \bigcup_{k' \in \mathcal{K}_\varepsilon} B_\varepsilon^{k'} \times \left(-\frac{h_\varepsilon}{2}, \frac{h_\varepsilon}{2}\right),$$

where

$$\mathcal{K}_\varepsilon = \left\{l' \in \mathbb{Z}^{N-1} : l' + \left[-\frac{1}{2}, \frac{1}{2}\right]^{N-1} \subset \varepsilon^{-1}\Sigma\right\}, \qquad \text{and} \quad B_\varepsilon^{k'} = (\varepsilon k' + r_\varepsilon Y).$$

Then we set (see Fig. 2)

$$\begin{aligned}
&S_\varepsilon = \Sigma_\varepsilon \setminus \overline{T_\varepsilon}, \qquad \Omega_\varepsilon = \Omega \setminus \overline{S_\varepsilon}, \\
&\Gamma_\varepsilon^\pm = \left\{x = (x', x_N) \in \partial\Omega_\varepsilon : x_N = \pm\frac{h_\varepsilon}{2}\right\}, \qquad \Gamma_\varepsilon = \Gamma_\varepsilon^- \cup \Gamma_\varepsilon^+ \\
&\gamma_\varepsilon = \left\{x = (x', x_N) \in \partial\Omega_\varepsilon : x' \in \bigcup_{k' \in \mathbb{Z}^{N-1}} \partial B_\varepsilon^{k'},\ -\frac{h_\varepsilon}{2} \leq x_N \leq \frac{h_\varepsilon}{2}\right\},
\end{aligned}$$

where $\partial B_\varepsilon^{k'}$ denotes the $(N-2)$-dimensional boundary of $B_\varepsilon^{k'}$. The set S_ε represents a sieve; it is a thin perforated layer, $B_\varepsilon^{k'} \times (-\frac{h_\varepsilon}{2}, \frac{h_\varepsilon}{2})$ is a cylindrical hole with a cross-section $B_\varepsilon^{k'}$ (see Fig. 1). The thickness of this cylinder is of order ε and its height is h_ε.

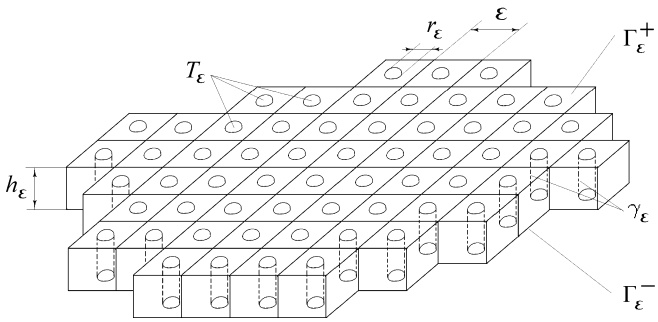

Fig. 1. The sieve S_ε.

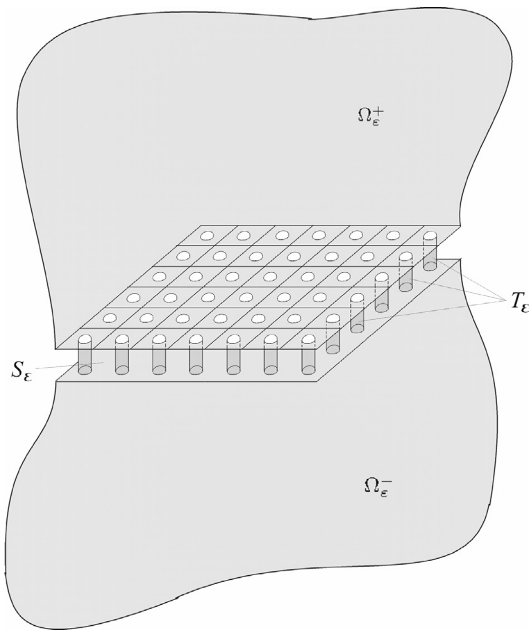

Fig. 2. The domain Ω_ε.

Notice that the $(\theta\varepsilon)$-neighborhood of $\partial\Omega$ does not intersect with T_ε where θ stands for the distance between Y and the boundary of the cube $Q = [-1/2, 1/2]^{N-1}$.

For a given function v such that $v^+ := v|_{\Omega^+} \in H^1(\Omega^+)$ and $v^- := v|_{\Omega^-} \in H^1(\Omega^-)$, we define the jump of v on Σ by $[v] = v^+(x', 0) - v^-(x', 0)$. We denote by n^- and n^+ the exterior unit normals to Ω^- and Ω^+ on Σ, and, for functions $v^\pm \in H^2(\Omega^\pm)$, $\frac{\partial v^-}{\partial n^-} = \frac{\partial v^-}{\partial x_N}$ and $\frac{\partial v^+}{\partial n^+} = -\frac{\partial v^+}{\partial x_N}$ stand for the corresponding normal derivatives. Given a function v defined a.e. in Ω_ε (see Fig. 2), we denote by $\widetilde{v}$ the zero extension of v to Ω, i.e.

$$\widetilde{v} = v \quad \text{in } \Omega_\varepsilon, \quad \widetilde{v} = 0 \quad \text{in } S_\varepsilon. \tag{2}$$

Let us denote $\Gamma_1 = \partial\Omega \setminus \Gamma_0$ and $\Gamma_0^\varepsilon = \partial\Omega_\varepsilon \cap \Gamma_0 = \{x \in \Gamma_0 : |x_N| > h_\varepsilon\}$. We consider the following spectral problem:

$$\begin{cases} \Delta u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \Gamma_1, \\ \frac{\partial u_\varepsilon}{\partial n} = 0 \quad \text{on } \Gamma_\varepsilon \cup \Gamma_0^\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial n} = \lambda_\varepsilon u_\varepsilon \quad \text{on } \gamma_\varepsilon, \end{cases} \tag{3}$$

where n denotes the outward unit normal to $\partial\Omega_\varepsilon$. We introduce the following Hilbert space:

$$H^1(\Omega_\varepsilon, \Gamma_1) = \{v \in H^1(\Omega_\varepsilon) : v|_{\Gamma_1} = 0\},$$

endowed with the scalar product $(v, w)_{H^1(\Omega_\varepsilon,\Gamma_1)} = \int_{\Omega_\varepsilon} \nabla v \cdot \nabla w \, dx$ and the corresponding norm $\|v\| = \left(\int_{\Omega_\varepsilon} |\nabla v|^2 \, dx\right)^{1/2}$ which is equivalent to the standard norm of $H^1(\Omega_\varepsilon)$. Variational formulation of problem (3) reads: find real numbers λ_ε such that problem

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v \, dx = \lambda_\varepsilon \int_{\gamma_\varepsilon} u_\varepsilon v \, ds, \quad \forall v \in H^1(\Omega_\varepsilon, \Gamma_1), \tag{4}$$

has a nonzero solution $u_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_1)$. Problem (3) can also be formulated in terms of the Dirichlet–Neumann map. Consider, for any $z \in H^{1/2}(\gamma_\varepsilon)$, the solution $v_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_1)$ of the boundary-value problem

$$\begin{cases} \Delta v_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \\ v_\varepsilon = 0 \quad \text{on } \Gamma_1, \\ \frac{\partial v_\varepsilon}{\partial n} = 0 \quad \text{on } \Gamma_\varepsilon \cup \Gamma_0^\varepsilon, \\ v_\varepsilon = z \quad \text{on } \gamma_\varepsilon, \end{cases} \tag{5}$$

then define the operator $\mathcal{L}^\varepsilon$ from $H^{1/2}(\gamma_\varepsilon)$ into $H^{-1/2}(\gamma_\varepsilon)$ by

$$\mathcal{L}^\varepsilon z = \left.\frac{\partial v_\varepsilon}{\partial n}\right|_{\gamma_\varepsilon}.$$

Problem (3) is equivalent to the following spectral problem: find real numbers λ_ε such that there is a nonzero function $z_\varepsilon \in H^{1/2}(\gamma_\varepsilon)$ satisfying

$$\mathcal{L}^\varepsilon z_\varepsilon = \lambda_\varepsilon z_\varepsilon. \tag{6}$$

The operator $\mathcal{L}^\varepsilon$ is invertible. Furthermore, $(\mathcal{L}^\varepsilon)^{-1}$ is compact and self-adjoint in $L^2(\gamma_\varepsilon)$, and $(z, (\mathcal{L}^\varepsilon)^{-1} z)_{L^2(\gamma_\varepsilon)} > 0$ for $z \neq 0$ (see [8]). Therefore, the spectrum of problem (6) consists of an increasing sequence of positive eigenvalues

$$0 < \lambda_{\varepsilon,1} \leq \lambda_{\varepsilon,2} \leq \cdots \leq \lambda_{\varepsilon j} \leq \cdots, \qquad \lambda_{\varepsilon,j} \to +\infty \text{ as } j \to \infty,$$

and there is an orthonormal sequence of the corresponding eigenvectors $(z_{\varepsilon,j})_{j\geq 1}$ in the space $L_2(\gamma_\varepsilon)$ endowed with the standard $(N-1)$-dimensional surface measure. If we substitute $(z_{\varepsilon,j})_{j\geq 1}$ for z in (5) and denote the corresponding solutions by $u_{\varepsilon j}$, then the sequence $\left(\frac{1}{\sqrt{\lambda_{\varepsilon,j}}} u_{\varepsilon,j}\right)_{j\geq 1}$ forms an orthonormal basis of eigenfunctions of problem (3) in $H^1(\Omega_\varepsilon, \Gamma_1)$ endowed with the norm $\int_{\Omega_\varepsilon} |\nabla v|^2 \, dx$.

Conversely, if $(u_{\varepsilon,j})_{j\geq 1}$ is an orthonormal sequence of eigenvectors of problem (3) then the family $(\sqrt{\lambda_{\varepsilon,j}}\, z_{\varepsilon,j})_{j\geq 1}$, with $z_{\varepsilon,j} = u_{\varepsilon,j}|_{\gamma_\varepsilon}$, is an orthonormal sequence of eigenvectors of (6). Moreover, the following variational principle holds. Introduce the Rayleigh quotient defined for $v \in H^1(\Omega_\varepsilon, \Gamma_1)\backslash\{0\}$, by

$$R_\varepsilon(v) = \frac{\int_{\Omega_\varepsilon} |\nabla v|^2 \, dx}{\int_{\gamma_\varepsilon} |v|^2 \, ds}. \tag{7}$$

Then,

$$\lambda_{\varepsilon,1} = \min\left\{R_\varepsilon(v) : v \in H^1(\Omega_\varepsilon, \Gamma_1)\right\}, \tag{8}$$

and for $j \geq 2$,

$$\lambda_{\varepsilon,j} = \min \left\{ R_\varepsilon(v) : v \in H^1(\Omega_\varepsilon, \Gamma_1), \int_{\gamma_\varepsilon} v\, u_{\varepsilon,i}\, ds = 0 \ \text{ for } i = 1, \cdots, j-1 \right\}. \tag{9}$$

Our aim is to investigate the asymptotic behavior of the eigenelements $(\lambda_{\varepsilon,j}, u_{\varepsilon,j})_{j\geq 1}$ of problem (3), as $\varepsilon \to 0$.

3. Convergence results

3.1. Homogenization theorem

As was mentioned above, we focus on the subcritical case, i.e.

$$r_\varepsilon = \varepsilon^{1+\delta}, \quad h_\varepsilon = \varepsilon^{1+\delta} h$$

with $0 \leq \delta < \frac{1}{N-2}$ if $N \geq 3$, and $\delta \in [0, +\infty)$ if $N = 2$.

We recall that the spectrum of problem (3) consists of an increasing sequence of positive eigenvalues

$$0 < \lambda_{\varepsilon,1} \leq \lambda_{\varepsilon,2} \leq \cdots \leq \lambda_{\varepsilon j} \leq \cdots, \qquad \lambda_{\varepsilon,j} \to +\infty \ \text{ as } \ j \to \infty,$$

and there is an orthonormal basis of the corresponding eigenfunctions in the space $H^1(\Omega_\varepsilon, \Gamma_1)$.

Here we formulate the main homogenization result.

We should choose a normalization condition for the eigenfunctions of problem (3). It is convenient to assume here and in what follows that the eigenfunctions $u_{\varepsilon,j}$ satisfy the following condition:

$$\int_{\Omega_\varepsilon} |\nabla u_{\varepsilon,j}|^2\, dx = 1, \ \text{ for any } j \geq 1. \tag{10}$$

Recall also that $\widetilde{u}_{\varepsilon,j}$ stands for the extension of $u_{\varepsilon,j}$ to Ω as defined in (2).

Our goal is to show that the limits Steklov-type spectral problem takes the form

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega^- \cup \Omega^+, \\ [u] = 0 \quad \text{on } \Sigma, \\ \left[\frac{\partial u}{\partial x_N}\right] = -\lambda_j K u \quad \text{on } \Sigma, \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_0, \\ u = 0 \quad \text{on } \Gamma_1, \end{cases} \tag{11}$$

where n denotes the outward unit normal to Γ_0, and

$$K = h\, \text{meas}_{N-2}(\partial Y) \ \text{ for } N \geq 3, \quad K = 2h \ \text{ for } N = 2.$$

Lemma 3.1. *Problem (11) has a real discrete spectrum*

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \qquad \lambda_j \to +\infty \quad \textit{as } j \to \infty.$$

There exists an orthonormal basis of eigenfunctions $\{u_j\}_{j\geq 1}$ in $L^2(\Sigma)$.

Proof. Consider two boundary value problems

$$\begin{cases} \Delta v^{\pm} = 0 & \text{in } \Omega^{\pm}, \\ v^{\pm} = 0 & \text{on } \Gamma_1 \cap \overline{\Omega}^{\pm}, \\ \dfrac{\partial v^{\pm}}{\partial n} = 0 & \text{on } \Gamma_0 \cap \overline{\Omega}^{\pm}, \\ v^{\pm} = z & \text{on } \Sigma, \end{cases}$$

and define the Dirichlet–Neumann operators $\mathcal{L}^{\pm}$ that associate to $z \in H^{1/2}(\Sigma)$ the function $\frac{\partial v^{\pm}}{\partial n} \in H^{-1/2}(\Sigma)$. The operators $\mathcal{L}^{\pm}$ are invertible and positive, $(\mathcal{L}^{\pm}z, z) > 0$ (see [8]). It is straightforward to check that the spectrum of $\mathcal{L}^- + \mathcal{L}^+$ coincides with the spectrum of problem (11). Since $(\mathcal{L}^- + \mathcal{L}^+)^{-1}$ is compact, self-adjoint and positive in $L^2(\Sigma)$, the desired statement follows. □

We proceed with the main result of this work.

Theorem 3.1. *Let $(\lambda_{\varepsilon,j}, u_{\varepsilon,j})_{j\geq 1}$ be the sequence of eigenpairs of problem (3).*

(i) If $\delta = 0$ then for any $j \geq 1$, $\lambda_{\varepsilon,j}$ converges, as $\varepsilon \to 0$, towards λ_j, where (λ_j, u_j) is the j-th eigenpair of problem (11). Furthermore, for a subsequence, $\widetilde{u}_{\varepsilon,j}$ converges in $L_2(\Omega)$ towards $u \in H^1(\Omega)$ being a linear combination of the eigenfunctions u_k related to the eigenvalue λ_j.

(ii) If $0 < \delta < \frac{1}{N-2}$ ($\delta < +\infty$ in dimension 2), then the sequence $\widehat{\lambda}_{\varepsilon,j} := \varepsilon^{(N-1)\delta}\lambda_{\varepsilon,j}$ converges, as $\varepsilon \to 0$, towards the eigenvalue λ_j of problem (11), and, for a subsequence, $\widetilde{u}_{\varepsilon,j}$ converges towards u in $L_2(\Omega)$. The function u is a linear combination of the eigenfunctions u_k related to the eigenvalue λ_j. Since $\delta > 0$, for any j the eigenvalue $\lambda_{\varepsilon,j}$ goes to infinity, as $\varepsilon \to 0$.

Remark 3.1. In the above theorem the whole sequence $\lambda_{\varepsilon,j}$ $(\widehat{\lambda}_{\varepsilon,j})$ converges, as $\varepsilon \to 0$. We do not need to choose a subsequence. However, if the eigenvalue λ_j of the homogenized problem is not simple, then the whole sequence of the corresponding eigenfunctions $\widetilde{u}_{\varepsilon,j}$ need not converge. We can only state the convergence of the eigenspaces related to λ_j. More precisely, let $\lambda_j, \lambda_{j+1}, \dots, \lambda_{j+m-1}$ be an eigenvalue of (11) of multiplicity m. Then the m-dimensional spaces generated by $\{\widetilde{u}_{\varepsilon,k}\}_{k=j}^{j+m-1}$ converge in $L^2(\Omega)$, as $\varepsilon \to 0$, to the space generated by $\{u_k\}_{k=j}^{j+m-1}$.

Remark 3.2. Instead of the interface with uniform thickness and cylindrical perforation one can consider more general family of perforated thin interfaces with non-uniform thickness and periodic microstructure like in [15]. We also assume that Steklov boundary condition is imposed on the periodically situated spots on the interfaces surface. In this case the statement of Theorem 3.1 remains valid if the following two conditions are satisfied: (i) An appropriate capacity type characteristics of the interfaces does not vanish, as $\varepsilon \to 0$. (ii) The scaled $N-1$-dimensional volume of the spots converges.

The first condition ensures that the limit functions do not have a jump on the interface. The second one allows us to derive the homogenized problem similar to (11). Of course, this statement is given in rather vague form. More accurate formulation would require some technical work.

3.2. Proof of Theorem 3.1 in the case $\delta = 0$

The variational formulation of spectral problem (11) reads

$$\lambda_1 = \min\left\{R(v) : v \in H^1(\Omega, \Gamma_1)\right\}, \qquad R(v) = \frac{\int_\Omega |\nabla v|^2\,dx}{\int_\Sigma |v|^2\,dx'}, \tag{12}$$

and for $j \geq 2$,

$$\lambda_j = \min\left\{ R(v) : v \in H^1(\Omega, \Gamma_1),\ \int\limits_{\Sigma} v\, u_i\, ds = 0 \ \text{ for } i = 1, \cdots, j-1 \right\}. \tag{13}$$

We begin by proving a priori estimates for the first eigenpair $(\lambda_{\varepsilon,1}, u_{\varepsilon,1})$. For brevity we denote it by $(\lambda_\varepsilon, u_\varepsilon)$. Let us first show that

$$0 < C_1 \leq \lambda_\varepsilon \leq C_2, \tag{14}$$

where constants C_1 and C_2 do not depend on ε. The upper bound relies on the following statement.

Lemma 3.2. *For any $\varepsilon > 0$ there is a function $w_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_1)$ such that $R_\varepsilon(w_\varepsilon) \leq C$ with a constant C that does not depend on ε; the functional R_ε being defined in (7).*

Proof. Let $\varphi = \varphi(x')$ be a $C_0^\infty(\Sigma)$ function such that $\varphi = 1$ on some $\Sigma_1 \subset \Sigma$ with $\text{meas}_{N-1}(\Sigma_1) > 0$, and denote by $\chi(x_N)$ a $C_0^\infty(-m, m)$ function such that $\chi = 1$ in the vicinity of 0. It is straightforward to check that, for sufficiently small $\varepsilon > 0$,

$$\int\limits_{\gamma_\varepsilon} (\varphi(x')\chi(x_n))^2 ds = \int\limits_{\gamma_\varepsilon} (\varphi(x'))^2 ds \geq C\, \text{meas}_{N-1}(\Sigma_1)\, h,$$

where C does not depend on ε. Since

$$\int\limits_{\Omega_\varepsilon} |\nabla(\varphi(x')\chi(x_N))|^2 dx \leq \int\limits_{\Omega} |\nabla(\varphi(x')\chi(x_N))|^2 dx,$$

this implies the desired inequality. $\square$

By (7) and Lemma 3.2 we obtain the upper bound in (14). In a similar way, using (9), one can prove that

$$\lambda_{\varepsilon,j} \leq C_{2,j} \quad \text{for all } j = 1,\, 2,\, \ldots \tag{15}$$

The proof of lower bound in (14) relies on the following statement.

Lemma 3.3. *There exists a constant $C > 0$ such that for any $v_1 \in H^1(\Omega_\varepsilon)$ and $v_2 \in H^1(\Omega_\varepsilon)$ and any $\varkappa \geq h$ we have*

$$\Big| \int\limits_{\gamma_\varepsilon} v_1 v_2\ ds - h\, \text{meas}_{N-2}(\partial Y) \int\limits_{\Sigma} v_1(x', \varkappa\varepsilon) v_2(x', \varkappa\varepsilon)\ dx' \Big| \leq C\varepsilon^{1/2} \|v_1\|_{H^1(\Omega_\varepsilon)} \|v_2\|_{H^1(\Omega_\varepsilon)}.$$

Proof. It is sufficient to prove the result in the case $v_1 = v_2$. Denote $\Pi_\varkappa = \big(Y \times [-\frac{h}{2}, \frac{h}{2}]\big) \cup \big(Q \times [\frac{h}{2}, \varkappa]\big)$ with $Q = [-1/2, 1/2]^{N-1}$, and $\Pi_\varkappa^\varepsilon = \varepsilon \Pi_\varkappa$. Let $\langle v \rangle_{\gamma_0}$ be the mean value of v over $\gamma_0 := \partial Y \times [-\frac{h}{2}, \frac{h}{2}]$ that is

$$\langle v \rangle_{\gamma_0} = \big|\gamma_0\big|_{N-1}^{-1} \int\limits_{\gamma_0} v\, ds.$$

The following two inequalities hold.

$$\int_{\gamma_0} \big(v - \langle v\rangle_{\gamma_0}\big)^2\, ds \le C \int_{\Pi_\varkappa} |\nabla v|^2\, dx, \qquad \int_{Q} \big(v(x',\varkappa) - \langle v\rangle_{\gamma_0}\big)^2\, dx' \le C \int_{\Pi_\varkappa} |\nabla v|^2\, dx. \tag{16}$$

We first prove the second inequality. Since both sides of this inequality are invariant with respect to adding an additive constant to a function v, we can assume without loss of generality that $\int_{\Pi_\varkappa} v\, dx = 0$. Then, by the Poincaré inequality, $\|v\|_{L^2(\Pi_\varkappa)} \le C\|\nabla v\|_{L^2(\Pi_\varkappa)}$. Finally, we have

$$\int_{Q} \big(v(x',\varkappa) - \langle v\rangle_{\gamma_0}\big)^2\, dx' \le 2 \int_{\partial\Pi_\varkappa} v^2 ds + C(Q)\langle v\rangle_{\gamma_0}^2 \le C(Q,\gamma_0) \int_{\partial\Pi_\varkappa} v^2 ds \le C_1(Q,\gamma_0) \int_{\Pi_\varkappa} |\nabla v|^2 dx;$$

the last inequality here follows from the trace theorem. The first estimate in (16) can be proved in the same way.

In the domain $\Pi_\varkappa^\varepsilon$ inequalities (16) take the form

$$\int_{\varepsilon\gamma_0} \big(v - \langle v\rangle_{\varepsilon\gamma_0}\big)^2\, ds \le C\varepsilon \int_{\Pi_\varkappa^\varepsilon} |\nabla v|^2\, dx, \qquad \int_{\varepsilon Q} \big(v(x',\varepsilon\varkappa) - \langle v\rangle_{\varepsilon\gamma_0}\big)^2\, dx' \le C\varepsilon \int_{\Pi_\varkappa^\varepsilon} |\nabla v|^2\, dx.$$

Similar inequalities hold for the sets $\Pi_\varkappa^\varepsilon + \varepsilon(j',0)$, $j' \in \mathcal{K}_\varepsilon$. Summing up over $j' \in \mathcal{K}_\varepsilon$, we obtain

$$\int_{\gamma_\varepsilon} \big(v - \widehat{\langle v\rangle}^\varepsilon\big)^2\, ds \le C\varepsilon \int_{\Omega_\varepsilon} |\nabla v|^2\, dx, \qquad \int_{\Sigma} \big(v(x',\varepsilon\varkappa) - \widehat{\langle v\rangle}^\varepsilon\big)^2\, dx' \le C\varepsilon \int_{\Omega_\varepsilon} |\nabla v|^2\, dx,$$

where $\widehat{\langle v\rangle}^\varepsilon$ denotes the piece-wise constant function equal to the mean value of v over $\varepsilon\gamma_0 + \varepsilon(j',0)$ in each $\Pi_\varkappa^\varepsilon + \varepsilon(j',0)$. Letting $K = h\,\mathrm{meas}_{N-2}(\partial Y)$, we have

$$\Big|\int_{\gamma_\varepsilon} v^2\, ds - K\int_{\Sigma} v^2(x',\varepsilon\varkappa)\, dx'\Big| = \Big|\int_{\gamma_\varepsilon} (v - \widehat{\langle v\rangle}^\varepsilon + \widehat{\langle v\rangle}^\varepsilon)^2\, ds - K\int_{\Sigma} (v(x',\varepsilon\varkappa) - \widehat{\langle v\rangle}^\varepsilon + \widehat{\langle v\rangle}^\varepsilon)^2\, dx'\Big|$$

$$= \Big|\int_{\gamma_\varepsilon} \big[(v - \widehat{\langle v\rangle}^\varepsilon)^2 + 2(v - \widehat{\langle v\rangle}^\varepsilon)\widehat{\langle v\rangle}^\varepsilon\big]\, ds - K\int_{\Sigma} \big[(v(x',\varepsilon\varkappa) - \widehat{\langle v\rangle}^\varepsilon)^2 + 2(v(x',\varepsilon\varkappa) - \widehat{\langle v\rangle}^\varepsilon)\widehat{\langle v\rangle}^\varepsilon\big]\, dx'\Big|$$

$$\le C\varepsilon \int_{\Omega_\varepsilon} |\nabla v|^2\, dx + C\varepsilon^{1/2}\|v\|_{L_2(\gamma_\varepsilon)}\Big(\int_{\Omega_\varepsilon} |\nabla v|^2\, dx\Big)^{\frac{1}{2}} \le C\varepsilon^{1/2}\|v\|^2_{H^1(\Omega_\varepsilon)}.$$

This completes the proof of lemma. □

According to Lemma 3.3,

$$\big|\, \|u_\varepsilon\|^2_{L^2(\gamma_\varepsilon)} - K\|u_\varepsilon(\cdot,\varepsilon h)\|^2_{L^2(\Sigma)}\,\big| \le C\varepsilon^{1/2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = C\varepsilon^{1/2}.$$

By the trace theorem, $\|u_\varepsilon(\cdot,\varepsilon h)\|_{L^2(\Sigma)} \le C\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}$. Combining the last two estimates yields the lower bound in (14).

As an immediate consequence of (10) we obtain

$$\int_{\Omega} |\widetilde{u}_\varepsilon|^2\, dx \le C_2. \tag{17}$$

Therefore, for a subsequence,

$$\lambda_\varepsilon \to \lambda, \qquad \widetilde{u}_\varepsilon \rightharpoonup u \quad \text{in } L_2(\Omega) \text{ weakly, as } \varepsilon \to 0, \tag{18}$$

here and in what follows we do not relabel subsequences of ε if it does not lead to an ambiguity.

In fact, $\widetilde{u}_\varepsilon$ converges strongly in $L_2(\Omega)$. Indeed, if we denote by I_ε the characteristic function of $\Omega \setminus \Sigma_\varepsilon$, then it easily follows from (10) that $I_\varepsilon \widetilde{u}_\varepsilon$ is compact in $L_2(\Omega)$. Combining the trace inequality with the Friedrichs inequality yields

$$\int_\Omega ((1 - I_\varepsilon)\widetilde{u}_\varepsilon)^2 dx = \int_{T_\varepsilon} (\widetilde{u}_\varepsilon)^2 dx$$

$$\leq C\varepsilon \int_{\{x_N = \pm \frac{h_\varepsilon}{2}\} \cap \Omega} (u_\varepsilon(x', \pm \frac{h_\varepsilon}{2}))^2 dx' + C\varepsilon^2 \int_{T_\varepsilon} |\nabla u_\varepsilon|^2 dx \leq C_1(\varepsilon + \varepsilon^2).$$

This implies the desired strong convergence.

According to (10), $u^+ := u\big|_{\Omega^+} \in H^1(\Omega^+)$, $u^- := u\big|_{\Omega^-} \in H^1(\Omega^-)$, and

$$\begin{cases} \Delta u^\pm = 0 \quad \text{in } \Omega^\pm, \\ u^\pm = 0 \quad \text{on } \Gamma_1 \cap \partial\Omega^\pm, \\ \frac{\partial u^\pm}{\partial n} = 0 \quad \text{on } \Gamma_0 \cap \partial\Omega^\pm. \end{cases} \tag{19}$$

From (17) we also have

$$\begin{aligned} \widetilde{\nabla u_\varepsilon^+} &\rightharpoonup \nabla u^+ \quad \text{in } (L_2(\Omega^+))^N \text{ weakly}, \\ \widetilde{\nabla u_\varepsilon^-} &\rightharpoonup \nabla u^- \quad \text{in } (L_2(\Omega^-))^N \text{ weakly}. \end{aligned} \tag{20}$$

We are going to use these relations as well as (18) in order to pass to the limit in (4).

It remains to derive the transmission conditions satisfied by u on Σ. Let us first show that $[u] = 0$ on Σ which implies that $u \in H^1(\Omega, \Gamma_1)$, here $H^1(\Omega, \Gamma_1)$ stands for the space of $H^1(\Omega)$ functions vanishing on Γ_1.

Lemma 3.4. *The jump of u on Σ is equal to zero, that is $[u] = 0$.*

Proof. We argue by contradiction. Assume that the jump set of u on Σ has positive $(N-1)$-dimensional measure. Then there is $\alpha > 0$ such that

$$\mathcal{R} := \text{meas}_{N-1}\{x' \in \Sigma \, : \, \big|[u]\big| \geq \alpha\} > 0.$$

Denote

$$T_\varepsilon^0 = \bigcup_{k' \in \mathcal{K}_\varepsilon} B_\varepsilon^{k'} \times \{0\}, \tag{21}$$

and let $\mathbf{1}_{T_\varepsilon^0}$ be the characteristic function of T_ε^0 on Σ. By the definition of T_ε^0 we have

$$\mathbf{1}_{T_\varepsilon^0} \quad \rightharpoonup \quad \text{meas}_{N-1}(Y) \qquad \text{weakly in } L_2(\Sigma),$$

as $\varepsilon \to 0$. Then, denoting $\mathcal{A} = \{x' \in \Sigma \, : \, |[u]| \geq \alpha\}$, we get

$$\mathbf{1}_{T_\varepsilon^0} \mathbf{1}_{\mathcal{A}} \ \rightharpoonup \ \text{meas}_{N-1}(Y) \mathbf{1}_{\mathcal{A}} \qquad \text{weakly in } L_2(\Sigma).$$

In particular,

$$\lim_{\varepsilon\to 0} \mathrm{meas}_{N-1}(\mathcal{A}\cap T_\varepsilon^0) = \mathcal{R}\,\mathrm{meas}_{N-1}(Y),$$

and, for all sufficiently small ε,

$$\mathrm{meas}_{N-1}(\mathcal{A}\cap T_\varepsilon^0) \geq \frac{1}{2}\mathcal{R}\,\mathrm{meas}_{N-1}(Y) \;=: \mathcal{R}_1. \tag{22}$$

Considering the L_2-continuity of trace of a H^1 function, we conclude that for sufficiently small ε it holds

$$\Big\| u(\cdot, \pm 0) - u(\cdot, \pm\frac{h_\varepsilon}{2}) \Big\|_{L_2(\Sigma)} \leq \frac{1}{20}\sqrt{\mathcal{R}_1}\alpha. \tag{23}$$

Since $\widetilde{u}_\varepsilon^\pm$ converges to u in $L_2(\Omega)$ and $\|u_\varepsilon\|_{H^1(\Omega\setminus\Sigma_\varepsilon)} \leq C$, for all sufficiently small ε we have

$$\Big\| u(\cdot, \pm\frac{h_\varepsilon}{2}) - u_\varepsilon(\cdot, \pm\frac{h_\varepsilon}{2}) \Big\|_{L_2(\Sigma)} \leq \frac{1}{20}\sqrt{\mathcal{R}_1}\alpha. \tag{24}$$

Combining (22)–(24) by means of triangle inequality we get

$$\Big\| u_\varepsilon(\cdot, \frac{h_\varepsilon}{2}) - u_\varepsilon(\cdot, -\frac{h_\varepsilon}{2}) \Big\|^2_{L_2(T_\varepsilon^0)} \geq \frac{1}{2}\mathcal{R}_1\alpha^2. \tag{25}$$

Now, writing

$$u_\varepsilon(x', \frac{h_\varepsilon}{2}) - u_\varepsilon(x', -\frac{h_\varepsilon}{2}) = \int_{-\frac{h_\varepsilon}{2}}^{\frac{h_\varepsilon}{2}} \frac{\partial u_\varepsilon}{\partial x_N}(x', t)\, dt$$

and using the Cauchy–Schwarz inequality we have

$$\Big| u_\varepsilon(x', \frac{h_\varepsilon}{2}) - u_\varepsilon(x', -\frac{h_\varepsilon}{2}) \Big|^2 \leq h_\varepsilon \int_{-\frac{h_\varepsilon}{2}}^{\frac{h_\varepsilon}{2}} \Big| \frac{\partial u_\varepsilon}{\partial x_N}(x', t) \Big|^2 dt.$$

Integrating this relation over T_ε^0 yields

$$\Big\| u_\varepsilon(\cdot, \frac{h_\varepsilon}{2}) - u_\varepsilon(\cdot, -\frac{h_\varepsilon}{2}) \Big\|^2_{L_2(T_\varepsilon^0)} \leq \varepsilon h \|\nabla u_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} \leq C\varepsilon.$$

For sufficiently small ε this contradicts (25). □

Considering (10), (14), (15) and Lemma 3.3, one can justify the following statement:

Lemma 3.5. *Under normalization condition (10) there exist constants $c_j > 0$, $j = 1, 2, \ldots,$ such that*

$$\|u_{\varepsilon,j}\|_{L_2(\Omega_\varepsilon)} \geq c_j.$$

Proof. From (10), (14) and (15) we obtain $\|u_{\varepsilon,j}\|^2_{L^2(\gamma_\varepsilon)} \geq C_j$. Then, by Lemma 3.3 below, we have $\|u_{\varepsilon,j}(\cdot,\varepsilon\frac{h}{2})\|^2_{L^2(\Sigma)} \geq \frac{1}{2}h\,\mathrm{meas}_{N-2}(\partial Y)C_j$. In view of (10) the $L^2(\Sigma)$ norm of function $u_{\varepsilon,j}(\cdot,s)$ is continuous in s uniformly in ε. This implies the desired lower bound. □

Let us now derive the Steklov type boundary condition satisfied by u on Σ. To this end we pass to the limit, as $\varepsilon \to 0$, in (4). Let $v \in H^1(\Omega,\Gamma_1)$. It is clear that $v|_{\Omega_\varepsilon} \in H^1(\Omega_\varepsilon,\Gamma_1)$, then according to (4) we have

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v\,dx = \lambda_\varepsilon \int_{\gamma_\varepsilon} u_\varepsilon v\,ds. \tag{26}$$

Writing

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v\,dx = \int_{\Omega} \widetilde{\nabla u_\varepsilon} \cdot \nabla v\,dx = \int_{\Omega^+} \widetilde{\nabla u_\varepsilon^+} \cdot \nabla v\,dx + \int_{\Omega^-} \widetilde{\nabla u_\varepsilon^-} \cdot \nabla v\,dx,$$

and using (20), we obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v\,dx = \int_{\Omega^+} \nabla u^+ \cdot \nabla v\,dx + \int_{\Omega^-} \nabla u^- \cdot \nabla v\,dx. \tag{27}$$

By Lemma 3.4, $u \in H^1(\Omega,\Gamma_1)$. Since u^+ and u^- satisfy (19), employing Green's formula we deduce from (27) that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v\,dx = -\left\langle \left[\frac{\partial u}{\partial x_N}\right], v\right\rangle_\Sigma. \tag{28}$$

Denote

$$\mathcal{M}_N = \frac{\mathrm{meas}_{N-2}(\partial Y)}{\mathrm{meas}_{N-1}(Y)};$$

here and in what follows for $N = 2$ we set $\mathrm{meas}_{N-2}(\partial Y) = 2$. Passage to the limit on the right-hand side of (26) relies on the following lemma.

Lemma 3.6. *Let $v \in H^1(\Omega,\Gamma_1) \cap C^1(\overline{\Omega})$. There exists $C > 0$ such that*

$$\left|\int_{\gamma_\varepsilon} u_\varepsilon v\,ds - h\mathcal{M}_N \int_{T_\varepsilon\cap\Sigma} u_\varepsilon(x',0)v(x',0)\,dx'\right| \leq C\sqrt{\varepsilon}. \tag{29}$$

Proof. In the cylinder $Y \times (0,h/2)$ consider the following problem

$$\begin{cases} \Delta\Upsilon = 0 & \text{in } Y \times (0,h/2), \\ \dfrac{\partial\Upsilon}{\partial n} = -h\mathcal{M}_N & \text{on } Y \times \{0\}, \\ \dfrac{\partial\Upsilon}{\partial n} = 0 & \text{on } Y \times \{h/2\}, \\ \dfrac{\partial\Upsilon}{\partial n} = 2 & \text{on } \partial Y \times (0,h/2). \end{cases}$$

Denoting $\Upsilon^\varepsilon = \varepsilon\Upsilon(x/\varepsilon)$, extending Υ^ε periodically in $(\varepsilon k' + \varepsilon Y) \times (0, \varepsilon h/2)$, $k' \in \mathcal{K}_\varepsilon$, integrating by parts and recalling the definition of T_ε^0 in (21), we get

$$0 = \int\limits_{T_\varepsilon \cap \{x_N > 0\}} u_\varepsilon v \Delta \Upsilon^\varepsilon \, dx = -h\mathcal{M}_N \int\limits_{T_\varepsilon^0} u_\varepsilon v \, dx' + 2 \int\limits_{\gamma_\varepsilon \cap \{x_N > 0\}} u_\varepsilon v \, ds - \int\limits_{T_\varepsilon \cap \{x_N > 0\}} \nabla(u_\varepsilon v) \cdot \nabla \Upsilon^\varepsilon \, dx. \tag{30}$$

From the definition of Υ^ε it easily follows that

$$\|\nabla \Upsilon^\varepsilon\|_{L_2(T_\varepsilon \cap \{x_N > 0\})} \le C\sqrt{\varepsilon}. \tag{31}$$

Indeed, by construction, $\|\nabla\Upsilon\|^2_{L^2(Y \times (0,h/2))} < \infty$. After dilatation we get $\|\nabla\Upsilon^\varepsilon\|^2_{L^2(\varepsilon Y \times (0,\varepsilon h/2)} \le C\varepsilon^N$. Summing up over $k' \in \mathcal{K}_\varepsilon$ yields $\|\nabla\Upsilon^\varepsilon\|^2_{L^2(T_\varepsilon \cap \{x_n > 0\}} \le C\varepsilon$, and (31) follows.

Combining (30) with (31) we obtain the desired inequality (29). □

In a similar way one can show that

$$\left| \frac{\operatorname{meas}_{N-2}(\partial Y)}{\operatorname{meas}_{N-1}(Y)} \int\limits_{T_\varepsilon^0} u_\varepsilon v \, dx' - \operatorname{meas}_{N-2}(\partial Y) \int\limits_{\Sigma} u_\varepsilon v \, dx' \right| \le C\sqrt{\varepsilon}. \tag{32}$$

Combining this estimates with (29) yields

$$\lim_{\varepsilon \to 0} \int\limits_{\gamma_\varepsilon} u_\varepsilon v \, ds = h \operatorname{meas}_{N-2}(\partial Y) \int\limits_{\Sigma} uv \, dx' = K \int\limits_{\Sigma} uv \, dx'. \tag{33}$$

From (26), (28) and (33) we deduce the spectral condition on Σ. The limit integral identity reads

$$\int\limits_{\Omega} \nabla u \nabla v \, dx = \lambda K \int\limits_{\Sigma} uv \, dx' \qquad \text{for any } v \in H^1(\Omega, \Gamma_1).$$

By Lemma 3.4, $u \in H^1(\Omega, \Gamma_1)$. From (10) and Lemma 3.5 it follows that $u \ne 0$. Therefore, (λ, u) is an eigenpair of (11).

Let us now show that the multiplicity of λ is at least k if there are k eigenvalues $\lambda_{\varepsilon,j_1}, \dots, \lambda_{\varepsilon,j_k}$, $j_i \ne j_m$ for $i \ne m$, converging (probably for a subsequence) to λ.

Assume that (for a subsequence)

$$\lambda_{\varepsilon,j_i} \to \lambda, \qquad i = 1, \dots, k, \qquad j_i \ne j_m \text{ if } i \ne m.$$

Choosing a subsequence once again we can assume that

$$\widetilde{u}_{\varepsilon,j_i} \rightharpoonup u_i \quad \text{weakly in } L_2(\Omega), \qquad i = 1, \dots, k, \quad \text{as } \varepsilon \to 0. \tag{34}$$

With the help of (10) and (17) one can easily show that $u^+ \in H^1(\Omega^+)$, $u^- \in H^1(\Omega^-)$ and u^+, u^- satisfy (19), and

$$\begin{aligned} &\widetilde{\nabla u_\varepsilon^+} \rightharpoonup \nabla u^+ \text{ in } (L_2(\Omega^+))^N \text{ weakly}, \\ &\widetilde{\nabla u_\varepsilon^-} \rightharpoonup \nabla u^- \text{ in } (L_2(\Omega^-))^N \text{ weakly}, \\ &\widetilde{u}_\varepsilon \to u \text{ in } L_2(\Omega) \text{ strongly}. \end{aligned} \tag{35}$$

Due to our normalization conditions for $u_{\varepsilon,j}|_{\gamma_\varepsilon}$ and by Lemma 3.3 we get

$$\left(\widetilde{u}_{\varepsilon,j_i}, \widetilde{u}_{\varepsilon,j_m}\right)_{L_2(\Sigma)} = K\delta_i^m + o(1), \quad \text{as } \varepsilon \to 0. \tag{36}$$

According to (34) and (35), $\|u_{\varepsilon,j_i} - u_i\|_{L_2(\gamma_\varepsilon)} \to 0$. Passing to the limit as $\varepsilon \to 0$ in (36) yields

$$\left(u_i, u_m\right)_{L_2(\Sigma)} = K\delta_i^m. \tag{37}$$

Therefore, $u_1, \dots, u_k$ are linearly independent and thus the multiplicity of λ is greater than or equal to k.

Let us now check that any eigenvalue of the homogenized problem (11) is a limit point of the eigenvalues of the original problem (3).

Lemma 3.7. *Let (λ_j, u_j) be the j-th eigenpair of problem (11). Then*

$$\limsup_{\varepsilon\to 0} R_\varepsilon(u_j) = \lambda_j, \qquad \lim_{\varepsilon\to 0} \|u_j\|^2_{L_2(\gamma_\varepsilon)} = K,$$
$$\lim_{\varepsilon\to 0} (u_j, u_m)_{L_2(\gamma_\varepsilon)} = 0 \text{ if } j \neq m. \tag{38}$$

Proof. The second and the third relations in (38) are straightforward consequences of Lemma 3.3. The first one easily follows from the second one. □

Combining (38) with variational formulae (7), (8), (9) and (12), (13), one concludes that

$$\limsup_{\varepsilon\to 0} \lambda_{\varepsilon,j} \le \lambda_j.$$

Assume that for a subsequence

$$\limsup_{\varepsilon\to 0} \lambda_{\varepsilon,j} < \lambda_j.$$

As was proved above in this case there exist at least j eigenvalues of problem (11) which are strictly less than λ_j. This contradiction shows that

$$\limsup_{\varepsilon\to 0} \lambda_{\varepsilon,j} = \lambda_j.$$

The convergence of the corresponding eigenspaces has already been justified.

3.3. Proof of Theorem 3.1 in the case $0 < \delta < \frac{1}{N-2}$

Consider the sequence of eigenpairs $(\lambda_{\varepsilon,j}, u_{\varepsilon,j})$ of problem (3) satisfying the normalization condition (10). As for the case $\delta = 0$, we denote $(\lambda_\varepsilon, u_\varepsilon)$ the first eigenpair $(\lambda_{\varepsilon,1}, u_{\varepsilon,1})$. It follows from the definition of $H^1(\Omega_\varepsilon, \Gamma_1)$ and Γ_1 that there exists a function $w \in H^1(\Omega_\varepsilon, \Gamma_1)$ such that

$$\int_{\Omega_\varepsilon} |\nabla w|^2 \, dx \le C, \tag{39}$$

and $w \equiv 1$ in the vicinity of S_ε for all sufficiently small ε; the constant C in (39) does not depend on ε. Since $\text{meas}_{N-1}\left(\partial B_\varepsilon^{k'} \times [-\frac{h_\varepsilon}{2}, \frac{h_\varepsilon}{2}]\right) = K\varepsilon^{(1+\delta)(N-1)}$ with $K = h\,\text{meas}_{N-2}(\partial Y)$, and the number of channels $|\mathcal{K}_\varepsilon|$ admits the estimate $|\mathcal{K}_\varepsilon| = \varepsilon^{1-N}\text{meas}_{N-1}(\Sigma)(1 + o(1))$, then

$$\int_{\gamma_\varepsilon} |w|^2 \, d\sigma = \text{meas}_{N-1}(\gamma_\varepsilon) \simeq \varepsilon^{(N-1)\delta}. \tag{40}$$

From (8), (39) and (40) we derive that $\varepsilon^{(N-1)\delta}\lambda_\varepsilon \le C$, i.e. $\widehat{\lambda}_\varepsilon \le C$. Similarly, using variational formulation for higher order eigenpairs, one can show that

$$\widehat{\lambda}_{\varepsilon,j} \le C_j. \tag{41}$$

Using the Poincaré inequality, we deduce that $(\widetilde{u}_\varepsilon)_{\varepsilon>0}$ is bounded in $L_2(\Omega)$, so we can extract a subsequence, not relabeled for convenience, such that

$$\begin{aligned}&\widehat{\lambda}_\varepsilon \to \widehat{\lambda},\\ &\widetilde{u}_\varepsilon \rightharpoonup u \quad \text{in } L_2(\Omega) \text{ weakly}, \quad \text{as } \varepsilon \to 0.\end{aligned} \tag{42}$$

With the help of (10) and (17) one can easily show that $u^+ \in H^1(\Omega^+)$, $u^- \in H^1(\Omega^-)$ and u^+, u^- satisfy (19), and

$$\begin{aligned}&\widetilde{\nabla u_\varepsilon^+} \rightharpoonup \nabla u^+ \text{ in } (L_2(\Omega^+))^N \text{ weakly},\\ &\widetilde{\nabla u_\varepsilon^-} \rightharpoonup \nabla u^- \text{ in } (L_2(\Omega^-))^N \text{ weakly},\\ &\widetilde{u}_\varepsilon \to u \text{ in } L_2(\Omega) \text{ strongly}.\end{aligned} \tag{43}$$

Clearly, the functions $u^\pm$ satisfy the equation and the boundary conditions in (19). It remains to derive the interface conditions satisfied by u on Σ. Let us first show that $[u] = 0$ on Σ so that $u \in H^1(\Omega, \Gamma_1)$. Reasoning as in the case $\delta = 0$, we assume, by contradiction, that u admits a jump through Σ. Then, for any $\varkappa > 0$ there exits a sequence $\{\varepsilon_k\}_{k=1}^\infty$, $\varepsilon_k \to 0$, and a set $X_{\varepsilon_k} \subset \Sigma$, and constants $c_1 > 0$, $c_2 > 0$, such that

$$\begin{aligned}&\text{meas}_{N-1}(X_{\varepsilon_k}) \ge c_1,\\ &\big|u_{\varepsilon_k}(x', \tfrac{\varepsilon_k\varkappa}{2}) - u_{\varepsilon_k}(x', -\tfrac{\varepsilon_k\varkappa}{2})\big| \ge c_2, \quad \text{for a.e. } x' \in X_{\varepsilon_k}.\end{aligned} \tag{44}$$

Without loss of generality we can assume that the origin belongs to Y. Then there is a cube $Q_\varpi = [-\varpi/2, \varpi/2]^{N-1}$, $\varpi > 0$, such that $Q_\varpi \subset Y$ for some ϖ.

Let $q < \varpi$ and introduce the following sets

$$\begin{aligned}\Phi_\varepsilon^+ &= \Big\{x \in \mathbb{R}^{N-1} \times \big[h\varepsilon^{1+\delta}, \frac{\varepsilon h}{2q}\big],\ x' \in 2(x_N/h)Q_q\Big\},\\ \Phi_\varepsilon^- &= \Big\{x \in \mathbb{R}^{N-1} \times \big[-\frac{\varepsilon h}{2q}, -h\varepsilon^{1+\delta}\big],\ x' \in -2(x_N/h)Q_q\Big\},\\ \Phi_\varepsilon^0 &= \varepsilon^{1+\delta}Q_q \times \big[-\varepsilon^{1+\delta}h, \varepsilon^{1+\delta}h\big], \quad \Phi_\varepsilon = \Phi_\varepsilon^+ \cup \Phi_\varepsilon^0 \cup \Phi_\varepsilon^-.\end{aligned}$$

Observe that $\{x \in \Phi_\varepsilon^+ : x_N = \varepsilon h/(2q)\} = \{x' \in Q_\varepsilon,\ x_N = \varepsilon h/(2q)\}$. Letting $\varkappa = h/q$ and choosing $\varepsilon < \varpi$, denote

$$\begin{aligned}X_{\varepsilon,0}^+ &= \Big\{x' \in Q_\varepsilon\ :\ \Big|u_\varepsilon(x', \frac{\varepsilon h}{2q}) - u_\varepsilon(\varepsilon^\delta x', h\varepsilon^{1+\delta})\Big| \ge c_2/3\Big\}\\ X_{\varepsilon,0}^- &= \Big\{x' \in Q_\varepsilon\ :\ \Big|u_\varepsilon(x', -\frac{\varepsilon h}{2q}) - u_\varepsilon(\varepsilon^\delta x', -h\varepsilon^{1+\delta})\Big| \ge c_2/3\Big\}\\ X_{\varepsilon,0}^0 &= \Big\{x' \in Q_\varepsilon\ :\ \big|u_\varepsilon(\varepsilon^\delta x', h\varepsilon^{1+\delta}) - u_\varepsilon(\varepsilon^\delta x', -h\varepsilon^{1+\delta})\big| \ge c_2/3\Big\}\end{aligned}$$

It is clear that the measure of at least one of these sets $X_{\varepsilon,0}^+$, $X_{\varepsilon,0}^-$ and $X_{\varepsilon,0}^0$ is greater than or equal to $\text{meas}_{N-1}(X_\varepsilon \cap Q_\varepsilon)/3$. Denote $c_{1,\varepsilon}^0 = \text{meas}_{N-1}(X_\varepsilon \cap Q_\varepsilon)/3$.

First we consider the case $N \geq 3$. If $\text{meas}_{N-1}(X^{\pm}_{\varepsilon,0}) \geq c^0_{1,\varepsilon}$, then by capacity arguments we obtain

$$c^0_{1,\varepsilon} c_2^2/9 \leq \int\limits_{Q_\varepsilon} \big(u_\varepsilon(x', \pm\varepsilon h/(2q)) - u_\varepsilon(\varepsilon^\delta x', \pm\varepsilon^{(1+\delta)})\big)^2 dx'$$
$$\leq C\varepsilon^{(1+\delta)(2-N)}\varepsilon^{N-1} \int\limits_{\Phi^{\pm}_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx = C\varepsilon^{1-(N-2)\delta} \int\limits_{\Phi^{\pm}_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx. \tag{45}$$

If $\text{meas}_{N-1}(X^0_{\varepsilon_k,0}) \geq c^0_{1,\varepsilon_k}$, then

$$c^0_{1,\varepsilon} c_2^2/9 \leq \int\limits_{Q_\varepsilon} \big(u_\varepsilon(\varepsilon^\delta x', \varepsilon^{(1+\delta)}) - u_\varepsilon(\varepsilon^\delta x', -\varepsilon^{(1+\delta)})\big)^2 dx'$$
$$\leq C\varepsilon^{(1+\delta)}\varepsilon^{(1-N)\delta} \int\limits_{\Phi^0_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx = C\varepsilon^{1-(N-2)\delta} \int\limits_{\Phi^0_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx.$$

In both cases

$$\int\limits_{\Phi_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx \geq C^{-1} c_2^2 \text{meas}_{N-1}(X_\varepsilon \cap Q_\varepsilon)\varepsilon^{\delta(N-2)-1}$$

Similarly,

$$\int\limits_{\Phi_\varepsilon + \varepsilon(j,0)} |\nabla u_\varepsilon(x)|^2 dx \geq C^{-1} c_2^2 \text{meas}_{N-1}(X_\varepsilon \cap \varepsilon(Q+j))\varepsilon^{\delta(N-2)-1}$$

for all $j \in \mathcal{K}_\varepsilon$. Summing up the last relations in j we obtain

$$\int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx \geq C^{-1} c_2^2 \text{meas}_{N-1}(X_\varepsilon)\varepsilon^{\delta(N-2)-1} \geq C^{-1} c_2^2 c_1 \varepsilon^{\delta(N-2)-1}.$$

If (44) holds then the integral on the left hand side tends to ∞ which contradicts (10). Therefore, $[u] = 0$ on Σ.

In the case $N = 2$, if $|X^{\pm}_{\varepsilon,0}| \geq c^0_{1,\varepsilon}$, then

$$c^0_{1,\varepsilon} c_2^2/9 \leq \int\limits_{Q_\varepsilon} \big(u_\varepsilon(x', \pm\varepsilon h/(2q)) - u_\varepsilon(\varepsilon^\delta x', \pm\varepsilon^{(1+\delta)})\big)^2 dx' \leq C\frac{\delta\varepsilon}{|\log\varepsilon|} \int\limits_{\Phi^{\pm}_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx. \tag{46}$$

If $|X^0_{\varepsilon_k,0}| \geq c^0_{1,\varepsilon_k}$, then

$$c^0_{1,\varepsilon} c_2^2/9 \leq \int\limits_{Q_\varepsilon} \big(u_\varepsilon(\varepsilon^\delta x', \varepsilon^{(1+\delta)}) - u_\varepsilon(\varepsilon^\delta x', -\varepsilon^{(1+\delta)})\big)^2 dx' \leq C\varepsilon \int\limits_{\Phi^0_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx.$$

In both cases

$$\int\limits_{\Phi_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx \geq C^{-1} c_2^2 |X_\varepsilon \cap Q_\varepsilon| (\delta\varepsilon)^{-1}.$$

The rest of the proof is the same as in the case $N \geq 3$.

Let us now derive the spectral boundary condition satisfied by u on Σ. Let $v \in C^\infty(\Omega) \cap H^1(\Omega, \Gamma_1)$. We have

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v \, dx = \varepsilon^{-(N-1)\delta} \widehat{\lambda}_\varepsilon \int\limits_{\gamma_\varepsilon} u_\varepsilon v \, ds. \tag{47}$$

As in the case $\delta = 0$, we have

$$\lim_{\varepsilon\to 0} \int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v \, dx = -\left\langle \left[\frac{\partial u}{\partial x_N}\right], v \right\rangle_\Sigma . \tag{48}$$

In order to pass to the limit on the right-hand side of (47) we make use of the following inequality: for any $w \in H^1(Y \times [-\frac{h}{2}, \frac{h}{2}])$

$$\int\limits_{Q_q} \big(w(x', h/2) - \langle w\rangle_\gamma\big)^2 \, dx' \le C \int\limits_{Y\times(-h/2,h/2)} |\nabla w|^2 \, dx;$$

here

$$\gamma = \partial Y \times [-\frac{h}{2}, \frac{h}{2}], \qquad \langle w\rangle_\gamma = |\gamma|_{N-1} \int\limits_\gamma w \, ds.$$

This inequality can be easily derived from the Poincaré and the trace inequalities. Its scaled version reads

$$\int\limits_{\varepsilon^{1+\delta}Q_q} \big(w(x', \varepsilon^{1+\delta}h/2) - \langle w\rangle_{\varepsilon^{1+\delta}\gamma}\big)^2 \, dx' \le C\varepsilon^{1+\delta} \int\limits_{\varepsilon^{1+\delta}(Y\times[-\frac{h}{2},\frac{h}{2}])} |\nabla w|^2 \, dx.$$

In the case $N \ge 3$, combining this estimate with the second inequality in (45), we obtain

$$\int\limits_{Q_\varepsilon} \big(u_\varepsilon(x', \varepsilon h/(2q)) - \langle u_\varepsilon\rangle_{\varepsilon^{1+\delta}\gamma}\big)^2 \, dx' \le C\varepsilon^{1-\delta(N-2)} \int\limits_{\mathcal{G}_\varepsilon} |\nabla u_\varepsilon|^2 \, dx$$

with $\mathcal{G}_\varepsilon = \varepsilon\big(Q \times [\varepsilon^\delta \frac{h}{2}, \frac{h}{2q}]\big) \cup \varepsilon^{1+\delta}\big(Y \times [-\frac{h}{2}, \frac{h}{2}]\big) \cup \varepsilon\big(Q \times [-\frac{h}{2q}, -\varepsilon^\delta \frac{h}{2}]\big)$. Similar inequalities hold in $\mathcal{G}_\varepsilon + \varepsilon(j, 0)$ for all $j \in \mathcal{K}_\varepsilon$. Summing up these inequalities in j yields

$$\int\limits_\Sigma \big(u_\varepsilon(x', \varepsilon h/(2q)) - \widehat{\langle u_\varepsilon\rangle}_{\varepsilon^{1+\delta}\gamma}\big)^2 \, dx' \le C\varepsilon^{1-\delta(N-2)} \int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx,$$

where $\widehat{\langle u_\varepsilon\rangle}_{\varepsilon^{1+\delta}\gamma}$ is a piece-wise constant function equal to $\langle u_\varepsilon\rangle_{(\varepsilon^{1+\delta}\gamma+\varepsilon(j,0))}$ on $\mathcal{G}_\varepsilon + \varepsilon(j, 0)$. Under our assumptions on δ the right-hand side in the last inequality tends to zero. Therefore,

$$\Big|\int\limits_\Sigma u_\varepsilon(x', \varepsilon h/(2q)) v(x', \varepsilon h/(2q)) \, dx' - \int\limits_\Sigma \widehat{\langle u_\varepsilon\rangle}_{\varepsilon^{1+\delta}\gamma} v(x', \varepsilon h/(2q)) \, dx'\Big| \longrightarrow 0.$$

For the first integral we have

$$\int\limits_\Sigma u_\varepsilon(x', \varepsilon h/(2q)) v(x', \varepsilon h/(2q)) \, dx' \longrightarrow \int\limits_\Sigma u(x', 0) v(x', 0) \, dx'.$$

For the second

$$\int\limits_{\Sigma} \widehat{\langle u_\varepsilon\rangle}_{\varepsilon^{1+\delta}\gamma} v(x', \varepsilon h/(2q))\, dx' = \varepsilon^{N-1} \sum_{j\in J_\varepsilon} \widehat{\langle u_\varepsilon\rangle}_{\varepsilon^{1+\delta}\gamma+\varepsilon(j,0)} v(\varepsilon j, 0) + o(1)$$

$$= \frac{\varepsilon^{\delta(1-N)}}{h|\partial Y|_{N-1}} \int\limits_{\varepsilon^{1+\delta}\gamma+\varepsilon(j,0)} u_\varepsilon(x) v(\varepsilon j, 0)\, ds + o(1) = \frac{\varepsilon^{\delta(1-N)}}{h|\partial Y|_{N-1}} \int\limits_{\gamma_\varepsilon} u_\varepsilon(x) v(x)\, ds + o(1);$$

here $o(1)$ tends to zero as $\varepsilon \to 0$. Finally, combining the above estimates, we obtain

$$\varepsilon^{\delta(1-N)} \int\limits_{\gamma_\varepsilon} u_\varepsilon(x) v(x)\, ds \longrightarrow h|\partial Y|_{N-1} \int\limits_{\Sigma} u(x', 0) v(x', 0)\, dx'$$

for any $v \in C^\infty(\Omega)$. Therefore,

$$\int\limits_{\Omega} \nabla u \nabla v\, dx = \widehat{\lambda} K \int\limits_{\Sigma} uv\, dx' \qquad \text{for any } v \in H^1(\Omega, \Gamma_1) \cap C^\infty(\Omega).$$

By the density arguments this limit relation also holds for any $v \in H^1(\Omega, \Gamma_1)$.

The proof in the case $N = 2$ is similar.

The fact that $u \neq 0$ relies on the following statement.

Lemma 3.8. *There exists a constant $C > 0$ such that for any $v_1 \in H^1(\Omega_\varepsilon)$ and $v_2 \in H^1(\Omega_\varepsilon)$ and any $\varkappa \geq h$ we have*

$$\Big| \varepsilon^{(1-N)\delta} \int\limits_{\gamma_\varepsilon} v_1 v_2\, ds - K \int\limits_{\Sigma} v_1(x', \varkappa\varepsilon) v_2(x', \varkappa\varepsilon)\, dx' \Big| \leq C \varepsilon^{\frac{1}{2}(1-(N-2)\delta)} \|v_1\|_{H^1(\Omega_\varepsilon)} \|v_2\|_{H^1(\Omega_\varepsilon)}.$$

Proof. It suffices to prove the statement of the lemma in the case $v_1 = v_2 = v$. Using capacity arguments, we first estimate

$$\begin{aligned}
&\Big| \int\limits_{Q_\varepsilon} v^2(x', \frac{\varepsilon h}{2q}) dx' - \varepsilon^{(1-N)\delta} q^{1-N} \int\limits_{\varepsilon^{1+\delta} Q_q} v^2\big(x', \frac{\varepsilon^{1+\delta} h}{2}\big)\, dx' \Big| \\
&\leq \int\limits_{Q_\varepsilon} \big(v^2(x', \frac{\varepsilon h}{2q}) - v^2(\varepsilon^\delta x', \frac{\varepsilon^{1+\delta} h}{2})\big)^2 dx' \\
&\leq \Big[\int\limits_{Q_\varepsilon} \Big(v(x', \frac{\varepsilon h}{2q}) - v(\varepsilon^\delta x', \frac{\varepsilon^{1+\delta} h}{2})\Big)^2 dx' \Big]^{\frac{1}{2}} \Big[\int\limits_{Q_\varepsilon} \Big(v(x', \frac{\varepsilon h}{2q}) + v(\varepsilon^\delta x', \frac{\varepsilon^{1+\delta} h}{2})\Big)^2 dx' \Big]^{\frac{1}{2}} \\
&\leq \Big(C\varepsilon^{1-(N-2)\delta} \int\limits_{\Phi_\varepsilon^+} |\nabla v|^2\, dx \Big)^{\frac{1}{2}} \Big[2\Big(\int\limits_{Q_\varepsilon} v^2(x', \frac{\varepsilon h}{2q})\, dx' \Big)^{\frac{1}{2}} + \Big(C\varepsilon^{1-(N-2)\delta} \int\limits_{\Phi_\varepsilon^+} |\nabla v|^2\, dx \Big)^{\frac{1}{2}} \Big] \\
&\leq C \varepsilon^{\frac{1}{2}(1-(N-2)\delta)} \Big[\int\limits_{\Phi_\varepsilon^+} |\nabla v|^2\, dx + \int\limits_{Q_\varepsilon} v^2(x', \frac{\varepsilon h}{2q})\, dx' \Big] \qquad (49)
\end{aligned}$$

Similar inequality holds in any set $\Phi_\varepsilon^+ + \varepsilon(j, 0)$, $j \in \mathcal{K}_\varepsilon$. Summing up these inequalities in $j \in \mathcal{K}_\varepsilon$ and letting $\mathcal{Q}_\varepsilon = \bigcup\limits_{j\in\mathcal{K}_\varepsilon} (\varepsilon^{1+\delta} Q_q + \varepsilon j)$ yields

$$\Big|\int\limits_{\Sigma} v^2(x',\frac{\varepsilon h}{2q})dx' - \varepsilon^{(1-N)\delta}q^{1-N}\int\limits_{\mathcal{Q}_\varepsilon} v^2\big(x',\frac{\varepsilon^{1+\delta}h}{2}\big)\,dx'\Big|$$
$$\le C\varepsilon^{\frac12(1-(N-2)\delta)}\Big[\int\limits_{\Omega}|\nabla v|^2\,dx + \int\limits_{\Sigma} v^2(x',\frac{\varepsilon h}{2q})\,dx'\Big] \tag{50}$$

Since $\|v\|_{L_2(\Sigma)} \le C\|v\|_{H^1(\Omega)}$, we deduce from (50) that

$$\varepsilon^{(1-N)\delta}\int\limits_{\mathcal{Q}_\varepsilon} v^2\big(x',\frac{\varepsilon^{1+\delta}h}{2}\big)\,dx' \le C\|v\|^2_{H^1(\Omega_\varepsilon)} \tag{51}$$

In the same way as in the proof of Lemma 3.3 one can prove that

$$\varepsilon^{(1-N)\delta}\Big|\int\limits_{\gamma_\varepsilon} v^2\,ds - \frac{K}{q^{N-1}}\int\limits_{\mathcal{Q}_\varepsilon} v^2(x',\varepsilon^{1+\delta}\frac{h}{2})\,dx'\Big|$$
$$\le C\varepsilon^{(1-(N-2)\delta)}\Big[\int\limits_{\Omega_\varepsilon}|\nabla v|^2\,dx + \varepsilon^{(1-N)\delta}\int\limits_{\mathcal{Q}_\varepsilon} v^2(x',\frac{\varepsilon^{1+\delta}h}{2})\,dx'\Big]$$
$$\le C\varepsilon^{(1-(N-2)\delta)}\|v\|^2_{H^1(\Omega_\varepsilon)}$$

Combining the last inequality with (49), we obtain

$$\Big|\varepsilon^{\delta(1-N)}\int\limits_{\gamma_\varepsilon} v^2\,ds - K\int\limits_{\Sigma} v^2(x',\frac{\varepsilon h}{2q})dx'\Big| \le C\varepsilon^{\frac12(1-(N-2)\delta)}\|v\|^2_{H^1(\Omega_\varepsilon)} \tag{52}$$

This gives the desired estimate for $\varkappa = h/(2q)$. For other values of $\varkappa \ge h$ it can be easily derived from (52) with the help of the trace theorem.

The case $N = 2$ can be considered in the same way. □

From Lemma 3.8 it follows that, under our normalization conditions, for sufficiently small ε the estimate holds $\|u_\varepsilon(\cdot, h\varepsilon)\|_{L^2(\Sigma)} \ge C$. This implies that $u \ne 0$.

The remaining part of the proof follows the line of the proof in the case $\delta = 0$. We should just use Lemma 3.8 instead of Lemma 3.3. The proof of Theorem 3.1 is completed.

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References

[1] H. Attouch, C. Picard, Comportement limite de problèmes de transmission unilatéraux à travers des grilles de forme quelconque, Rend. Semin. Mat. Univ. Politec. Torino 45 (1987) 71–85.
[2] G.A. Chechkin, R.R. Gadyl'shin, Yu.O. Koroleva, On the asymptotics of a simple eigenvalue to boundary value problem in domain perforated along the boundary, Differ. Equ. 47 (6) (2011) 822–831, Translated from Differ. Uravn. 47 (6) (2011) 819–828.
[3] A.G. Chechkina, Convergence of solutions and eigenelements of Steklov type boundary value problems with boundary conditions of rapidly varying type, Probl. Mat. Anal. 42 (2009) 129–143, English translation: Math. Sci. (N. Y.) 162 (3) (2009) 443–458.

[4] D. Cioranescu, A. Damlamian, G. Griso, D. Onofrei, The periodic unfolding method for perforated domains and Neumann sieve models, J. Math. Pures Appl. 89 (2008) 248–277.

[5] C. Conca, Étude d'un fluide traversant une paroi perforée I, II. Comportement limite près de la paroi, J. Math. Pures Appl. 66 (1987) 1–43, 45–69.

[6] A. Damlamian, La passoire de Neumann, Rend. Semin. Mat. Univ. Politec. Torino 43 (3) (1985) 427–450.

[7] T. Del Vecchio, The thick Neumann's sieve, Ann. Mat. Pura Appl. 147 (1987) 363–402.

[8] G.I. Eskin, Boundary Value Problems for Elliptic Pseudodifferential Equations, AMS, 1981.

[9] J.L. Lions, Some Methods in the Mathematical Analysis of Systems and Their Control, Gordon and Breach, Beijing, 1981.

[10] V.A. Marchenko, E.Ya. Khruslov, Boundary-Value Problems in Domains with Fine-Grained Boundaries, Naukova Dumka, Kiev, 1974 (in Russian).

[11] T.A. Mel'nyk, Asymptotic behavior of eigenvalues and eigenfunctions of the Steklov problem in a thick periodic junction, Nonlinear Oscill. 4 (1) (2001) 91–105, Translated from Nelin. Koliv. 4 (1) (2001) 91–105.

[12] F. Murat, The Neumann sieve, in: A. Marino (Ed.), Non Linear Variational Problems, in: Research Notes in Mathematics, vol. 127, Pitman, London, 1985, pp. 24–32.

[13] S.A. Nazarov, J. Taskinen, On the spectrum of the Steklov problem in a domain with a peak, Vestnik St. Petersburg Univ. Math. 41 (1) (2008) 45–52.

[14] O.A. Oleinik, A.S. Shamaev, G.A. Yosifian, Mathematical Problems in Elastity and Homogenization, Studies in Mathematics and its Applications, vol. 26, North-Holland Publishing Co., Amsterdam, 1992.

[15] D. Onofrei, The unfolding operator near a hyperplane and its applications to the Neumann Sieve model, Adv. Math. Sci. Appl. 16 (1) (2006) 239–258.

[16] D. Onofrei, B. Vernescu, Asymptotics of a spectral problem associated with the Neumann Sieve, Anal. Appl. 3 (1) (2005) 69–87.

[17] M.E. Pérez, On periodic Steklov type eigenvalue problems on half-bands and the spectral homogenization problem, Discrete Contin. Dyn. Syst. Ser. B 7 (4) (2007) 859–883.

[18] E. Sánchez-Palencia, Boundary-value problems in domains containing perforated walls, in: Nonlinear Partial Differential Equations and Their Applications, Coll. de France Semin., vol. 3, in: Res. Notes Math., vol. 70, 1982, pp. 309–325.

[19] E. Sánchez-Palencia, Un problème d'écoulement lent d'un fluide visqueux incompressible au travers d'une paroi finement perforée, in: D. Bergman, et al. (Eds.), Les méthodes de l'homogénéisation: théorie et applications en physique, in: Collection de la Direction des Études et Recherches d'Électricité de France, vol. 57, Eyrolles, Paris, 1985, pp. 371–400.

[20] M. Vanninathan, Homogenization of eigenvalue problems in perforated domains, Proc. Indian Acad. Sci. Math. Sci. 90 (3) (1981) 239–271.