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Homogenization of spectral problem for locally periodic elliptic operators with sign-changing density function

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ABSTRACT

The paper deals with homogenization of a spectral problem for a second order self-adjoint elliptic operator stated in a thin cylinder with homogeneous Neumann boundary condition on the lateral boundary and Dirichlet condition on the bases of the cylinder. We assume that the operator coefficients and the spectral density function are locally periodic in the axial direction of the cylinder, and that the spectral density function changes sign. We show that the behavior of the spectrum depends essentially on whether the average of the density function is zero or not. In both cases we construct the effective 1-dimensional spectral problem and prove the convergence of spectra.

1. Introduction

The paper is aimed at homogenization of a spectral problem for a second order divergence form elliptic operator defined in a thin cylinder of finite length with homogeneous Neumann boundary condition on the lateral boundary of the cylinder and Dirichlet conditions on the cylinder bases. We make a crucial assumption that the spectral weight function changes sign and assume that both operator coefficients and the weight function are locally periodic in the axial direction of the cylinder.

Under the said conditions we show that the asymptotic behavior of the spectrum depends essentially on whether the average of the weight function over the period is equal to zero or not. In both cases we construct an effective model and prove the convergence result; the estimates for the rate of convergence are also obtained.

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The studied spectral problem might have interesting and important applications in the modern theory of metamaterials, that is artificial composite materials engineered to produce a desired electromagnetic behavior with significantly enhanced performance over "natural" structures. For example, when the world is observed through conventional lenses, the sharpness of the image is determined by and limited to the wavelength of light. Metamaterials with negative refractive index aimed at creation of "perfect" lenses, that is lenses with capabilities beyond conventional (positive index) ones.

First initiated by L.S. Pontrgyagin in [15], the qualitative theory of spectral problems in spaces with indefinite metric was further developed by M.G. Krein [7], I.S. Iokhvidov [4] and other mathematicians. The detailed presentation of this theory can be found, for example, in books [1,16].

The homogenization of spectral problems in the case of positive weight functions was considered in [5,6,17], then in [13] for elasticity system and then in many other works. However, the presence of sign-changing weight function makes the problem nonstandard and leads to new interesting phenomena. For operators with pure periodic coefficients defined in a fixed (not asymptotically thin) domains similar problems have been studied in the recent works [11,12]. In contrast with problems investigated in these works, for the model considered in the present paper the limit spectral problem is one-dimensional, so that dimension reduction arguments are to be used. We combine the asymptotic expansion technique with the singular measure approach developed in [20] and [2].

For the density function having positive average the effective spectral problem happens to be a Sturm–Liouville problem. In this case the convergence of the positive part of the spectrum is justified by means of convergence in variable spaces with singular measures.

In the case of zero average weight function the limit spectral problem is that for a quadratic operator pencil. To study this operator pencil we apply the results from [8] combined with usual arguments used when studying Sturm–Liouville problems. It should be noted that in contrast with [12], the presence of slow variable in the coefficients makes the limit operator pencil nontrivial, so that it cannot be reduced to the standard Sturm–Liouville problem.

The fact that the considered operator is defined in a thin cylinder allows us to build boundary layer correctors in the neighborhood of the cylinder bases and, as a result, improve essentially the asymptotics. As a matter of fact, if the coefficients are sufficiently regular, then arbitrary many terms in the asymptotic expansion can be constructed. This allows one to approximate the eigenpairs of the studied problem up to an arbitrary large power of the small parameter characterizing the microstructure period. The existence of exponentially decaying boundary layer correctors is assured by the results obtained in [14].

In the last section we address the case when the local average of the weight function changes sign. In this case the convergence of both, positive and negative parts of the spectrum is justified.

The asymptotics of negative part of the spectrum in the case of positive average of the density function will be treated in a separate publication.

The paper is organized as follows. Section 2 contains the statement of the problem together with some preliminary results concerning the structure of the spectrum of the original operator. In Section 3.1 we construct the formal asymptotic expansion in the case when the average of the weight function over the period is positive. The justification of the homogenization procedure is given in Section 3.2. Section 4 is devoted to the case when the average of the weight function is equal to zero. In Section 5 the case when the average of the weight function changes sign is considered.

2. Problem setup and main results

Let Q be a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^{d-1}$ with a boundary ∂Q. The points in $\mathbb{R}^d$ are denoted $x=(x_1,x')$, where $x'=x_2,\dots,x_d$. Denote by G_ε a thin rod $[-1,1]\times\varepsilon Q$ with the lateral boundary $\Sigma_\varepsilon=(-1,1)\times\partial(\varepsilon Q)$ and the bases $S_{\pm 1}=\{\pm 1\}\times\varepsilon Q$. In the cylinder G_ε we consider the following spectral problem:

$$\begin{cases} \mathcal{A}^\varepsilon u^\varepsilon(x)\equiv -\operatorname{div}\big(a^\varepsilon(x)\nabla u^\varepsilon(x)\big)=\lambda^\varepsilon\rho^\varepsilon(x)u^\varepsilon(x), & x\in G_\varepsilon,\\ \mathcal{B}^\varepsilon u^\varepsilon(x)\equiv \big(a^\varepsilon\nabla u^\varepsilon,n\big)=0, & x\in\Sigma_\varepsilon,\\ u^\varepsilon\big(-1,x'\big)=u^\varepsilon\big(1,x'\big)=0, & x\in\partial(\varepsilon Q), \end{cases} \tag{2.1}$$

with

$$a^\varepsilon(x) = a\left(x_1, \frac{x}{\varepsilon}\right), \qquad \rho^\varepsilon(x) = \rho\left(x_1, \frac{x}{\varepsilon}\right),$$

where $a(x_1, y)$ is a symmetric $d \times d$ matrix and $\rho(x_1, y)$ is a scalar function; $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^d$. We assume the following conditions to hold:

(H0) $a_{ij}(x_1, y), \rho(x_1, y) \in C^{1,\alpha}([-1,1]; C^\alpha(\overline{Y}))$ for some $\alpha > 0$. Here $Y = \mathcal{S}_1 \times Q$ denotes the periodicity cell, $\mathcal{S}_1$ is a unit circle;
(H1) Functions $a_{ij}(x_1, y)$ and $\rho(x_1, y)$, are 1-periodic in y_1;
(H2) The matrix $a(x_1, y)$ satisfies the uniform ellipticity condition, that is for any $x_1 \in [-1,1]$ and $y \in Y$

$$\sum_{i,j=1}^{d} a_{ij}(x_1, y)\xi_i\xi_j \geqslant \Lambda|\xi|^2, \quad \xi \in \mathbb{R}^d,\ \Lambda > 0;$$

(H3) The weight function $\rho(x_1, y)$ changes sign, that is for any $x_1 \in [-1,1]$ the sets $\{y \in Y\colon \rho(x_1, y) < 0\}$ and $\{y \in Y\colon \rho(x_1, y) > 0\}$ have positive Lebesgue measures, i.e.

$$\left|\left\{y \in Y\colon \rho(x_1, y) \lessgtr 0\right\}\right| > 0.$$

Also, for presentation simplicity we assume that

$$\varepsilon = 1/L, \quad L = 1, 2, \ldots. \tag{2.2}$$

The general case can be treated in the same way, see Remark 3.2 in Section 3 for further discussion.

Remark 2.1. It follows from condition (H3) that, for sufficiently small ε, the sets $\{x \in G_\varepsilon\colon \rho(x_1, \frac{x}{\varepsilon}) < 0\}$ and $\{x \in G_\varepsilon\colon \rho(x_1, \frac{x}{\varepsilon}) > 0\}$ have positive Lebesgue measures.

The weak formulation of problem (2.1) is as follows: find $\lambda^\varepsilon \in \mathbb{C}$ (eigenvalues) and $u^\varepsilon \in H^1(G_\varepsilon) \setminus \{0\}$ (eigenfunctions) such that $u^\varepsilon(\pm 1, x') = 0$ and

$$\left(a^\varepsilon \nabla u^\varepsilon, \nabla v\right)_{L^2(G_\varepsilon)} = \lambda^\varepsilon \left(\rho^\varepsilon u^\varepsilon, v\right)_{L^2(G_\varepsilon)}, \tag{2.3}$$

where $v \in C^\infty(G_\varepsilon)$ such that $v(\pm 1, x') = 0$, $(\cdot,\cdot)_{L^2(G_\varepsilon)}$ denotes the usual scalar product in $L^2(G_\varepsilon)$.

First we study the qualitative properties of problem (2.1) for a fixed value of ε. For this aim, following the ideas in [12], we are going to reduce the problem under consideration to an equivalent spectral problem for a compact self-adjoint operator. To this end let us introduce the space

$$\mathcal{H}^\varepsilon = \left\{u \in H^1(G_\varepsilon)\colon u|_{S_{\pm 1}} = 0\right\}$$

equipped with the norm

$$\|u\|^2_{\mathcal{H}^\varepsilon} = (u, u)_{\mathcal{H}^\varepsilon} = \left(a^\varepsilon \nabla u, \nabla u\right)_{L^2(G_\varepsilon)}.$$

Thanks to the Friedrichs inequality

$$\|v\|_{L^2(G^\varepsilon)} \leqslant 2\|\nabla v\|_{L^2(G_\varepsilon)}, \quad v \in \mathcal{H}^\varepsilon,$$

the quadratic form $(a^\varepsilon \nabla u, \nabla u)_{L^2(G_\varepsilon)}$ defines a norm in $\mathcal{H}^\varepsilon$, which is equivalent to the standard $H^1(G_\varepsilon)$ norm.

In view of condition (H0), the bilinear form $(\rho^\varepsilon u, v)_{L^2(G_\varepsilon)}$ defines on $\mathcal{H}^\varepsilon$ a bounded linear operator $\mathcal{K}^\varepsilon : \mathcal{H}^\varepsilon \to \mathcal{H}^\varepsilon$ by the following rule:

$$\left(\mathcal{K}^\varepsilon u, v\right)_{\mathcal{H}^\varepsilon} = \left(\rho^\varepsilon u, v\right)_{L^2(G_\varepsilon)}.$$

By definition, the operator $\mathcal{K}^\varepsilon$ is symmetric and, since it is bounded, it is self-adjoint. Notice that $\mathcal{K}^\varepsilon u$ can be also introduced as a solution of the boundary value problem

$$\begin{cases} \mathcal{A}^\varepsilon\left(\mathcal{K}^\varepsilon u(x)\right) = \rho^\varepsilon(x) u(x), & x \in G_\varepsilon, \\ B^\varepsilon\left(\mathcal{K}^\varepsilon u(x)\right) = 0, & x \in \Sigma_\varepsilon, \\ \mathcal{K}^\varepsilon u(x) = 0, & x \in S_{\pm 1}. \end{cases} \tag{2.4}$$

Considering this representation and the compactness of the imbedding $H^1(G_\varepsilon)$ in $L^2(G_\varepsilon)$, one can see that $\mathcal{K}^\varepsilon$ is a compact operator, both in $\mathcal{H}^\varepsilon$ and in $L^2(G_\varepsilon)$.

Remark 2.2. Since for any $u \in L^2(G_\varepsilon)$ the function $\mathcal{K}^\varepsilon u$ belongs to $\mathcal{H}^\varepsilon$, then the spectrum of $\mathcal{K}^\varepsilon$ in $L^2(G_\varepsilon)$ coincides with that in $\mathcal{H}^\varepsilon$. We prefer to study the spectrum of $\mathcal{K}^\varepsilon$ in the space $\mathcal{H}^\varepsilon$ because in this space $\mathcal{K}^\varepsilon$ is self-adjoint.

In terms of the operator $\mathcal{K}^\varepsilon$ problem (2.1) takes the form

$$\mathcal{K}^\varepsilon u^\varepsilon = \mu^\varepsilon u^\varepsilon, \quad \mu^\varepsilon = 1/\lambda^\varepsilon. \tag{2.5}$$

Exactly in the same way as in [12] (see Lemma 2.1) one can show that the discrete spectrum of the operator $\mathcal{K}^\varepsilon$ consists of two infinite sequences. The following statement holds.

Lemma 2.1. *Suppose that conditions* (H0)–(H3) *are fulfilled. Then the spectrum $\sigma(\mathcal{K}^\varepsilon)$ of the operator $\mathcal{K}^\varepsilon$ belongs to the interval $[-k^\varepsilon, k^\varepsilon]$, $k^\varepsilon = \|\mathcal{K}^\varepsilon\|$; the point $\mu = 0$ is the only element of the essential spectrum $\sigma_e(\mathcal{K}^\varepsilon)$. Moreover, the discrete spectrum of the operator $\mathcal{K}^\varepsilon$ consists of two infinite sequences*

$$\mu_1^{\varepsilon,+} \geqslant \mu_2^{\varepsilon,+} \geqslant \cdots \geqslant \mu_j^{\varepsilon,+} \geqslant \cdots \to +0,$$
$$\mu_1^{\varepsilon,-} \leqslant \mu_2^{\varepsilon,-} \leqslant \cdots \leqslant \mu_j^{\varepsilon,-} \leqslant \cdots \to -0.$$

Taking into account (2.5), we conclude that problem (2.1) has a discrete spectrum which consists also of two infinite sequences. More precisely, we have proved the following result.

Theorem 2.1. *Under the assumptions* (H0)–(H3) *spectral problem* (2.1) *has a discrete spectrum which consists of two sequences*

$$0 < \lambda_1^{\varepsilon,+} \leqslant \lambda_2^{\varepsilon,+} \leqslant \cdots \leqslant \lambda_j^{\varepsilon,+} \leqslant \cdots \to +\infty,$$
$$0 > \lambda_1^{\varepsilon,-} \geqslant \lambda_2^{\varepsilon,-} \geqslant \cdots \geqslant \lambda_j^{\varepsilon,-} \geqslant \cdots \to -\infty.$$

Under proper normalization, the corresponding eigenfunctions $u_j^{\varepsilon,\pm}$ satisfy the orthogonality condition

$$\left(u_i^{\varepsilon,\pm}, u_j^{\varepsilon,\pm}\right)_{\mathcal{H}^\varepsilon} = \varepsilon^{d-1} |Q| \delta_{ij}, \tag{2.6}$$

where $|Q|$ is the Lebesgue measure of Q and δ_{ij} is the Kronecker delta.

The goal of the present work is to study the asymptotic behavior of the spectrum of problem (2.1), as $\varepsilon \to 0$. As was already pointed out, the asymptotic behavior of the spectrum depends crucially on whether the local average of $\rho(x_1,\cdot)$ is zero on $[-1,1]$ or not. To avoid the technicalities for the moment, we formulate here the main result of the paper in a slightly reduced form, without specifying the rate of convergence. More detailed formulation can be found in Sections 3–5.

Theorem 2.2. *Let conditions* (H0)–(H3) *be fulfilled. If* $\lambda_j^{\varepsilon,+}$ ($\lambda_j^{\varepsilon,-}$) *stands for the* j*th positive* (*negative*) *eigenvalue of problem* (2.1), *and* $u_j^{\varepsilon,+}$ ($u_j^{\varepsilon,-}$) *for the corresponding eigenfunction, then the following convergence results hold*:

1. *If* $\langle\rho(x_1,\cdot)\rangle > 0$ *for all* $x_1 \in [-1,1]$, *then, for any* j,

$$\lambda_j^{\varepsilon,+} \to \lambda_j^{0,+}, \quad \varepsilon \to 0,$$
$$\varepsilon^{\frac{d-1}{2}} \big\| u_j^{\varepsilon,+} - u_j^{0,+} \big\|_{L^2(G_\varepsilon)} \to 0, \quad \varepsilon \to 0,$$

where $(\lambda_j^{0,+}, u_j^{0,+})$ *is the* j*th eigenpair of the effective Sturm–Liouville problem*

$$\begin{cases} -\dfrac{d}{dx_1}\left(a^{\mathrm{eff}}(x_1)\dfrac{du^0(x_1)}{dx_1}\right) = \lambda^0\big\langle\rho(x_1,\cdot)\big\rangle u^0(x_1), \quad x_1 \in (-1,1), \\ u^0(\pm 1) = 0, \end{cases} \tag{2.7}$$

with a strictly positive continuous function $a^{\mathrm{eff}}(x_1)$ (*see* (3.3) *for detailed definition*).

2. *If* $\langle\rho(x_1,\cdot)\rangle = 0$ *for all* $x_1 \in [-1,1]$, *then, for any* j,

$$\varepsilon\lambda_j^{\varepsilon,\pm} - \nu_j^{0,\pm} \to 0, \quad \varepsilon \to 0,$$
$$\varepsilon^{\frac{d-1}{2}} \big\| u_j^{\varepsilon,\pm} - v_j^{0,\pm} \big\|_{L^2(G_\varepsilon)} \to 0, \quad \varepsilon \to 0,$$

where $(\nu_j^{0,\pm}, v_j^{0,\pm})$ *are the* j*th eigenpairs of the following quadratic operator pencil*:

$$\begin{cases} -\dfrac{d}{dx_1}\left(a^{\mathrm{eff}}(x_1)\dfrac{dv^0(x_1)}{dx_1}\right) + \nu^0\mathbf{B}(x_1)v^0(x_1) \\ \quad - \left(\nu^0\right)^2\mathbf{C}(x_1)v^0(x_1) = 0, \quad x_1 \in (-1,1), \\ v^0(-1) = v^0(1) = 0, \end{cases} \tag{2.8}$$

with the functions $\mathbf{B}(x_1)$, $\mathbf{C}(x_1) > 0$ *defined by* (4.8) *and* (4.7), *respectively. The spectrum of this operator pencil is discrete and real, it consists of two infinite series*

$$0 < \nu_1^{0,+} \leqslant \nu_2^{0,+} \leqslant \cdots \leqslant \nu_j^{0,+} \leqslant \cdots \to +\infty,$$
$$0 > \nu_1^{0,-} \geqslant \nu_2^{0,-} \geqslant \cdots \geqslant \nu_j^{0,-} \geqslant \cdots \to -\infty.$$

Moreover, all the eigenvalues $\nu_j^{0,\pm}$ *are simple.*

3. *If $\langle\rho(x_1,\cdot)\rangle$ changes sign, then, for any j,*

$$\lambda_j^{\varepsilon,\pm} \to \lambda_j^{0,\pm}, \quad \varepsilon \to 0,$$
$$\varepsilon^{\frac{d-1}{2}} \big\| u_j^{\varepsilon,\pm} - u_j^{0,\pm} \big\|_{L^2(G_\varepsilon)} \to 0, \quad \varepsilon \to 0,$$

where $(\lambda_j^{0,\pm}, u_j^{0,\pm})$ are the jth eigenpairs of the effective spectral problem

$$\begin{cases} -\dfrac{d}{dx_1}\left(a^{\mathrm{eff}}(x_1)\dfrac{du^0(x_1)}{dx_1}\right) = \lambda^0 \big\langle \rho(x_1,\cdot)\big\rangle u^0(x_1), \quad x_1 \in (-1,1), \\ u^0(\pm 1) = 0, \end{cases} \tag{2.9}$$

with the function $a^{\mathrm{eff}}(x_1) > 0$ defined by (3.3)*. The spectrum of the effective problem is discrete and consists of two infinite series*

$$0 < \lambda_1^{0,+} \leqslant \lambda_2^{0,+} \leqslant \cdots \leqslant \lambda_j^{0,+} \leqslant \cdots \to +\infty,$$
$$0 > \lambda_1^{0,-} \geqslant \lambda_2^{0,-} \geqslant \cdots \geqslant \lambda_j^{0,-} \geqslant \cdots \to -\infty.$$

All the eigenvalues $\lambda_j^{0,\pm}$ are simple.

Notice that in the case $\langle\rho(x_1,\cdot)\rangle > 0$ the eigenvalues of the effective problem form a monotone sequence $\lambda_j^{0,+} \to +\infty$, as $j \to +\infty$, while in the cases $\langle\rho(x_1,\cdot)\rangle = 0$ and when $\langle\rho(x_1,\cdot)\rangle$ changes sign the spectra of the effective spectral problems (2.8) and (2.9) consist of two infinite sequences, tending to $+\infty$ and $-\infty$ (see Theorems 3.1, 4.1 and Section 5). Thus, one cannot characterize the asymptotic behavior of the negative part of the spectrum in the case $\langle\rho(x_1,\cdot)\rangle > 0$ in terms of the effective problem (2.7). The negative part of the spectrum will be considered elsewhere.

Theorem 2.2 follows from stronger results given in Sections 3–5 (see Theorems 3.2, 4.3, 5.1). In all cases we construct interior correctors, boundary layer correctors in the vicinity of the cylinder bases, and obtain estimates for the rate of convergence.

3. The case $\langle\rho(x_1,\cdot)\rangle > 0$

3.1. Formal asymptotic expansion

In what follows we denote $\nabla_y = \{\partial_{y_1}, \ldots, \partial_{y_d}\}^T$,

$$\big\langle\rho(x_1,\cdot)\big\rangle = \int_Y \rho(x_1,y)\,dy,$$
$$\mathcal{A}_y u \equiv -\mathrm{div}_y\big(a(x_1,y)\nabla_y u\big), \qquad \mathcal{B}_y u \equiv \big(a(x_1,y)\nabla_y u, n\big).$$

We are looking for a solution $(\lambda^\varepsilon, u^\varepsilon)$ of problem (2.1) in the form

$$u^\varepsilon(x) = u^0(x_1) + \varepsilon u^1(x_1,y) + \varepsilon^2 u^2(x_1,y) + \varepsilon^3 u^3(x_1,y) + \cdots,$$
$$\lambda^\varepsilon = \lambda^0 + \varepsilon\lambda^1 + \cdots, \qquad y = \frac{x}{\varepsilon}, \tag{3.1}$$

where unknown functions $u^k(x_1, y)$ are 1-periodic in y_1. Let us substitute ansatz (3.1) into (2.1) and collect power-like with respect to ε terms. Equating the coefficient in front of ε^{-1} to 0, we obtain an equation for $u^1(x_1,\cdot)$, $x_1 \in (-1,1)$:

$$\begin{cases} \mathcal{A}_y u^1(x_1,y) = \operatorname{div}_y a_{\cdot 1}(x_1,y)\dfrac{du^0}{dx_1}, & y \in Y, \\ \mathcal{B}_y u^1(x_1,y) = -\big(a_{\cdot 1}(x_1,y),n\big)\dfrac{du^0}{dx_1}, & y \in \partial Y, \\ u^1(x_1,\cdot) - y_1\text{-periodic}, \end{cases}$$

where $a_{\cdot k}$ is a kth column of the matrix $a(y)$. Note that $\partial Y = \mathcal{S}_1 \times \partial Q$. Particular form of the right-hand side in the last equation suggests the representation for u^1,

$$u^1(x_1,y) = N^{1,1}(x_1,y)\frac{du^0(x_1)}{dx_1} + v^1(x_1),$$

with $N^{1,1}$ being, for any $x_1 \in (-1,1)$, a solution of the problem

$$\begin{cases} \mathcal{A}_y N^{1,1}(x_1,y) = \operatorname{div}_y a_{\cdot 1}(x_1,y), & y \in Y, \\ \mathcal{B}_y N^{1,1}(x_1,y) = -\big(a_{\cdot 1}(x_1,y),n\big), & y \in \partial Y, \\ N^{1,1}(x_1,\cdot) - y_1\text{-periodic}. \end{cases} \tag{3.2}$$

Under assumption (H0), $N^{1,1}(x_1,y) \in C^{1,\alpha}([-1,1]; C^{1,\alpha}(\overline{Y}))$.

Similarly, collecting the terms of order ε^0 we obtain the problem for u^2:

$$\begin{cases} \mathcal{A}_y u^2(x_1,y) = \dfrac{\partial}{\partial x_1}\left(a_{1\cdot}(x_1,y)\nabla_y N^{1,1}(x_1,y)\dfrac{du^0(x_1)}{dx_1}\right) + \dfrac{\partial}{\partial x_1}\left(a_{11}(x_1,y)\dfrac{du^0(x_1)}{dx_1}\right) \\ \qquad + \dfrac{\partial}{\partial x_1}\left(a_{11}(x_1,y)\dfrac{du^0(x_1)}{dx_1}\right) + \operatorname{div}_y\left(a_{\cdot 1}(x_1,y)\dfrac{\partial}{\partial x_1}\big(N^{1,1}(x_1,y)u^0(x_1)\big)\right) \\ \qquad + \operatorname{div}_y a_{\cdot 1}(x_1,y)\dfrac{dv^1(x_1)}{dx_1} + \lambda^0\rho(x_1,y)u^0(x_1), \\ \quad x_1 \in (-1,1),\ y \in Y, \\ \mathcal{B}_y u^2(x_1,y) = -\big(a_{\cdot 1}(x_1,y),n\big)\dfrac{\partial}{\partial x_1}\big(N^{1,1}(x_1,y)u^0(x_1)\big) - \big(a_{\cdot 1}(x_1,y),n\big)\dfrac{dv^1}{dx_1}, \\ \quad x_1 \in (-1,1),\ y \in \partial Y, \\ u^2(x_1,\cdot) - y_1\text{-periodic}. \end{cases}$$

The compatibility condition for the last problem reads

$$\frac{d}{dx_1}\int_Y \big(a_{11}(x_1,y) + a_{1\cdot}(x_1,y)\nabla_y N^{1,1}(x_1,y)\big)\,dy\frac{du^0(x_1)}{dx_1} + \lambda^0\int_Y \rho(x_1,y)\,dy u^0(x_1) = 0,$$
$$x_1 \in (-1,1).$$

Denoting

$$a^{\text{eff}}(x_1) = \int_Y a_{1j}(x_1,y)\big(\delta_{1j} + \partial_{y_j}N^{1,1}(x_1,y)\big)\,dy, \tag{3.3}$$

we derive the following problem for u^0:

$$\begin{cases} \mathcal{A}^0 u^0(x_1) \equiv -\dfrac{d}{dx_1}\left(a^{\mathrm{eff}}(x_1)\dfrac{du^0(x_1)}{dx_1}\right) = \lambda^0 \langle \rho(x_1,\cdot)\rangle u^0(x_1), \quad x_1 \in (-1,1), \\ u^0(\pm 1) = 0. \end{cases} \tag{3.4}$$

Lemma 3.1. *The effective coefficient $a^{\mathrm{eff}}(x_1) \in C^{1,\alpha}[-1,1]$ is positive for all $x_1 \in [-1,1]$.*

Proof. Obviously, $a^{\mathrm{eff}}(x_1)$ is an element $\{A^{\mathrm{eff}}(x_1)\}_{11}$ of the matrix $A^{\mathrm{eff}}(x_1)$ given by

$$A^{\mathrm{eff}}_{ij}(x_1) = \int\limits_Y \bigl(a_{ij}(x_1,y) + a_{ik}\partial_{y_k} N_k^{1,1}(x_1,y)\bigr)\,dy,$$

where functions $N_k^{1,1}$ solve the problems

$$\begin{cases} \mathcal{A}_y N_k^{1,1}(x_1,y) = \operatorname{div}_y a_{\cdot k}(x_1,y), & k = 2,\dots,d,\ y \in Y, \\ \mathcal{B}_y N_k^{1,1}(x_1,y) = -\bigl(a_{\cdot k}(x_1,y), n\bigr), & y \in \partial Y, \\ N_k^{1,1} - y_1\text{-periodic}. \end{cases}$$

Let us show that the matrix $A^{\mathrm{eff}}(x_1)$ is positive definite. Notice that

$$0 = \int\limits_Y \partial_{y_m}\bigl(a_{jm} N_i^{1,1}\bigr)\,dy - \int\limits_{\partial Y} a_{jm} N_i^{1,1} n_m\,d\sigma.$$

Reorganizing the last expression yields

$$\begin{aligned} 0 &= \int\limits_Y \partial_{y_m}\bigl(a_{jm} N_i^{1,1}\bigr)\,dy - \int\limits_{\partial Y} a_{jm} N_i^{1,1} n_m\,d\sigma \\ &= \int\limits_Y \bigl(a_{jm}\partial_{y_m} N_i^{1,1} + \partial_{y_m} a_{mj} N_i^{1,1}\bigr)\,dy - \int\limits_{\partial Y} a_{jm} N_i^{1,1} n_m\,d\sigma \\ &= \int\limits_Y \bigl(a_{jm}\partial_{y_m} N_i^{1,1} - \partial_{y_m}\bigl(a_{mk}\partial_{y_k} N_j^{1,1}\bigr) N_i^{1,1}\bigr)\,dy - \int\limits_{\partial Y} a_{jm} N_i^{1,1} n_m\,d\sigma \\ &= \int\limits_Y \bigl(a_{jm}\partial_{y_m} N_i^{1,1} + a_{mk}\partial_{y_k} N_j^{1,1}\partial_{y_m} N_i^{1,1}\bigr)\,dy. \end{aligned}$$

Consequently,

$$\begin{aligned} A^{\mathrm{eff}}(x_1) &= \int\limits_Y \bigl(a_{ij}(x_1,y) + a_{ik}\partial_{y_k} N_k^{1,1}(x_1,y)\bigr)\,dy + \int\limits_Y \bigl(a_{jm}\partial_{y_m} N_i^{1,1} + a_{mk}\partial_{y_k} N_j^{1,1}\partial_{y_m} N_i^{1,1}\bigr)\,dy \\ &= \int\limits_Y \bigl(\delta_{im} + \partial_{y_m} N_i^{1,1}\bigr) a_{mk} \bigl(\delta_{kj} + \partial_{y_k} N_j^{1,1}\bigr)\,dy, \end{aligned}$$

thus, the matrix A^{eff} is nonnegative. Let us show that $a^{\mathrm{eff}} > 0$. For an arbitrary nonnegative matrix C we state that if $C_{11} = 0$, then $C_{1k} = 0$, $k = 2, \dots, d$, and, consequently, $Ce_1 = 0$. Assuming that $(\delta_{1j} + \partial_{y_1} N_j^{1,1}) = 0$ we arrive at contradiction with the periodicity of $N_1^{1,1}$ in y_1. Thus, $a^{\mathrm{eff}} > 0$. $\square$

For the reader's convenience we formulate here the classical result on Sturm–Liouville spectral problem (see, for instance, [10]).

Theorem 3.1. *The eigenvalues of the Sturm–Liouville problem* (3.4) *are real and form a monotone sequence*

$$0 < \lambda_1^{0,+} < \lambda_2^{0,+} < \dots < \lambda_j^{0,+} \dots \to +\infty.$$

Moreover, all the eigenvalues are simple.

Remark 3.1. The corresponding eigenfunctions $u_i^{0,+} \in C^{2,\alpha}[-1,1]$ of problem (3.4) can be normalized by

$$\int_{-1}^{1} a^{\mathrm{eff}}(x_1) \frac{du_i^{0,+}}{dx_1} \frac{du_j^{0,+}}{dx_1} \, dx_1 = \delta_{ij}. \tag{3.5}$$

Our next goal is to derive the equation for the unknown function $v^1(x_1)$. To this end we analyze the right-hand side of the equation for $u^2(x_1, y)$. The structure of the right-hand side suggests the following representation:

$$\begin{aligned} u^2(x_1, y) = {} & N^{2,2}(x_1, y) \frac{d^2 u^0(x_1)}{dx_1^2} + N^{2,1}(x_1, y) \frac{du^0(x_1)}{dx_1} + N^{2,0}(x_1, y) u^0(x_1) \\ & + N^{1,1}(x_1, y) \frac{dv^1(x_1)}{dx_1} + v^2(x_1), \end{aligned} \tag{3.6}$$

where $N^{2,2}$, $N^{2,1}$ and $N^{2,0}$ are y_1-periodic solutions of the problems

$$\begin{cases} \mathcal{A}_y N^{2,2}(x_1, y) = \operatorname{div}_y\big(a(x_1, y) N^{1,1}(x_1, y)\big) \\ \qquad\qquad + a_{1j}(x_1, y)\big(\delta_{1j} + \partial_{y_j} N^{1,1}(x_1, y)\big) - a^{\mathrm{eff}}(x_1), & y \in Y, \\ \mathcal{B}_y N^{2,2}(x_1, y) = -\big(a_{\cdot 1}(x_1, y), n\big) N^{1,1}(x_1, y), & y \in \partial Y, \end{cases} \tag{3.7}$$

$$\begin{cases} \mathcal{A}_y N^{2,1}(x_1, y) = \operatorname{div}_y\left(a_{\cdot 1}(x_1, y) \dfrac{\partial}{\partial x_1} N^{1,1}(x_1, y)\right) \\ \qquad\qquad + \dfrac{\partial}{\partial x_1}\big[a_{1j}(x_1, y)\big(\delta_{1j} + \partial_{y_j} N^{1,1}(x_1, y)\big)\big] - \dfrac{da^{\mathrm{eff}}(x_1)}{dx_1}, & y \in Y, \\ \mathcal{B}_y N^{2,1}(x_1, y) = -\big(a_{\cdot 1}(x_1, y), n\big) \dfrac{\partial}{\partial x_1} N^{1,1}(x_1, y), & y \in \partial Y, \end{cases} \tag{3.8}$$

$$\begin{cases} \mathcal{A}_y N^{2,0}(x_1, y) = \lambda^0\big(\rho(x_1, y) - \langle \rho(x_1, \cdot)\rangle\big), & y \in Y, \\ \mathcal{B}_y N^{2,0}(x_1, y) = 0, & y \in \partial Y. \end{cases} \tag{3.9}$$

Equating the coefficients in front of ε^1, we get the equation for u^3:

$$\begin{cases} \mathcal{A}_y u^3(x_1, y) = \operatorname{div}_y\left(a_{\cdot 1}(x_1, y)\dfrac{\partial u^2}{\partial x_1}\right) + \dfrac{\partial}{\partial x_1}\left(a_{11}(x_1, y)\dfrac{\partial u^1}{\partial x_1}\right) \\ \qquad\qquad + \lambda^0 \rho(x_1, y) u^1(x_1, y) + \lambda^1 \rho(x_1, y) u^0(x_1), \quad y \in Y, \\ \mathcal{B}_y u^3(x_1, y) = -\big(a_{\cdot 1}(x_1, y), n\big)\dfrac{\partial u^2}{\partial x_1}. \end{cases}$$

The compatibility condition for the last equation reads

$$-\frac{d}{dx_1}\left(a^{\mathrm{eff}}(x_1)\frac{dv^1}{dx_1}\right) - \lambda^0 \big\langle \rho(x_1, \cdot)\big\rangle v^1(x_1) = F(x_1) + \lambda^1 \big\langle \rho(x_1, \cdot)\big\rangle u^0, \tag{3.10}$$

where

$$F(x_1) = \sum_{k=0}^{2} \frac{d}{dx_1} \int_Y a_{1\cdot}(x_1, y) \nabla_y N^{2,k}(x_1, y) \frac{d^k u^0(x_1)}{dx_1^k}\, dy \\ + \lambda^0 \int_Y \rho(x_1, y) N^{1,1}(x_1, y) \frac{du^0(x_1)}{dx_1}\, dy. \tag{3.11}$$

Determining the boundary conditions for $v^1(x_1)$ at the points $x_1 = \pm 1$ requires constructing boundary layer correctors in the vicinity of these points.

Let $G^- = (0, +\infty) \times Q$ and $G^+ = (-\infty, 0) \times Q$ be semi-infinite cylinders with the axis directed along y_1 and lateral boundaries $\Sigma^- = (0, +\infty) \times \partial Q$ and $\Sigma^+ = (-\infty, 0) \times \partial Q$. We denote by $w^\pm(y)$ solutions to the following boundary value problems:

$$\begin{cases} -\operatorname{div}_y\big(a\big(\pm 1, y_1 \pm \delta, y'\big)\nabla_y w^\pm\big) = 0, & y \in G^\pm, \\ \big(a\big(\pm 1, y_1 \pm \delta, y'\big)\nabla_y w^\pm, n\big) = 0, & y \in \Sigma^\pm, \\ w^\pm(0, y') = -N^{1,1}\big(\pm 1, \pm\delta, y'\big)\dfrac{du^0}{dx_1}(\pm 1), & y' \in Q, \end{cases} \tag{3.12}$$

where $\delta = \delta(\varepsilon)$ is the fractional part of ε^{-1}. Due to our assumption (2.2) we have $\delta = 0$ so that problem (3.12) reads

$$\begin{cases} -\operatorname{div}_y\big(a(\pm 1, y)\nabla_y w^\pm\big) = 0, & y \in G^\pm, \\ \big(a(\pm 1, y)\nabla_y w^\pm, n\big) = 0, & y \in \Sigma^\pm, \\ w^\pm\big(0, y'\big) = -N^{1,1}\big(\pm 1, 0, y'\big)\dfrac{du^0}{dx_1}(\pm 1), & y' \in Q. \end{cases} \tag{3.13}$$

According to [14] there exists a unique bounded solution $w^\pm \in H^1_{loc}(G^\pm) \cap C^{1,\alpha}(\overline{G^\pm})$ of problem (3.13). It stabilizes to some constant $\hat{w}^\pm$, as $|y_1| \to +\infty$:

$$\big|w^\pm\big(y_1, y'\big) - \hat{w}^\pm\big| \leqslant C_0 e^{-\gamma|y_1|}, \quad C_0, \gamma > 0, \\ \big\|\nabla w^+\big\|_{L^2((n,n+1)\times Q)} \leqslant Ce^{-\gamma n}, \quad \forall n > 0, \\ \big\|\nabla w^-\big\|_{L^2((-(n+1),-n)\times Q)} \leqslant Ce^{-\gamma n}, \quad \forall n > 0, \tag{3.14}$$

for some $\gamma > 0$. As a boundary condition for $v^1(x_1)$ we choose the uniquely defined constants $\hat{w}^{\pm}$: $v^1(\pm 1) = \hat{w}^{\pm}$. Thus, the problem for v^1 takes the form

$$\begin{cases} -\dfrac{d}{dx_1}\left(a^{\mathrm{eff}}(x_1)\dfrac{dv^1}{dx_1}\right) - \lambda^0 \langle \rho(x_1,\cdot)\rangle v^1(x_1) = F(x_1) + \lambda^1 \langle \rho(x_1,\cdot)\rangle u^0, \quad x_1 \in (-1,1), \\ v^1(\pm 1) = \hat{w}^{\pm}, \end{cases} \tag{3.15}$$

where $F(x_1)$ is defined by (3.11).

Due to the Fredholm alternative, problem (3.15) is solvable in $H^1(-1,1)$ if and only if the right-hand side is orthogonal to the kernel of the adjoint operator, that is to the function $u^0(x_1)$ (see (3.4)). Thus, taking into account the normalization condition (3.5), we have

$$\lambda^1 = -\lambda^0 \int_{-1}^{1} F(x_1) u^0(x_1)\, dx_1 + \lambda^0 \left(a^{\mathrm{eff}}(1) \frac{du^0}{dx_1}(1) \hat{w}^+ - a^{\mathrm{eff}}(-1) \frac{du^0}{dx_1}(-1) \hat{w}^- \right). \tag{3.16}$$

Under our standing assumptions $v^1 \in C^{2,\alpha}[-1,1]$. Notice that $v^1(x_1)$ is defined up to a function of the form $Cu^0(x_1)$, where C is a constant. We fix the choice of v^1 setting

$$\int_{-1}^{1} v^1(x_1) u^0(x_1)\, dx_1 = 0.$$

In this way the function

$$\begin{aligned} u^0(x_1) &+ \varepsilon\left[N^{1,1}\left(x_1, \frac{x'}{\varepsilon}\right) \frac{du^0(x_1)}{dx_1} + v^1(x_1)\right] + \varepsilon\left[w^+\left(\frac{x_1 - 1}{\varepsilon}, \frac{x'}{\varepsilon}\right) - \hat{w}^+\right] \\ &+ \varepsilon\left[w^-\left(\frac{x_1 + 1}{\varepsilon}, \frac{x'}{\varepsilon}\right) - \hat{w}^-\right] \end{aligned}$$

satisfies the homogeneous Dirichlet boundary conditions on $S_{\pm 1}$. We denote

$$u_{\mathrm{bl}}^{\varepsilon,\pm}(x) \equiv \tilde{u}_{\mathrm{bl}}^{\varepsilon,\pm}(y)|_{y=\frac{x}{\varepsilon}} = w^{\pm}\left(\frac{x_1 \mp 1}{\varepsilon}, \frac{x'}{\varepsilon}\right) - \hat{w}^{\pm}, \tag{3.17}$$

where

$$\tilde{u}_{\mathrm{bl}}^{\varepsilon,\pm}(y) = w^{\pm}\left(y_1 \mp \frac{1}{\varepsilon}, y'\right) - \hat{w}^{\pm}.$$

Remark 3.2. If assumption (2.2) does not hold, then problem (3.12) depends on a parameter $\delta = \delta(\varepsilon) \in [0,1)$ being the fractional part of $1/\varepsilon$. In this case the boundary layer functions $w^{\pm}(y)$ also depend on δ, so do $\hat{w}^{\pm}$, v^1 and λ^1. Nevertheless, all the results of Theorem 2.2 remain valid. We assume (2.2) just for presentation simplicity. The dependence on $\delta(\varepsilon)$ does not create any additional technical difficulties.

Remark 3.3. We succeeded in constructing exponential boundary layer correctors $u_{\mathrm{bl}}^{\varepsilon,\pm}$ owing to the special structure of the domain G_ε. This allowed us to define v^1, λ^1 and other higher order terms of the asymptotic expansion (3.1). In the case of a generic smooth bounded domain one is unable to construct such a boundary layer due to the disagreement between the periodic structure and the domain boundary. By this reason in [11] and [12] only two leading terms of the expansion have been constructed.

3.2. Justification procedure in the case $\langle\rho(x_1,\cdot)\rangle > 0$

Let $\lambda_j^{0,+}$ be the jth eigenvalue and $u_j^{0,+}$ the corresponding eigenfunction of problem (3.4). For any $j\in\mathbb{N}$ we denote

$$U_j^{\varepsilon,+}(x) = u_j^{0,+}(x_1) + \varepsilon N^{1,1}\left(x_1, \frac{x}{\varepsilon}\right)\frac{du_j^{0,+}(x_1)}{dx_1} + \varepsilon v_j^{1,+}(x_1) + \varepsilon\left(u_{\rm bl}^{\varepsilon,+}(x) + u_{\rm bl}^{\varepsilon,-}(x)\right), \qquad (3.18)$$

where $u_j^{0,+}$, $N^{1,1}$ and $v_j^{1,+}$ solve problems (3.4), (3.2) and (3.15), respectively (with $u^0 = u_j^{0,+}$ and $\lambda^0 = \lambda_j^{0,+}$). The boundary layer functions $u_{\rm bl}^{\varepsilon,\pm}$ are defined by (3.17) and (3.13). Let us emphasize that, due to the presence of the boundary layer terms, the function $U_j^{\varepsilon,+}$ satisfies the homogeneous Dirichlet boundary conditions on $S_{\pm1}$, and, as a consequence, belong to the space $\mathcal{H}^\varepsilon$.

The goal of this section is to prove the following result.

Theorem 3.2. *Let conditions* (H0)–(H3) *be fulfilled, and suppose that* $\langle\rho(x_1,\cdot)\rangle > 0$ *for any* $x_1\in[-1,1]$. *If* $\lambda_j^{\varepsilon,+}$ *is the* j*th positive eigenvalue of problem* (2.1) *and* $u_j^{\varepsilon,+}$ *is the corresponding eigenfunction, then the following statements hold*:

(i) *For any* $j\in\mathbb{N}$, *there exist* ε_j *and* $C_j>0$ *such that*

$$\left|\lambda_j^{\varepsilon,+} - \lambda_j^{0,+}\right| \leqslant C_j\varepsilon, \quad \forall\varepsilon\in(0,\varepsilon_j].$$

(ii) *For any* $j\in\mathbb{N}$,

$$\left\|u_j^{\varepsilon,+} - U_j^{\varepsilon,+}\right\|_{H^1(G_\varepsilon)} \leqslant C_j\varepsilon\varepsilon^{\frac{d-1}{2}}$$

where $U_j^{\varepsilon,+}$ *is defined by* (3.18), *and* $(\lambda_j^{0,+}, u_j^{0,+})$ *is the* j*th eigenpair of the limit problem* (3.4). *Moreover, the "almost eigenfunctions" are almost orthonormal, that is*

$$\left|\frac{\varepsilon^{-(d-1)}}{|Q|}\left(a^\varepsilon\nabla U_i^{\varepsilon,+}, \nabla U_j^{\varepsilon,+}\right)_{L^2(G_\varepsilon)} - \delta_{ij}\right| \leqslant C_j\varepsilon.$$

(iii) *For any* $j\in\mathbb{N}$, $\lambda_j^{\varepsilon,+}$ *is simple, for sufficiently small* $\varepsilon>0$.

Remark 3.4. The estimates of Theorem 3.2 rely on the presence of the boundary layer correctors in the asymptotics of $u_j^{\varepsilon,+}$. The estimates obtained in [11] and [12] for a generic smooth domain are of order $\sqrt{\varepsilon}$.

Proof of Theorem 3.2. We make use of the following statement about "almost eigenvalues and eigenfunctions" (see [18,19]).

Lemma 3.2. *Given a compact self-adjoint operator* $\mathcal{K}^\varepsilon : \mathcal{H}^\varepsilon \to \mathcal{H}^\varepsilon$, *let* $\nu\in\mathbb{R}$ *and* $v\in\mathcal{H}^\varepsilon$ *be such that*

$$\|v\|_{\mathcal{H}^\varepsilon} = 1, \qquad \delta \equiv \left\|\mathcal{K}^\varepsilon v - \nu v\right\|_{\mathcal{H}^\varepsilon} < |\nu|.$$

Then there exists an eigenvalue μ_l^ε of the operator $\mathcal{K}^\varepsilon$ such that

$$\left|\mu_l^\varepsilon - \nu\right| \leqslant \delta.$$

Moreover, for any $\delta_1 \in (\delta, |\nu|)$ there exist coefficients $\{b_j^\varepsilon\} \in \mathbb{R}$ satisfying

$$\left\| v - \sum b_j^\varepsilon u_j^\varepsilon \right\|_{\mathcal{H}^\varepsilon} \leqslant 2\frac{\delta}{\delta_1},$$

where the sum is taken over all the eigenvalues of the operator $\mathcal{K}^\varepsilon$ in the segment $[\nu-\delta_1, \nu+\delta_1]$, and $\{u_j^\varepsilon\}$ are the corresponding orthonormalized in $\mathcal{H}^\varepsilon$ eigenfunctions. The coefficients b_j^ε are normalized by $\sum |b_j^\varepsilon|^2 = 1$.

As $v \in \mathcal{H}^\varepsilon$ and $\nu \in \mathbb{R}$ in Lemma 3.2 we use the normalized ansatz (3.18)

$$\mathcal{U}_j^{\varepsilon,+} = \frac{U_j^{\varepsilon,+}}{\|U_j^{\varepsilon,+}\|_{\mathcal{H}^\varepsilon}}$$

and the numbers $(\lambda_j^{0,+} + \varepsilon\lambda_j^{1,+})^{-1}$, respectively. Here $\lambda_j^{1,+}$ is defined by formula (3.16) with $u^0 = u_j^{0,+}$.

Lemma 3.3. *For any $j \in \mathbb{N}$ there is $\varepsilon_j > 0$ such that*

$$\left\|\mathcal{K}^\varepsilon \mathcal{U}_j^{\varepsilon,+} - \left(\lambda_j^{0,+} + \varepsilon\lambda_j^{1,+}\right)^{-1} \mathcal{U}_j^{\varepsilon,+}\right\|_{\mathcal{H}^\varepsilon} \leqslant C_j\varepsilon, \quad \varepsilon < \varepsilon_j, \tag{3.19}$$

for some constant C_j that does not depend on ε.

Proof. Letting

$$I^\varepsilon \equiv \left\|\mathcal{K}^\varepsilon \mathcal{U}_j^{\varepsilon,+} - \left(\lambda_j^{0,+} + \varepsilon\lambda_j^{1,+}\right)^{-1} \mathcal{U}_j^{\varepsilon,+}\right\|_{\mathcal{H}^\varepsilon},$$

after straightforward rearrangements we have

$$\begin{aligned}
I^\varepsilon &= \sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}} \left|\left(\mathcal{K}^\varepsilon \mathcal{U}_j^{\varepsilon,+} - \left(\lambda_j^{0,+} + \varepsilon\lambda_j^{1,+}\right)^{-1} \mathcal{U}_j^{\varepsilon,+}, w\right)_{\mathcal{H}^\varepsilon}\right| \\
&= \frac{\|U_j^{\varepsilon,+}\|_{\mathcal{H}^\varepsilon}^{-1}}{(\lambda_j^{0,+} + \varepsilon\lambda_j^{1,+})} \sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}} \left|\left(\left(\lambda_j^{0,+} + \varepsilon\lambda_j^{1,+}\right)\mathcal{K}^\varepsilon U_j^{\varepsilon,+} - U_j^{\varepsilon,+}, w\right)_{\mathcal{H}^\varepsilon}\right| \\
&= \frac{\|U_j^{\varepsilon,+}\|_{\mathcal{H}^\varepsilon}^{-1}}{(\lambda_j^{0,+} + \varepsilon\lambda_j^{1,+})} \sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}} \left|\left(\lambda_j^{0,+} + \varepsilon\lambda_j^{1,+}\right)\left(\rho^\varepsilon U_j^{\varepsilon,+}, w\right)_{L^2(G_\varepsilon)} - \left(a^\varepsilon \nabla U_j^{\varepsilon,+}, \nabla w\right)_{L^2(G_\varepsilon)}\right|.
\end{aligned}$$

Integrating by parts and using the boundary conditions for $N^{1,1}$ yield

$$I^\varepsilon = \frac{\|U_j^{\varepsilon,+}\|_{\mathcal{H}^\varepsilon}^{-1}}{(\lambda_j^{0,+} + \varepsilon\lambda_j^{1,+})} \sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}} \left|\left(\mathcal{A}^\varepsilon U_j^{\varepsilon,+} - \left(\lambda_j^{0,+} + \varepsilon\lambda_j^{1,+}\right)\rho^\varepsilon U_j^{\varepsilon,+}, w\right)_{L^2(G_\varepsilon)}\right.$$

$$+\varepsilon\int_{\Sigma_\varepsilon}\bigl(a^\varepsilon_{\cdot 1}(x),n\bigr)\frac{\partial}{\partial x_1}\left(N^{1,1}(x_1,y)\frac{du_j^{0,+}(x_1)}{dx_1}+v_j^{1,+}(x_1)\right)\bigg|_{y=x/\varepsilon}d\sigma$$

$$+\varepsilon\int_{\Sigma_\varepsilon}\bigl(a^\varepsilon\nabla_y\bigl(\tilde u_{\mathrm{bl}}^{\varepsilon,-}+\tilde u_{\mathrm{bl}}^{\varepsilon,+}\bigr),n\bigr)w\,d\sigma\bigg|$$

$$=\frac{\|U_j^{\varepsilon,+}\|^{-1}_{\mathcal{H}^\varepsilon}}{(\lambda_j^{0,+}+\varepsilon\lambda_j^{1,+})}\sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}}\bigg|\bigl(\varepsilon^0 I_0^\varepsilon+\varepsilon^1 I_1^\varepsilon+\varepsilon^2 I_2^\varepsilon,w\bigr)_{L^2(G_\varepsilon)}$$

$$+\varepsilon\int_{\Sigma_\varepsilon}\bigl(a^\varepsilon_{\cdot 1}(x),n\bigr)\frac{\partial}{\partial x_1}\left(N^{1,1}(x_1,y)\frac{du_j^{0,+}(x_1)}{dx_1}+v_j^{1,+}(x_1)\right)\bigg|_{y=x/\varepsilon}d\sigma$$

$$+\varepsilon\int_{\Sigma_\varepsilon}\bigl(a^\varepsilon\nabla_y\bigl(\tilde u_{\mathrm{bl}}^{\varepsilon,-}+\tilde u_{\mathrm{bl}}^{\varepsilon,+}\bigr),n\bigr)w\,d\sigma\bigg|.$$

Here

$$I_0^\varepsilon(x)=I_0(x_1,y)|_{y=x/\varepsilon}=-\frac{\partial}{\partial x_1}\left(a_{1\cdot}\nabla_y N^{1,1}\frac{du_j^{0,+}}{dx_1}\right)-\frac{\partial}{\partial x_1}\left(a_{11}\frac{du_j^{0,+}}{dx_1}\right)-\lambda_j^{0,+}\rho u_j^{0,+}\big|_{y=x/\varepsilon},$$

$$I_1^\varepsilon(x)=I_{\mathrm{bl}}^\varepsilon(x)-\left\{\operatorname{div}_x+\frac{1}{\varepsilon}\operatorname{div}_y\right\}\left(a_{\cdot 1}\frac{\partial}{\partial x_1}\left(N^{1,1}\frac{du_j^{0,+}}{dx_1}+v_j^{1,+}\right)\bigg|_{y=x/\varepsilon}\right)$$
$$-\lambda_j^{1,+}\rho\left(N^{1,1}\frac{du_j^{0,+}}{dx_1}+v_j^{1,+}\right)\bigg|_{y=x/\varepsilon}-\lambda_j^{1,+}\rho^\varepsilon u_j^{0,+},$$

$$I_{\mathrm{bl}}^\varepsilon(x)=\mathcal{A}^\varepsilon\bigl(u_{\mathrm{bl}}^{\varepsilon,-}+u_{\mathrm{bl}}^{\varepsilon,+}\bigr)-\lambda_j^{0,+}\rho^\varepsilon\bigl(u_{\mathrm{bl}}^{\varepsilon,-}+u_{\mathrm{bl}}^{\varepsilon,+}\bigr),$$

$$I_2^\varepsilon(x)=\lambda_j^{1,+}\rho(x_1,y)N^{1,1}(x_1,y)\frac{du_j^{0,+}}{dx_1}\bigg|_{y=x/\varepsilon}+\lambda_j^{1,+}\rho(x_1,y)v_j^{1,+}(x_1)|_{y=x/\varepsilon}$$
$$+\lambda_j^{1,+}\bigl(u_{\mathrm{bl}}^{\varepsilon,-}(x)+u_{\mathrm{bl}}^{\varepsilon,+}(x)\bigr).$$

Proposition 3.1. *The boundary layer functions $u_{\mathrm{bl}}^{\varepsilon,\pm}$ satisfy the estimate*

$$\bigl|\varepsilon\bigl(\mathcal{A}^\varepsilon u_{\mathrm{bl}}^{\varepsilon,\pm},v\bigr)_{L^2(G_\varepsilon)}+\varepsilon\bigl(a^\varepsilon\nabla u_{\mathrm{bl}}^{\varepsilon,\pm}v,n\bigr)_{L^2(\Sigma_\varepsilon)}-\varepsilon\lambda_j^{0,+}\bigl(\rho^\varepsilon u_{\mathrm{bl}}^{\varepsilon,\pm},v\bigr)_{L^2(G_\varepsilon)}\bigr|$$
$$\leqslant C\varepsilon\varepsilon^{(d-1)/2}\|v\|_{H^1(G_\varepsilon)},\quad v\in\mathcal{H}^\varepsilon.$$

Proof. We prove the proposition for $u_{\mathrm{bl}}^{\varepsilon,-}$, a similar proof can be performed for $u_{\mathrm{bl}}^{\varepsilon,+}$. Due to the definition of $u_{\mathrm{bl}}^{\varepsilon,-}$, up to the terms of higher order,

$$\varepsilon\mathcal{A}^\varepsilon u_{\mathrm{bl}}^{\varepsilon,-}(x)=-\left(\operatorname{div}_x+\frac{1}{\varepsilon}\operatorname{div}_y\right)\left((x_1+1)\frac{\partial a}{\partial x_1}(x_1,y)\nabla_y\tilde u_{\mathrm{bl}}^{\varepsilon,-}(y)\right)\bigg|_{y=x/\varepsilon}.$$

Integrating by parts yields

$$\varepsilon\big(\mathcal{A}^\varepsilon u_{\mathrm{bl}}^{\varepsilon,-}, v\big)_{L^2(G_\varepsilon)} + \varepsilon\big(a^\varepsilon \nabla_y u_{\mathrm{bl}}^{\varepsilon,-} v, n\big)_{L^2(\Sigma_\varepsilon)}$$
$$= \varepsilon \int\limits_{G_\varepsilon} \left(y_1 + \frac{1}{\varepsilon}\right) \frac{\partial a}{\partial x_1}(-1, y)\big(\nabla_y \tilde{u}_{\mathrm{bl}}^{\varepsilon,-}(y), \nabla v(x)\big)\Big|_{y=x/\varepsilon}\, dx.$$

Schwartz inequality and the exponential decay of u_{bl}^- give

$$\Big|\varepsilon\big(\mathcal{A}^\varepsilon u_{\mathrm{bl}}^{\varepsilon,-}, v\big)_{L^2(G_\varepsilon)} + \varepsilon\big(a^\varepsilon \nabla_y u_{\mathrm{bl}}^{\varepsilon,-} v, n\big)_{L^2(\Sigma_\varepsilon)}\Big| \leqslant C\varepsilon\varepsilon^{(d-1)/2}\|v\|_{H^1(G_\varepsilon)}$$

with the constant C depending only on Λ and Q. Then, due to the boundedness of ρ and the Schwartz inequality,

$$\big|\varepsilon\lambda_j^{0,+}\big(\rho^\varepsilon u_{\mathrm{bl}}^{\varepsilon,\pm}, v\big)_{L^2(G_\varepsilon)}\big| \leqslant C\varepsilon \int\limits_{G_\varepsilon} \big|u_{\mathrm{bl}}^{\varepsilon,-}\big||v|\, dx.$$

By the exponential decay property of u_{bl}^-,

$$\big\|u_{\mathrm{bl}}^{\varepsilon,-}\big\|_{L^2(G_\varepsilon)} \leqslant C\sqrt{\varepsilon}\varepsilon^{\frac{d-1}{2}}.$$

The last estimate completes the proof. □

Further analysis essentially relies on the following statement.

Lemma 3.4. *Let $g(x_1, y) \in C^{1,\alpha}([-1,1]; C^\alpha(\overline{Y}))$ be such that*

$$\big\langle g(x_1,\cdot)\big\rangle = \int\limits_Y g(x_1, y)\, dy = 0.$$

Then, for any $w \in H^1(G_\varepsilon)$, the following estimate is valid:

$$\left|\int\limits_{G_\varepsilon} g\left(x_1, \frac{x}{\varepsilon}\right) w(x)\, dx\right| \leqslant C\varepsilon\varepsilon^{\frac{d-1}{2}}\|w\|_{H^1(G_\varepsilon)}$$

with a constant C independent of ε.

Proof. Since $\langle g(x_1,\cdot)\rangle = 0$, then there exists a y_1-periodic function $\psi(x_1, y) \in C^{1,\alpha}([-1,1]; C^{2,\alpha}(\overline{Y}))$ being a solution of the problem

$$\begin{cases} -\Delta_y \psi(x_1, y) = g(x_1, y), & y \in Y, \\ \big(\nabla_y \Psi(x_1, y), n\big) = 0, & y \in \partial Y. \end{cases}$$

Then we have

$$\int\limits_{G_\varepsilon} g(x_1,y)w(x)\,dx = \varepsilon \int\limits_{G_\varepsilon} \big(\nabla_y\psi(x_1,y), \nabla w(x)\big)\big|_{y=x/\varepsilon}\,dx + \varepsilon \int\limits_{G_\varepsilon} w(x)\,\mathrm{div}_x\big(\nabla_y\psi(x_1,y)\big)\big|_{y=x/\varepsilon}\,dx \leqslant C\varepsilon\varepsilon^{\frac{d-1}{2}}\|w\|_{H^1(G_\varepsilon)}. \qquad \square$$

Let us turn back to the proof of Lemma 3.3. Since $u_j^{0,+}$ is a solution of problem (3.4), then $I_0(x_1,y)\in C^{1,\alpha}([-1,1];C^\alpha(\overline{Y}))$ and

$$\int\limits_Y I_0(x_1,y)\,dy = 0.$$

Thus, by Lemma 3.4,

$$\left|\int\limits_{G_\varepsilon} I_0^\varepsilon(x)w(x)\,dx\right| \leqslant C\varepsilon\varepsilon^{\frac{d-1}{2}}\|w\|_{H^1(G_\varepsilon)}. \tag{3.20}$$

The terms containing $u_{\mathrm{bl}}^\varepsilon$ have been estimated in Proposition 3.1. Integrating by parts the remaining terms of $(I_1^\varepsilon,w)_{L^2(G_\varepsilon)}$, using (H0) and the regularity properties of $u_j^{0,+}$, $N^{1,1}$ and $v_j^{1,+}$, one can show that

$$\left|\varepsilon\big(I_1^\varepsilon,w\big)_{L^2(G_\varepsilon)} + \varepsilon\int\limits_{\Sigma_\varepsilon}\big(a_{\cdot 1}^\varepsilon,n\big)\frac{\partial}{\partial x_1}\left(N^{1,1}\frac{du_j^{0,+}}{dx_1}+v_j^{1,+}\right)\bigg|_{y=x/\varepsilon} w\,d\sigma\right| \leqslant C\varepsilon\varepsilon^{\frac{d-1}{2}}\|w\|_{H^1(G_\varepsilon)}. \tag{3.21}$$

The quantity $(I_2^\varepsilon,w)_{L^2(G_\varepsilon)}$ is estimated in a similar way:

$$\varepsilon^2\big|\big(I_2^\varepsilon,w\big)_{L^2(G_\varepsilon)}\big| \leqslant C\varepsilon^2\varepsilon^{\frac{d-1}{2}}\|w\|_{H^1(G_\varepsilon)}. \tag{3.22}$$

It remains to estimate the norm $\|U_j^{\varepsilon,+}\|_{\mathcal{H}^\varepsilon}$. To this end we compute first the gradient of $U_j^{\varepsilon,+}$:

$$\frac{\partial}{\partial x_1}U_j^{\varepsilon,+} = \frac{du_j^{0,+}(x_1)}{dx_1} + \varepsilon\frac{\partial u_j^{1,+}}{\partial x_1}(x_1,y) + \frac{\partial}{\partial y_1}N^{1,1}(x_1,y)\frac{du_j^{0,+}(x_1)}{dx_1} + \frac{\partial}{\partial y_1}\big(\tilde{u}_{\mathrm{bl}}^{\varepsilon,+}(y)+\tilde{u}_{\mathrm{bl}}^{\varepsilon,-}(y)\big)\big|_{y=\frac{x}{\varepsilon}},$$

$$\frac{\partial}{\partial x_k}U_j^{\varepsilon,+} = \frac{\partial}{\partial y_k}N^{1,1}(x_1,y)\frac{du_j^{0,+}(x_1)}{dx_1} + \frac{\partial}{\partial y_k}\big(\tilde{u}_{\mathrm{bl}}^{\varepsilon,+}(y)+\tilde{u}_{\mathrm{bl}}^{\varepsilon,-}(y)\big)\big|_{y=\frac{x}{\varepsilon}}, \quad k\neq 1,$$

where

$$u_j^{1,+}(x_1,y) = N^{1,1}(x_1,y)\frac{du_j^{0,+}(x_1)}{dx_1} + v_j^{1,+}(x_1).$$

It is easy to see that

$$\begin{aligned}
\big(a^\varepsilon \nabla U_i^{\varepsilon,+}, \nabla U_j^{\varepsilon,+}\big) &= \big[a_{11}(x_1,y) + a_{\cdot 1}(x_1,y)\nabla_y N^{1,1}(x_1,y)\big]\frac{du_i^{0,+}}{dx_1}\frac{du_j^{0,+}}{dx_1} \\
&\quad + \big[a_{1\cdot}(x_1,y) + a(x_1,y)\nabla_y N^{1,1}(x_1,y)\big]\nabla_y N^{1,1}(x_1,y)\frac{du_i^{0,+}}{dx_1}\frac{du_j^{0,+}}{dx_1} \\
&\quad + J_{xx}^\varepsilon(x_1,y) + J_{xy}^\varepsilon(x_1,y) + J_{yy}^\varepsilon(x_1,y), \quad y = \frac{x}{\varepsilon}
\end{aligned}$$

where

$$\begin{aligned}
J_{xx}^\varepsilon(x_1,y) &= \varepsilon a_{11}\frac{du_i^{0,+}}{dx_1}\frac{du_j^{1,+}}{dx_1} + \varepsilon a_{11}\frac{\partial u_i^{1,+}}{\partial x_1}\frac{du_j^{0,+}}{dx_1} + \varepsilon^2 a_{11}\frac{\partial u_i^{1,+}}{\partial x_1}\frac{\partial u_j^{1,+}}{\partial x_1}, \\
J_{xy}^\varepsilon(x_1,y) &= \varepsilon a_{\cdot 1}\nabla_y N^{1,1}\frac{\partial u_i^{1,+}}{\partial x_1}\frac{du_j^{0,+}}{dx_1} + \varepsilon a_{1\cdot}\nabla_y N^{1,1}\frac{\partial u_j^{1,+}}{\partial x_1}\frac{du_i^{0,+}}{dx_1}, \\
J_{yy}^\varepsilon(x_1,y) &= a_{\cdot 1}\nabla_y\big(\tilde u_{\mathrm{bl}}^{\varepsilon,+} + \tilde u_{\mathrm{bl}}^{\varepsilon,-}\big)\frac{du_i^{0,+}}{dx_1} + \varepsilon a_{\cdot 1}\nabla_y\big(\tilde u_{\mathrm{bl}}^{\varepsilon,+} + \tilde u_{\mathrm{bl}}^{\varepsilon,-}\big)\frac{\partial u_i^{1,+}}{\partial x_1} \\
&\quad + \varepsilon a_{1\cdot}\nabla_y\big(\tilde u_{\mathrm{bl}}^{\varepsilon,+} + \tilde u_{\mathrm{bl}}^{\varepsilon,-}\big)\frac{\partial u_j^{1,+}}{\partial x_1} + \big(a\nabla_y N^{1,1}, \nabla_y\big(\tilde u_{\mathrm{bl}}^{\varepsilon,+} + \tilde u_{\mathrm{bl}}^{\varepsilon,-}\big)\big)\frac{du_i^{0,+}}{dx_1} \\
&\quad + \big(a\nabla_y N^{1,1}, \nabla_y\big(\tilde u_{\mathrm{bl}}^{\varepsilon,+} + \tilde u_{\mathrm{bl}}^{\varepsilon,-}\big)\big)\frac{du_j^{0,+}}{dx_1} + \big(a\nabla_y\big(\tilde u_{\mathrm{bl}}^{\varepsilon,+} + \tilde u_{\mathrm{bl}}^{\varepsilon,-}\big), \nabla_y\big(\tilde u_{\mathrm{bl}}^{\varepsilon,+} + \tilde u_{\mathrm{bl}}^{\varepsilon,-}\big)\big).
\end{aligned}$$

Using the regularity properties of $u_i^{0,+}$ and $N^{1,1}$ one can easily see that

$$\bigg|\int\limits_{G_\varepsilon} J_{xx}^\varepsilon\Big(x_1,\frac{x}{\varepsilon}\Big)\,dx\bigg| \leqslant C\varepsilon|G_\varepsilon| \leqslant C\varepsilon\varepsilon^{d-1}.$$

Then, by the periodicity of $N^{1,1}$ in y_1, we have

$$\bigg|\int\limits_{G_\varepsilon} J_{xy}^\varepsilon\Big(x_1,\frac{x}{\varepsilon}\Big)\,dx\bigg| \leqslant C\varepsilon\int\limits_{G_\varepsilon}\big|\nabla_y N^{1,1}\big|_{y=x/\varepsilon}\,dx = C\varepsilon\varepsilon^{-1}\varepsilon^d\int\limits_Y\big|\nabla_y N^{1,1}\big|\,dy \leqslant C\varepsilon\varepsilon^{d-1}.$$

Taking into account the exponential decay of $u_{\mathrm{bl}}^{\varepsilon,\pm}$ (see Proposition 3.1) we obtain the estimate

$$\bigg|\int\limits_{G_\varepsilon} J_{yy}^\varepsilon\Big(x_1,\frac{x}{\varepsilon}\Big)\,dx\bigg| \leqslant C\varepsilon\varepsilon^{d-1}.$$

Thus,

$$\biggl|\bigl(a^\varepsilon \nabla U_i^{\varepsilon,+}, \nabla U_j^{\varepsilon,+}\bigr)_{L^2(G_\varepsilon)} - \int\limits_{G_\varepsilon} \bigl\{a_{11} + a_{\cdot 1}\nabla_y N^{1,1}\bigr\}_{y=x/\varepsilon} \frac{du_i^{0,+}}{dx_1}\frac{du_j^{0,+}}{dx_1}\,dx$$
$$- \int\limits_{G_\varepsilon} \bigl\{a_{1\cdot} + a\nabla_y N^{1,1}\bigr\}\nabla_y N^{1,1}\big|_{y=x/\varepsilon} \frac{du_i^{0,+}}{dx_1}\frac{du_j^{0,+}}{dx_1}\,dx\biggr| \leqslant C\varepsilon\varepsilon^{d-1}.$$

Considering (3.3) and Lemma 3.4, we get

$$\biggl|\bigl(a^\varepsilon \nabla U_i^{\varepsilon,+}, \nabla U_j^{\varepsilon,+}\bigr)_{L^2(G_\varepsilon)} - \int\limits_{G_\varepsilon} a^{\mathrm{eff}}(x_1)\frac{du_i^{0,+}}{dx_1}\frac{du_j^{0,+}}{dx_1}\,dx\biggr| \leqslant C\varepsilon\varepsilon^{d-1}.$$

Consequently, in view of the normalization condition (3.5), one has

$$\bigl|\bigl(a^\varepsilon \nabla U_i^{\varepsilon,+}, \nabla U_j^{\varepsilon,+}\bigr)_{L^2(G_\varepsilon)} - |Q|\varepsilon^{d-1}\delta_{ij}\bigr| \leqslant C\varepsilon\varepsilon^{d-1}, \tag{3.23}$$

and, for sufficiently small ε,

$$\varepsilon^{-\frac{(d-1)}{2}}\bigl\|U_i^{\varepsilon,+}\bigr\|_{\mathcal{H}^\varepsilon} \geqslant \frac{|Q|^{1/2}}{2}, \quad \varepsilon < \varepsilon_i. \tag{3.24}$$

Combining estimates (3.20), (3.21), (3.22), (4.35) and Proposition 3.1 yields the desired bound (3.19). Lemma 3.3 is proved.

Combining Lemma 3.3 and Lemma 3.2, we conclude that for any eigenvalue $\lambda_j^{0,+}$ of problem (3.4) there exists an eigenvalue $\mu_q^{\varepsilon,+}$ of the operator $\mathcal{K}^\varepsilon$ such that

$$\bigl|\mu_q^{\varepsilon,+} - \bigl(\lambda_j^{0,+} + \varepsilon\lambda_j^{1,+}\bigr)^{-1}\bigr| \leqslant \tilde{c}_j\varepsilon.$$

Considering the fact that $\lambda_q^{\varepsilon,+} = (\mu_q^{\varepsilon,+})^{-1}$, we have

$$\bigl|\lambda_q^{\varepsilon,+} - \lambda_j^{0,+}\bigr| \leqslant c_j\varepsilon, \quad \varepsilon < \varepsilon_j. \tag{3.25}$$

Generally speaking, there might be more than one eigenvalue of the operator $\mathcal{A}^\varepsilon$ (problem (2.1)) satisfying inequality (5.14), but we will show that in the case under consideration such an eigenvalue $\lambda_j^{\varepsilon,+}$ is unique if $\varepsilon < \varepsilon_j$.

Lemma 3.5. *For any q, the estimate holds*

$$0 < m \leqslant \lambda_q^{\varepsilon,+} \leqslant M_q.$$

Proof. Let us first estimate the norm of the operator $\mathcal{K}^\varepsilon$,

$$\bigl\|\mathcal{K}^\varepsilon\bigr\| = \sup_{\|u\|_{\mathcal{H}^\varepsilon}=1}\bigl(\mathcal{K}^\varepsilon u, u\bigr)_{\mathcal{H}^\varepsilon} = \sup_{\|u\|_{\mathcal{H}^\varepsilon}=1}\bigl(\rho^\varepsilon u, u\bigr)_{L^2(G_\varepsilon)} \leqslant C\|u\|_{L^2(G_\varepsilon)}$$

where C does not depend on ε. Thus, $\mu_q^{\varepsilon,+} \leqslant C$, for any q, and, consequently, $\lambda_q^{\varepsilon,+} \geqslant m$ with m independent of ε.

In order to show that the inverse inequality is valid, we recall that for any $\lambda_j^{0,+}$ there is an eigenvalue of $\mathcal{K}^\varepsilon$ such that

$$\mu(\varepsilon, j) \to \left(\lambda_j^{0,+}\right)^{-1}, \quad \varepsilon \to 0.$$

It implies that $\mu(\varepsilon, j) \geqslant c_j$ and, moreover, $\mu_k^{\varepsilon,+} \geqslant c_j$ for all $k \geqslant j$. Lemma 3.5 is proved. □

It follows from Lemma 3.5 that, up to a subsequence, $\lambda_j^{\varepsilon,+}$ converges to some λ_*, as $\varepsilon \to 0$.

Lemma 3.6. *Suppose that* (*perhaps for a subsequence*)

$$\lambda_j^{\varepsilon,+} \to \lambda_*, \quad \varepsilon \to 0.$$

Then λ_* is an eigenvalue of problem (3.4).

There are several different ways of proving Lemma 3.6. Here we expose the proof based on the technique of convergence in variable spaces with singular measures.

Introduce the "universal domain" $K_d = [-1, 1]^d$. For ε small enough, $G_\varepsilon \subset K_d$. In what follows, for arbitrary Borel set $B \subset K_d$, we denote

$$\mu_\varepsilon(B) = \frac{\varepsilon^{-(d-1)}}{|Q|} \int\limits_B \chi(G_\varepsilon)\, dx, \tag{3.26}$$

where $\chi(G_\varepsilon)$ is the characteristic function of G_ε; dx is a usual d-dimensional Lebesgue measure. Then μ_ε converges weakly to a measure $\mu_* = dx_1 \times \delta(x')$, as $\varepsilon \to 0$. For any ε, the space of Borel measurable functions $g(x)$ such that

$$\int\limits_{K_d} \left(g(x)\right)^2 d\mu_\varepsilon(x) < \infty$$

is denoted $L^2(K_d, \mu_\varepsilon)$.

Let us also recall the definition of the Sobolev space with measure.

Definition 3.1. We say that a function $g \in L^2(K_d, \mu_\varepsilon)$ belongs to the space $H^1(K_d, \mu_\varepsilon)$ if there exists a vector function $z \in L^2(K_d, \mu_\varepsilon)^d$ and a sequence $\varphi_k \in C^\infty(K_d)$ such that

$$\varphi_k \to g \quad \text{in } L^2(K_d, \mu_\varepsilon), \quad k \to \infty,$$
$$\nabla\varphi_k \to z \quad \text{in } L^2(K_d, \mu_\varepsilon)^d, \quad k \to \infty.$$

In this case z is called the gradient of g and is denoted by $\nabla^{\mu_\varepsilon} g$.

Since in our case the measure μ_ε is a weighted Lebesgue measure, then $\nabla^{\mu_\varepsilon} g = \nabla g$ and the space $H^1(K_d, \mu_\varepsilon)$ coincides with the usual Sobolev space $H^1(G_\varepsilon)$.

Definition 3.2. We say that a sequence of functions $\{g^\varepsilon(x)\} \subset L^2(K_d, \mu_\varepsilon)$ weakly converges in $L^2(K_d, \mu_\varepsilon)$ to a function $g(x_1) \in L^2(K_d, \mu_*)$, as $\varepsilon \to 0$, if

(i) $\|g^\varepsilon\|_{L^2(K_d,\mu_\varepsilon)} \leqslant C$;
(ii) For any $\varphi \in C^\infty(\mathbb{R}^d)$ the following limit relation holds:

$$\lim_{\varepsilon\to 0}\int\limits_{K_d} g^\varepsilon(x)\varphi(x)\,d\mu_\varepsilon(x) = \int\limits_{K_d} g(x_1)\varphi(x)\,d\mu_*(x).$$

A sequence $\{g^\varepsilon\}$ is said to converge strongly to $g(x_1)$ in $L^2(K_d,\mu_\varepsilon)$, as $\varepsilon\to 0$, if it converges weakly and

$$\lim_{\varepsilon\to 0}\int\limits_{K_d} g^\varepsilon(x)\psi^\varepsilon(x)\,d\mu_\varepsilon(x) = \int\limits_{K_d} g(x_1)\psi(x_1)\,d\mu_*(x)$$

for any sequence $\{\psi^\varepsilon(x)\}$ weakly converging to $\psi(x_1)$ in $L^2(K_d,\mu_\varepsilon)$.

Notice that the property of weak compactness of a bounded sequence in a separable Hilbert space remains valid with respect to the convergence in variable spaces.

In order to prove Lemma 3.6 we use the technique of two-scale convergence in variable spaces with measure, so for the reader's convenience we recall the relevant definition.

Definition 3.3. We say that $g^\varepsilon \in L^2(K_d,\mu_\varepsilon)$ two-scale converges in $L^2(K_d,\mu_\varepsilon)$ to a function $\tilde g(x_1,y)\in L^2(K_d\times Y,\mu_*\times dy)$, as $\varepsilon\to 0$, if

(i)

$$\left\|g^\varepsilon\right\|_{L^2(K_d,\mu_\varepsilon)} \leqslant C, \quad \varepsilon>0;$$

(ii) The following limit relation holds:

$$\lim_{\varepsilon\to 0}\int\limits_{K_d} g^\varepsilon(x)\varphi(x)\psi\left(\frac{x}{\varepsilon}\right)d\mu_\varepsilon(x) = \int\limits_{K_d}\int\limits_{Y}\tilde g(x_1,y)\varphi(x)\psi(y)\,dy\,d\mu_*(x)$$

for any $\varphi\in C^\infty(K_d)$, and $\psi(y)\in C^\infty(Y)$ periodic in y_1.

Proof of Lemma 3.6. By the normalization condition (2.6)

$$\left\|u_j^{\varepsilon,+}\right\|_{L^2(K_d,\mu_\varepsilon)} + \left\|\nabla u_j^{\varepsilon,+}\right\|_{L^2(K_d,\mu_\varepsilon)^d} \leqslant C, \tag{3.27}$$

thus, up to a subsequence, $u_j^{\varepsilon,+}(x)$ converges weakly in $L^2(K_d,\mu_\varepsilon)$ to a function $u_*(x_1)\in L^2(K_d,\mu_*)$, as $\varepsilon\to 0$. Let us show that in fact the convergence is strong. Denote

$$\bar u_j^\varepsilon(x_1) = \int\limits_{\varepsilon Q} u_j^{\varepsilon,+}\big(x_1,x'\big)\,dx'.$$

Then, due to the Poincaré inequality,

$$\int\limits_{\varepsilon Q}\big(u_j^{\varepsilon,+}(x)-\bar u_j^\varepsilon(x_1)\big)^2\,dx' \leqslant C\varepsilon^2\int\limits_{\varepsilon Q}\big|\nabla\big(u_j^{\varepsilon,+}(x)-\bar u_j^\varepsilon(x_1)\big)\big|^2\,dx'.$$

Integrating with respect to x_1 and taking into account (3.27), we get

$$\int\limits_{K_d} \left(u_j^{\varepsilon,+}(x) - \overline{u}_j^{\varepsilon}(x_1)\right)^2 d\mu_\varepsilon \leqslant C\varepsilon.$$

On the other hand, $\overline{u}_j^{\varepsilon}(x_1)$ is uniformly bounded in $H^1(-1,1)$, thus there exists $\overline{u}(x_1)$ such that

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^{-(d-1)}}{|Q|} \int\limits_{G_\varepsilon} \left(\overline{u}_j^{\varepsilon}(x_1)\right)^2 dx = \int\limits_{-1}^{1} \left(\overline{u}(x_1)\right)^2 dx_1.$$

The strong convergence of $u_j^{\varepsilon,+}(x)$ to $\overline{u}(x_1) = u_*(x_1)$ in $L^2(K_d, \mu_\varepsilon)$ is the immediate consequence of the last two formulae.

By Lemma 3.4, $\rho^\varepsilon(x)$ converges weakly to $\langle \rho(x_1,\cdot)\rangle$ in $L^2(K_d,\mu_\varepsilon)$. Thus,

$$\lambda_j^{\varepsilon,+} \rho^\varepsilon(x) u_j^{\varepsilon,+}(x) \to \lambda_* \langle \rho(x_1,\cdot)\rangle u_*(x_1) \quad \text{weakly in } L^2(K_d,\mu_\varepsilon),\ \varepsilon \to 0.$$

Denoting

$$f^\varepsilon(x) = \lambda_j^{\varepsilon,+} \rho^\varepsilon(x) u_j^{\varepsilon,+}(x), \qquad f^0(x_1) = \lambda_* \langle \rho(x_1,\cdot)\rangle u_*(x_1),$$

we arrive at the following boundary value problem:

$$\begin{cases} \mathcal{A}^\varepsilon u_j^{\varepsilon,+}(x) = f^\varepsilon(x), & x \in G_\varepsilon, \\ \mathcal{B}^\varepsilon u_j^{\varepsilon,+}(x) = 0, & x \in \Sigma_\varepsilon, \\ u_j^{\varepsilon,+}(\pm 1, x') = 0, & x' \in \varepsilon Q. \end{cases} \tag{3.28}$$

The homogenization theorem for locally periodic elliptic equations in variable spaces (see [2,20]) implies that

$$u_j^{\varepsilon,+}(x) \to u_*(x_1) \quad \text{weakly in } L^2(K_d,\mu_\varepsilon),\ \varepsilon \to 0,$$
$$a^\varepsilon(x)\nabla u_j^{\varepsilon,+}(x) \to \left\{ a^{\mathrm{eff}}(x_1) \frac{du_*}{dx_1}(x_1), 0, \dots 0 \right\}^T \quad \text{weakly in } L^2(K_d,\mu_\varepsilon)^d,\ \varepsilon \to 0,$$

where $u_*(x_1) \in H_0^1(-1,1)$ is a solution of problem (3.4).

It follows from the normalization condition (2.6), boundedness of $\rho(x_1,y)$ and $\lambda_j^{\varepsilon,+}$ that

$$\begin{aligned} 1 &= \frac{\varepsilon^{-(d-1)}}{|Q|} \left(a^\varepsilon \nabla u_j^{\varepsilon,+}, \nabla u_j^{\varepsilon,+}\right)_{L^2(G_\varepsilon)} \\ &= \lambda_j^{\varepsilon,+} \int\limits_{K_d} \rho^\varepsilon \left(u_j^{\varepsilon,+}\right)^2 d\mu_\varepsilon \\ &\leqslant C_j \left\| u_j^{\varepsilon,+} \right\|^2_{L^2(K_d,\mu_\varepsilon)}. \end{aligned}$$

Considering the strong convergence of $u_j^{\varepsilon,+}$ to u_* in $L^2(K_d,\mu_\varepsilon)$, we conclude that $u_* \neq 0$. Thus, (λ_*, u_*) is an eigenpair of the effective problem (3.4). Lemma 3.6 is proved. □

Turning back to the proof of Theorem 3.2, suppose that there exist two different eigenvalues $\lambda_i^{\varepsilon,+} \neq \lambda_j^{\varepsilon,+}$ satisfying inequality (5.14) with $\lambda^{0,+}$ being an eigenvalue of the operator $\mathcal{A}^0$. As was proved in Lemma 3.6, in this case the corresponding eigenfunctions $u_i^{\varepsilon,+}$ and $u_j^{\varepsilon,+}$ converge strongly in $L^2(K_d,\mu_\varepsilon)$ to the eigenfunctions $u_i^{0,+}$ and $u_j^{0,+}$ of $\mathcal{A}^0$, which correspond to $\lambda^{0,+}$. Let us show that $u_i^{0,+}$ and $u_j^{0,+}$ are linearly independent. By the normalization condition

$$\lambda_i^{\varepsilon,+}\big(\rho^\varepsilon u_i^{\varepsilon,+}, u_j^{\varepsilon,+}\big)_{L^2(K_d,\mu_\varepsilon)} = \delta_{ij}.$$

Notice that, by Lemma 3.4, ρ^ε converges weakly in $L^2(K_d,\mu_\varepsilon)$ to its average $\langle\rho(x_1,\cdot)\rangle$. Thus, passing to the limit in the last identity, we obtain

$$\lambda^{0,+}\int\limits_{K_d}\langle\rho(x_1,\cdot)\rangle u_i^{0,+}(x_1)u_j^{0,+}(x_1)\,d\mu_* = \delta_{ij}$$

that implies the linear independence of $u_i^{0,+}$ and $u_j^{0,+}$. But $\lambda^{0,+}$ as an eigenvalue of $\mathcal{A}^0$ is simple by Theorem 3.1. We arrive at contradiction, thus, for any j there exists a unique $\lambda_j^{\varepsilon,+}$ satisfying (5.14). In particular, it means that for sufficiently small ε the eigenvalues $\lambda_j^{\varepsilon,+}$ are simple.

Combining Lemma 3.2, Lemma 3.5 and Lemma 3.6 one obtains the first statement of Theorem 3.2.

The second statement (ii) of Theorem 3.2 follows immediately from Lemma 3.2 and (i). This completes the proof. □

Theorem 3.2 might be formulated in terms of convergence in variable spaces with measure.

Corollary 3.1. *Suppose that conditions* (H0)–(H3) *hold true and* $\langle\rho(x_1,\cdot)\rangle > 0$. *Let* $(\lambda_j^{\varepsilon,+}, u_j^{\varepsilon,+})$ *and* $(\lambda_j^{0,+}, u_j^{0,+})$ *be eigenpairs of problems* (2.1) *and* (3.4), *respectively. Then*

(a) *For any* $j \in \mathbb{N}$, $\lambda_j^{\varepsilon,+} \to \lambda_j^{0,+}$, *as* $\varepsilon \to 0$, *and*

$$u_j^{\varepsilon,+}(x) \to u_j^{0,+}(x_1) \quad \text{strongly in } L^2(K_d,\mu_\varepsilon),\ \varepsilon \to 0$$

in terms of Definition 3.2.

(b) *The convergence of fluxes takes place, that is*

$$a^\varepsilon(x)\nabla u_j^{\varepsilon,+}(x) \to \left\{a^{\mathrm{eff}}(x_1)\frac{du_j^{0,+}}{dx_1}(x_1), 0, \ldots, 0\right\}^T$$

weakly in $L^2(K_d,\mu_\varepsilon)^d$, *as* $\varepsilon \to 0$.

Proof. The first statement follows from the normalization condition (2.6) (see proof of Lemma 3.6). The convergence of fluxes is a consequence of the homogenization result used while proving Lemma 3.6. □

4. The case $\langle \rho(x_1, \cdot)\rangle = 0$

4.1. Formal asymptotic expansion

Using the arguments similar to those in Section 3.4.1, [12], yields

$$c\varepsilon^{-1} \leqslant \lambda_1^{\varepsilon,\pm} \leqslant C\varepsilon^{-1},$$

for some constants c and C.

Considering the last estimate, we look for a solution of problem (2.1) in the form

$$\begin{aligned} u^\varepsilon(x) &= u^0(x_1) + \varepsilon u^1(x_1, y) + \varepsilon^2 u^2(x_1, y) + \cdots, \quad y = \frac{x}{\varepsilon}, \\ \lambda^\varepsilon &= \varepsilon^{-1}\nu^0 + \nu^1 + \cdots, \end{aligned} \tag{4.1}$$

where ν^0, ν^1, $u^0(x_1)$, $u^1(x_1, y)$ and $u^2(x_1, y)$ are to be determined. We suppose that $u^1(x_1, y)$ and $u^2(x_1, y)$ are 1-periodic in y_1. Substituting asymptotic ansatz (4.1) into (2.1) and collecting terms of order ε^{-1}, we obtain the following equation for the unknown function $u^1(x_1, y)$:

$$\begin{cases} \mathcal{A}_y u^1(x_1, y) = \mathrm{div}_y\, a_{\cdot 1}(x_1, y)\dfrac{du^0(x_1)}{dx_1} + \nu^0\rho(x_1, y)u^0(x_1), & y \in Y, \\ \mathcal{B}_y u^1(x_1, y) = -a_{i1}(x_1, y)n_i\dfrac{du^0(x_1)}{dx_1}, & y \in \partial Y, \\ u^1(x_1, y) \text{ is 1-periodic} & \text{in } y_1. \end{cases}$$

Note that, since $\langle \rho(x_1, \cdot)\rangle = 0$, the compatibility condition is satisfied. The structure of the right-hand side of the last equation suggests the following representation for $u^1(x_1, y)$:

$$u^1(x_1, y) = N^{1,1}(x_1, y)\frac{du^0(x_1)}{dx_1} + \nu^0 N^{1,0}(x_1, y)u^0(x_1) + v^1(x_1). \tag{4.2}$$

Then the functions $N^{1,1}$ and $N^{1,0}$ are 1-periodic in y_1 solutions of the problems

$$\begin{cases} \mathcal{A}_y N^{1,1}(x_1, y) = \mathrm{div}_y\, a_{\cdot 1}(x_1, y), & y \in Y, \\ \mathcal{B}_y N^{1,1}(x_1, y) = -a_{i1}(x_1, y)n_i, & y \in \partial Y, \\ N^{1,1}(x_1, y) \text{ is 1-periodic} & \text{in } y_1, \end{cases} \tag{4.3}$$

$$\begin{cases} \mathcal{A}_y N^{1,0}(x, y) = \rho(x_1, y), & y \in Y, \\ \mathcal{B}_y N^{1,0}(x_1, y) = 0, & y \in \partial Y, \\ N^{1,0}(x_1, y) \text{ is 1-periodic} & \text{in } y_1. \end{cases} \tag{4.4}$$

Under assumption (H0) the functions $N^{1,1}(x_1, y), N^{1,0}(x_1, y)$ belong to the space $C^{1,\alpha}([-1, 1] \times \overline{Y})$.

Similarly, substituting (4.1) into (2.1) and collecting the terms in front of ε^0, we have

$$\begin{cases} \mathcal{A}_y u^2(x_1,y) = \operatorname{div}_y\left(a_{\cdot 1}(x_1,y)\dfrac{\partial u^1}{\partial x_1}(x_1,y)\right) \\ \qquad + \dfrac{\partial}{\partial x_1}\big(a_{1\cdot}(x_1,y)\nabla_y u^1(x_1,y)\big) \\ \qquad + \dfrac{\partial}{\partial x_1}\left(a_{11}(x_1,y)\dfrac{du^0(x_1)}{dx_1}\right) \\ \qquad + \nu^1\rho(x_1,y)u^0(x_1) \\ \qquad + \nu^0\rho(x_1,y)u^1(x_1), & y\in Y, \\ \mathcal{B}_y u^2(x_1,y) = -a_{i1}(x_1,y)n_i\dfrac{\partial}{\partial x_1}u^1(x_1,y), & y\in\partial Y, \\ u^2(x_1,y) \text{ is 1-periodic} & \text{in } y_1. \end{cases} \tag{4.5}$$

The compatibility condition for the last problem reads

$$\begin{aligned} &\frac{d}{dx_1}\int_Y \big(a_{11}+a_{1\cdot}(x_1,y)\nabla_y N^{1,1}(x_1,y)\big)\frac{du^0(x_1)}{dx_1}\,dy \\ &\quad + \nu^0\frac{d}{dx_1}\int_Y a_{1\cdot}(x_1,y)\nabla_y N^{1,0}(x_1,y)u^0(x_1)\,dy \\ &\quad + \nu^0\int_Y \rho(x_1,y)N^{1,1}(x_1,y)\,dy\,\frac{du^0(x_1)}{dx_1} \\ &\quad + \big(\nu^0\big)^2\int_Y \rho(x_1,y)N^{1,0}(x_1,y)u^0(x_1)\,dy = 0. \end{aligned} \tag{4.6}$$

Rearranging the last three terms in (4.6) gives

$$\begin{aligned} &\nu^0\frac{d}{dx_1}\int_Y a_{1\cdot}(x_1,y)\nabla_y N^{1,0}(x_1,y))u^0(x_1)\,dy \\ &\quad + \nu^0\int_Y \rho(x_1,y)N^{1,1}(x_1,y)\,dy\,\frac{du^0(x_1)}{dx_1} \\ &\quad + \big(\nu^0\big)^2\int_Y \rho(x_1,y)N^{1,0}(x_1,y)u^0(x_1)\,dy \\ &= \big(\nu^0\big)^2 u^0(x_1)\int_Y \big(a(x_1,y)\nabla_y N^{1,0},\nabla_y N^{1,0}\big)\,dy \\ &\quad + \nu^0 u^0(x_1)\frac{d}{dx_1}\int_Y \big(a(x_1,y)\nabla_y N^{1,1},\nabla_y N^{1,0}\big)\,dy. \end{aligned}$$

Denote

$$\mathbf{C}(x_1) = \int_Y \big(a(x_1, y)\nabla_y N^{1,0}, \nabla_y N^{1,0}\big)\, dy, \tag{4.7}$$

$$\mathbf{B}(x_1) = \frac{\partial}{\partial x_1} \int_Y \big(a(x_1, y)\nabla_y N^{1,1}, \nabla_y N^{1,0}\big)\, dy. \tag{4.8}$$

In view of the regularity properties of $N^{1,1}$ and $N^{1,0}$, $\mathbf{C} \in C^{1,\alpha}[-1,1]$ and $\mathbf{B} \in C^{\alpha}[-1,1]$. Thus, (4.6) supplemented with an appropriate boundary condition takes the form of a quadratic operator pencil

$$\begin{cases} \boldsymbol{\Pi}\big(\nu^0\big)u^0(x_1) \equiv -\dfrac{d}{dx_1}\left(a^{\text{eff}}(x_1)\dfrac{du^0(x_1)}{dx_1}\right) + \nu^0 \mathbf{B}(x_1)u^0(x_1) \\ \qquad\qquad - \big(\nu^0\big)^2 \mathbf{C}(x_1)u^0(x_1) = 0, \quad x_1 \in (-1,1), \\ u^0(-1) = u^0(1) = 0. \end{cases} \tag{4.9}$$

The variational formulation of problem (4.9) reads: find $u^0 \in H_0^1(-1,1)$, $u^0 \neq 0$, such that

$$\int_{-1}^{1} a^{\text{eff}} \frac{du^0}{dx_1}\frac{dv}{dx_1}\, dx_1 + \nu^0 \int_{-1}^{1} \mathbf{B} u^0 v\, dx_1 - \big(\nu^0\big)^2 \int_{-1}^{1} \mathbf{C} u^0 v\, dx_1 = 0, \tag{4.10}$$

for any $v \in H_0^1(-1,1)$.

The next theorem characterizes the spectrum of the quadratic operator pencil (4.9).

Theorem 4.1. *The spectrum of problem* (4.9) *is discrete. The eigenvalues are real, algebraically and geometrically simple, and form two infinite sequences*

$$0 < \nu_1^{0,+} < \nu_2^{0,+} < \cdots < \nu_j^{0,+} \cdots \to +\infty,$$

$$0 > \nu_1^{0,-} > \nu_2^{0,-} > \cdots > \nu_j^{0,-} \cdots \to -\infty.$$

The corresponding eigenfunctions can be normalized by

$$\int_{-1}^{1} a^{\text{eff}} \frac{du_i^{0,\pm}}{dx_1}\frac{du_j^{0,\pm}}{dx_1}\, dx_1 + \nu_i^{0,\pm}\nu_j^{0,\pm} \int_{-1}^{1} \mathbf{C} u_i^{0,\pm} u_j^{0,\pm}\, dx_1 = \delta_{ij}, \tag{4.11}$$

where a^{eff} and $\mathbf{C}$ are defined by (3.3) *and* (4.7)*, respectively.*

Proof. The existence of infinite number of eigenvalues is given by the following classical theorem (see [3,8]).

Theorem 4.2 *(Keldysh theorem). Given compact operators T and H, such that H is a normal operator with $\mathrm{Ker}\, H = \{0\}$ ($HH^* = H^*H$) and H^2 is self-adjoint. Consider the Keldysh operator pencil*

$$B(\lambda) = Id - \lambda T H - \lambda^2 H^2,$$

where Id is the identity operator. The following statements hold:

1. *For any $\delta > 0$, there is only finite number of eigenvalues outside the angle*

$$\left\{\lambda:\ \left|\arg\lambda - \frac{k\pi}{2}\right| < \delta\right\}, \quad k = 0, 2;$$

2. *Denote $N_+(r)$ the number of eigenvalues counted according to their multiplicity of the operator H^2 in the interval $(1/r^2, +\infty)$. Let $N_k(r, B(\lambda))$ be a number of eigenvalues of the operator pencil $B(\lambda)$ contained in the sector*

$$\left\{\lambda:\ \left|\arg\lambda - \frac{k\pi}{2}\right| < \frac{\pi}{4},\ |\lambda| < r\right\}, \quad k = 0, 1, 2, 3.$$

If

$$\liminf_{r\to\infty} \frac{\log N_+(r)}{\log r} < \infty, \tag{4.12}$$

then

$$\liminf_{r\to\infty} \left|\frac{N_{2k}(r, B(\lambda))}{N_+(r)} - 1\right| = 0, \quad k = 0, 1.$$

In our case the operator pencil has the form

$$\boldsymbol{\Pi}\big(\nu^0\big) = \mathcal{A}^0 + \nu^0 \mathbf{B}(x_1) Id - \big(\nu^0\big)^2 \mathbf{C}(x_1) Id.$$

Since $(\mathcal{A}^0)^{-1}$ is a self-adjoint compact positive operator from $L^2(-1,1)$ into itself, then there exists a self-adjoint positive operator $S = (\mathcal{A}^0)^{-1/2}$. It is compact as an operator from $L^2(-1,1)$ into itself, bounded if we consider it as an operator from $L^2(-1,1)$ into $H_0^1(-1,1)$, and compact if it acts on $H_0^1(-1,1)$ with values in $H_0^1(-1,1)$. We apply the operator S to both sides of the operator pencil $\boldsymbol{\Pi}(\nu^0)$. As a result we obtain

$$\widetilde{\boldsymbol{\Pi}}\big(\nu^0\big) = Id + \nu^0 S\mathbf{B}(x_1)S - \big(\nu^0\big)^2 S\mathbf{C}(x_1)S. \tag{4.13}$$

One can check that $H^2 = S\mathbf{C}(x_1)S : L^2(-1,1) \to L^2(-1,1)$ is a self-adjoint compact positive operator. Then $H = (S\mathbf{C}(x_1)S)^{1/2}$ is also compact positive and self-adjoint with $\mathit{Ker}\, H = \{0\}$. Introducing

$$T = S\mathbf{B}(x_1)S\big(S\mathbf{C}(x_1)S\big)^{-1/2},$$

we see that T is a compact operator from $L^2(-1,1)$ into itself. Indeed, $S\mathbf{B}(x_1)S$ is a compact operator from $L^2(-1,1)$ into $H_0^1(-1,1)$, and $H^{-1} = (S\mathbf{C}(x_1)S)^{-1/2} : H_0^1(-1,1) \to L^2(-1,1)$ is bounded.

The spectrum of the quadratic operator pencil (4.13) is discrete and consists of eigenvalues of finite multiplicity possibly accumulating at ∞.

Let us estimate the number of eigenvalues of H^2 in the interval $(1/r^2, +\infty)$. Let L be a subspace of $L^2(-1,1)$. Then due to the minimax principle, the kth eigenvalue of H^2 can be found from the formula

$$\begin{aligned}\nu_k^+ &= \min_L \max_{x \in L\setminus\{0\}} \frac{(H^2 x, x)_{L^2(-1,1)}}{(x,x)_{L^2(-1,1)}} \\ &\leqslant \underline{C} \min_L \max_{x \in L\setminus\{0\}} \frac{(Sx, Sx)_{L^2(-1,1)}}{(x,x)_{L^2(-1,1)}} \\ &= \underline{C}\mu_k^+,\end{aligned}$$

where μ_k^+ is the kth eigenvalue of the operator $(\mathcal{A}^0)^{-1}$. Similarly, since $\mathbf{C}(x_1)$ is bounded from below, we get the lower bound for ν_k^+, and, consequently,

$$\overline{C}\mu_k^+ \leqslant \nu_k^+ \leqslant \underline{C}\mu_k^+.$$

Thus, we conclude that the number of eigenvalues of the operators H^2 and $(\mathcal{A}^0)^{-1}$ in $(1/r^2, +\infty)$ is asymptotically equivalent. The following inequality characterizes the growth of the eigenvalues of the Sturm–Liouville problem for the operator $\mathcal{A}^0$ (see, for example, [9,10]):

$$\frac{C_1\pi^2k^2}{4} \leqslant \frac{1}{\mu_k^+} \leqslant \frac{C_2\pi^2k^2}{4},$$

where the constants C_1 and C_2 are lower and upper bounds for $a^{\mathrm{eff}}(x_1)$, respectively.

Thus, we conclude that the number of eigenvalues of H^2 in the interval $(1/r^2, +\infty)$ is proportional to r, and, consequently, condition (4.12) is satisfied. By the Keldysh theorem, $N_0(r, \widetilde{\boldsymbol{\Pi}}(\nu^0))$, as well as $N_2(r, \widetilde{\boldsymbol{\Pi}}(\nu^0))$, goes to infinity, as $r \to \infty$, thus, it is true also for $\boldsymbol{\Pi}(\nu^0)$.

Let us show that the eigenvalues of problem (4.9) are real. Suppose

$$\nu^0 = \Re(\nu^0) + i\Im(\nu^0),$$

where $\Re(\nu^0)$ and $\Im(\nu^0)$ represent the real and imaginary parts of ν^0, respectively. Substituting the last expression in (4.10) and setting $v = \overline{u^0}$ we obtain

$$\begin{cases} \displaystyle\int_{-1}^{1} a^{\mathrm{eff}}\left|\frac{du^0}{dx_1}\right|^2 dx_1 + \Re(\nu^0)\int_{-1}^{1} \mathbf{B}|u^0|^2\,dx_1 - \left[\left(\Re(\nu^0)\right)^2 - \left(\Im(\nu^0)\right)^2\right]\int_{-1}^{1} \mathbf{C}|u^0|^2\,dx_1 = 0, \\ \displaystyle\Im(\nu^0)\int_{-1}^{1} \mathbf{B}|u^0|^2\,dx_1 - 2\Im(\nu^0)\Re(\nu^0)\int_{-1}^{1} \mathbf{C}|u^0|^2\,dx_1 = 0. \end{cases}$$

By our assumption $\Im(\nu^0) \neq 0$. Thus, it follows from the last equation that

$$\int_{-1}^{1} \mathbf{B}|u^0|^2\,dx_1 = 2\Re(\nu^0)\int_{-1}^{1} \mathbf{C}|u^0|^2\,dx_1,$$

and, therefore,

$$\int_{-1}^{1} a^{\mathrm{eff}}\left|\frac{du^0}{dx_1}\right|^2 dx_1 + \left[\left(\Re(\nu^0)\right)^2 + \left(\Im(\nu^0)\right)^2\right]\int_{-1}^{1} \mathbf{C}|u^0|^2\,dx_1 = 0$$

that contradicts the positiveness of a^{eff} and $\mathbf{C}$, and, consequently, ν^0 is real. In this way the existence of two infinite sequences of eigenvalues tending to $\pm\infty$ is proved.

Let us show that the algebraic multiplicity of ν^0 is equal to 1. Suppose there exists $\varphi^1 \in H_0^1(-1,1)$ such that

$$\boldsymbol{\Pi}\big(\nu^0\big)\varphi^1(x_1) = -\mathbf{B}(x_1)u^0(x_1) + 2\nu^0\mathbf{C}(x_1)u^0(x_1),$$

where $\boldsymbol{\Pi}$ is defined by (4.9). Using φ^1 as a test function in (4.10) and substituting the resulting equality into the last formula yields

$$2\nu^0\int_{-1}^{1}\mathbf{C}\big(u^0\big)^2\,dx_1 - \int_{-1}^{1}\mathbf{B}\big(u^0\big)^2\,dx_1 = 0.$$

In view of (4.9),

$$\begin{aligned}
0 &= 2\big(\nu^0\big)^2\int_{-1}^{1}\mathbf{C}\big(u^0\big)^2\,dx_1 - \nu^0\int_{-1}^{1}\mathbf{B}\big(u^0\big)^2\,dx_1 \\
&= \big(\nu^0\big)^2\int_{-1}^{1}\mathbf{C}\big(u^0\big)^2\,dx_1 + \int_{-1}^{1}a^{\mathrm{eff}}\left|\frac{du^0}{dx_1}\right|^2\,dx_1 > 0.
\end{aligned}$$

We arrive at contradiction. Thus, the eigenvalues of problem (4.9) are algebraically simple.

Suppose the geometric multiplicity of ν^0 is greater than 1, in other words, there exist two linearly independent eigenfunctions u_1^0 and u_2^0 corresponding to the same ν^0. Choosing C_1 and C_2 in such a way that the function $\tilde{u}^0 = C_1u_1^0 + C_2u_2^0$ satisfies the boundary conditions

$$\tilde{u}^0(-1) = \frac{d\tilde{u}^0}{dx_1}(-1) = 0,$$

we see that, by the uniqueness result for ordinary differential equations, $\tilde{u}^0 = 0$, that contradicts the linear independence of u_1^0 and u_2^0. $\square$

We turn back to constructing the asymptotic expansion. The specific form of the right-hand side of (3.6) suggests the following representation for $u^2(x_1,y)$:

$$\begin{aligned}
u^2(x_1,y) &= N^{2,2}(x_1,y)\frac{d^2u^0(x_1)}{dx_1^2} + N^{2,1}(x_1,y)\frac{du^0(x_1)}{dx_1} \\
&\quad + \nu^0q_2(x_1,y)\frac{du^0(x_1)}{dx_1} + \nu^0N^{2,0}(x_1,y)u^0(x_1) + \nu^1N^{1,0}u^0 \\
&\quad + \big(\nu^0\big)^2r_2(x_1,y)u^0(x_1) + N^{1,1}(x_1,y)\frac{dv^1(x_1)}{dx_1} \\
&\quad + \nu^0N^{1,0}(x_1,y)v^1(x_1) + v^2(x_1),
\end{aligned} \tag{4.14}$$

where $N^{2,2}$, $N^{2,1}$ and $N^{2,0}$ are y_1-periodic solutions of the problems

$$\begin{cases} \mathcal{A}_y N^{2,2}(x_1, y) = \operatorname{div}_y\big(a(x_1, y) N^{1,1}(x_1, y)\big) \\ \qquad\qquad + a_{1j}(x_1, y)\big(\delta_{1j} + \partial_{y_j} N^{1,1}(x_1, y)\big) - a^{\text{eff}}(x_1), & y \in Y, \\ \mathcal{B}_y N^{2,2}(x_1, y) = -\big(a_{\cdot 1}(x_1, y), n\big) N^{1,1}(x_1, y), & y \in \partial Y, \end{cases} \tag{4.15}$$

$$\begin{cases} \mathcal{A}_y N^{2,1}(x_1, y) = \operatorname{div}_y\left(a_{\cdot 1}(x_1, y) \frac{\partial}{\partial x_1} N^{1,1}(x_1, y)\right) \\ \qquad\qquad + \frac{\partial}{\partial x_1}\big[a_{1j}(x_1, y)\big(\delta_{1j} + \partial_{y_j} N^{1,1}(x_1, y)\big)\big] - \frac{da^{\text{eff}}(x_1)}{dx_1}, & y \in Y, \\ \mathcal{B}_y N^{2,1}(x_1, y) = -\big(a_{\cdot 1}(x_1, y), n\big) \frac{\partial}{\partial x_1} N^{1,1}(x_1, y), & y \in \partial Y, \end{cases} \tag{4.16}$$

$$\begin{cases} \mathcal{A}_y N^{2,0}(x_1, y) = \operatorname{div}_y\left(a_{\cdot 1}(x_1, y) \frac{\partial}{\partial x_1} N^{1,0}(x_1, y)\right) \\ \qquad\qquad + \frac{\partial}{\partial x_1}\big(a_{1\cdot}(x_1, y) \nabla_y N^{1,0}(x_1, y)\big) \\ \qquad\qquad - \frac{d}{dx_1} \int\limits_{\partial Y} a_{1\cdot}(x_1, y) \nabla_y N^{1,0}(x_1, y)\, dy, & y \in Y, \\ \mathcal{B}_y N^{2,0}(x_1, y) = \big(a_{\cdot 1}(x_1, y), n\big) \frac{\partial}{\partial x_1} N^{1,0}(x_1, y), & y \in \partial Y. \end{cases} \tag{4.17}$$

The y_1-periodic functions $q_2(x_1, y)$ and $r_2(x_1, y)$ solve the problems

$$\begin{cases} \mathcal{A}_y q_2(x_1, y) = \operatorname{div}_y\big(a_{\cdot 1}(x_1, y) N^{1,0}(x_1, y)\big) + a_{1\cdot}(x_1, y) \nabla_y N^{1,0}(x_1, y) \\ \qquad\qquad + \rho(x_1, y) N^{1,1}(x_1, y), & y \in Y, \\ \mathcal{B}_y q_2(x_1, y) = -\big(a_{\cdot 1}(x_1, y), n\big) N^{1,0}(x_1, y), & y \in \partial Y, \end{cases} \tag{4.18}$$

$$\begin{cases} \mathcal{A}_y r_2(x_1, y) = \rho(x_1, y) N^{1,0}(x_1, y) - \mathbf{C}(x_1), & y \in Y, \\ \mathcal{B}_y r_2(x_1, y) = 0, & y \in \partial Y. \end{cases} \tag{4.19}$$

Bearing in mind (4.3) and (4.4), we see that the compatibility condition for (4.18) is satisfied. Similarly, by (4.7), problem (4.19) is solvable.

Our next goal is to obtain an equation for $v^1(x_1)$. To this end we substitute (4.1) into (2.1) and collect terms of order ε^1 in the equation and of order ε^2 in the boundary condition. In this way we get the problem for $u^3(x_1, y)$,

$$\begin{cases} \mathcal{A}_y u^3(x_1, y) = \operatorname{div}_y\left(a_{\cdot 1}(x_1, y) \frac{\partial u^2}{\partial x_1}(x_1, y)\right) \\ \qquad\qquad + \frac{\partial}{\partial x_1}\big(a_{1\cdot}(x_1, y) \nabla_y u^2(x_1, y)\big) \\ \qquad\qquad + \frac{\partial}{\partial x_1}\left(a_{11}(x_1, y) \frac{du^1(x_1)}{dx_1}\right) \\ \qquad\qquad + \nu^1 \rho(x_1, y) u^1(x_1) + \nu^0 \rho(x_1, y) u^2(x_1), & y \in Y, \\ \mathcal{B}_y u^3(x_1, y) = -a_{i1}(x_1, y) n_i \frac{\partial}{\partial x_1} u^2(x_1, y), & y \in \partial Y, \\ u^3(x_1, y) \text{ is 1-periodic} & \text{in } y_1. \end{cases} \tag{4.20}$$

The compatibility condition for the last problem reads

$$-\frac{d}{dx_1}\left(a^{\mathrm{eff}}\frac{dv^1}{dx_1}\right)+\nu^0\mathbf{B}v^1-\left(\nu^0\right)^2\mathbf{C}v^1=F_1-\nu^1\mathbf{B}u^0+2\nu^1\nu^0\mathbf{C}u^0, \tag{4.21}$$

where $\mathbf{B}(x_1)$ and $\mathbf{C}(x_1)$ are defined by (4.8) and (4.7), respectively, and

$$\begin{aligned} F_1(x_1)=&\frac{d}{dx_1}\int\limits_Y a_{1\cdot}(x_1,y)\nabla_y\tilde{u}_2(x_1,y)\,dy\\ &+\frac{d}{dx_1}\int\limits_Y a_{11}(x_1,y)\frac{\partial\tilde{u}_1}{\partial x_1}(x_1,y)\,dy\\ &+\nu^0\int\limits_Y \rho(x_1,y)\tilde{u}_2(x_1,y)\,dy. \end{aligned}\tag{4.22}$$

Here for brevity we denote

$$\begin{aligned} \tilde{u}_1(x_1,y)&=N^{1,1}(x_1,y)\frac{du^0(x_1)}{dx_1}+\nu^0N^{1,0}(x_1,y)u^0(x_1),\\ \tilde{u}_2(x_1,y)&=N^{2,2}(x_1,y)\frac{d^2u^0(x_1)}{dx_1^2}+N^{2,1}(x_1,y)\frac{du^0(x_1)}{dx_1}\\ &\quad+\nu^0q_2(x_1,y)\frac{du^0(x_1)}{dx_1}+\nu^0N^{2,0}(x_1,y)u^0(x_1)\\ &\quad+\left(\nu^0\right)^2r_2(x_1,y)u^0(x_1) \end{aligned}$$

with the functions $N^{2,2},N^{2,1},N^{2,0},q_2,r_2$ defined in (3.7), (3.8), (3.9), (4.18), (4.19).

As in Section 3, determining the boundary conditions for $v^1(x_1)$ requires constructing the boundary layer correctors in the neighborhood of the points $x=\pm1$.

Denote, as before, $G^-=(0,+\infty)\times Q$ and $G^+=(-\infty,0)\times Q$ the semi-infinite cylinders with the axis directed along y_1 and lateral boundaries $\Sigma^-=(0,+\infty)\times\partial Q$ and $\Sigma^+=(-\infty,0)\times\partial Q$. Consider the following boundary value problem:

$$\begin{cases} -\mathrm{div}_y\left(a\left(\pm1,y_1+\delta,y'\right)\nabla_y w^\pm\right)=0, & y\in G^\pm,\\ \left(a\left(\pm1,y_1+\delta,y'\right)\nabla_y w^\pm,n\right)=0, & y\in\Sigma^\pm,\\ w^\pm\left(0,y'\right)=-N^{1,1}\left(\pm1,\delta,y'\right)\dfrac{du^0}{dx_1}(\pm1)-\nu^0N^{1,0}\left(\pm1,\delta,y'\right)u^0(\pm1), \end{cases}\tag{4.23}$$

with δ being the fractional part of ε^{-1}, which is equal to zero in view of condition (2.2). There exists a unique bounded solution $w^\pm\in C^{1,\alpha}(\overline{G^\pm})$ of problem (4.23) stabilizing to some constant $\hat{w}^\pm$, as $|y_1|\to+\infty$ (see [14]):

$$\begin{aligned} \left|w^\pm\left(y_1,y'\right)-\hat{w}^\pm\right|&\leqslant C_0e^{-\gamma|y_1|},\quad C_0,\gamma>0,\\ \left\|\nabla w^+\right\|_{L^2((n,n+1)\times Q)}&\leqslant Ce^{-\gamma n},\quad \forall n\geqslant0,\\ \left\|\nabla w^-\right\|_{L^2((-(n+1),-n)\times Q)}&\leqslant Ce^{-\gamma n},\quad \forall n\geqslant0, \end{aligned}\tag{4.24}$$

for some $\gamma > 0$. As a boundary condition for $v^1(x_1)$ we choose the uniquely defined constants $\hat{w}^{\pm}$: $v^1(\pm 1) = \hat{w}^{\pm}$. Thus, the problem for v^1 takes the form

$$\begin{cases} \boldsymbol{\Pi}(\nu^0)v^1(x_1) = F_1 - \nu^1\mathbf{B}u^0 + 2\nu^1\nu^0\mathbf{C}u^0, & x_1 \in (-1,1), \\ v^1(\pm 1) = \hat{w}^{\pm}. \end{cases} \tag{4.25}$$

Since $\boldsymbol{\Pi}(\nu^0)u^0 = 0$, problem (4.25) is solvable if the right-hand side is orthogonal to u^0, that is

$$\int_{-1}^{1} F_1 u^0 dx_1 = \nu^1 \int_{-1}^{1} \mathbf{B}\big(u^0\big)^2\, dx_1 - 2\nu^0\nu^1 \int_{-1}^{1} \mathbf{C}\big(u^0\big)^2\, dx_1 + \overline{F},$$

where the constant $\overline{F}$ is given by

$$\overline{F} = \left(a^{\mathrm{eff}}(1)\frac{du^0}{dx_1}(1)\hat{w}^+ - a^{\mathrm{eff}}(-1)\frac{du^0}{dx_1}(-1)\hat{w}^-\right). \tag{4.26}$$

It follows easily from (4.9) that

$$\int_{-1}^{1} \mathbf{B}(x_1)\big(u^0(x_1)\big)^2\, dx_1 - 2\nu^0 \int_{-1}^{1} \mathbf{C}(x_1)\big(u^0(x_1)\big)^2\, dx_1 \neq 0.$$

Thus, ν^1 can be defined in such a way that (4.25) possesses a solution. Namely,

$$\nu^1 = \left\{\int_{-1}^{1} F_1 u^0 dx_1 - \overline{F}\right\}\left\{\int_{-1}^{1} \big[\mathbf{B}(x_1) - 2\nu^0\mathbf{C}(x_1)\big]\big(u^0(x_1)\big)^2\, dx_1\right\}^{-1}. \tag{4.27}$$

We fix the choice of the function v^1 by setting

$$\int_{-1}^{1} v^1(x_1)u^0(x_1)\, dx_1 = 0.$$

Note that, in view of the regularity assumptions (H0), $v^1 \in C^{2,\alpha}[-1,1]$, $\alpha > 0$. In this way the function

$$u^0(x_1) + \varepsilon N^{1,1}\left(x_1, \frac{x}{\varepsilon}\right)\frac{du^0(x_1)}{dx_1} + \varepsilon\nu^0 N^{1,0}\left(x_1, \frac{x}{\varepsilon}\right)u^0(x_1) + \varepsilon v^1(x_1) + \varepsilon u^{\varepsilon}_{\mathrm{bl}}(x),$$

with

$$\begin{aligned} u^{\varepsilon}_{\mathrm{bl}}(x) &= \tilde{u}^{\varepsilon}_{\mathrm{bl}}(y)|_{y=x/\varepsilon} \\ &= \left(w^+\left(y_1 - \frac{1}{\varepsilon}, y'\right) - \hat{w}^+\right) + \left(w^-\left(y_1 + \frac{1}{\varepsilon}, y'\right) - \hat{w}^-\right)\bigg|_{y=x/\varepsilon}, \end{aligned} \tag{4.28}$$

satisfies the homogeneous Dirichlet boundary conditions at $x_1 = \pm 1$.

4.2. Justification procedure in the case $\langle \rho(x_1,\cdot)\rangle = 0$

Let $\nu_j^{0,\pm}$ be the eigenvalues and $u_j^{0,\pm}$ the corresponding eigenfunctions of problem (4.9). For any $j \in \mathbb{N}$ we denote

$$\begin{aligned} U_j^{\varepsilon,\pm}(x) = u_j^{0,\pm}(x_1) + \varepsilon N^{1,1}\left(x_1, \frac{x}{\varepsilon}\right)\frac{du_j^{0,\pm}(x_1)}{dx_1} + \varepsilon \nu_j^{0,\pm} N^{1,0}\left(x_1, \frac{x}{\varepsilon}\right) u_j^{0,\pm}(x_1) \\ + \varepsilon v_j^{1,\pm}(x_1) + \varepsilon u_{\mathrm{bl}}^{\varepsilon}(x), \end{aligned} \tag{4.29}$$

where $u_j^{0,\pm}$, $N^{1,1}$, $N^{1,0}$ and $v_j^{1,\pm}$ solve problems (4.9), (4.3), (4.4) and (4.25), respectively (with $u^0 = u_j^{0,\pm}$ and $\nu^0 = \nu_j^{0,\pm}$). The boundary layer corrector $u_{\mathrm{bl}}^{\varepsilon}$ is defined by (4.28) and (4.23).

Let us emphasize that, due to the presence of the boundary layer terms, the function $U_j^{\varepsilon,\pm}$ satisfies the homogeneous Dirichlet boundary conditions on $S_{\pm 1}$, and, as a consequence, belong to the space $\mathcal{H}^{\varepsilon}$.

We denote by $\nu_j^{1,\pm}$ a constant defined by (4.27) with $u^0 = u_j^{0,\pm}$ and $\nu^0 = \nu_j^{0,\pm}$. For the readers convenience we recall its definition.

$$\nu_j^{1,\pm} = \left\{ \int_{-1}^{1} F_1 u_j^{0,\pm}\, dx_1 - \overline{F} \right\} \left\{ \int_{-1}^{1} \left[\mathbf{B} - 2\nu_j^{0,\pm}\mathbf{C}\right]\left(u_j^{0,\pm}\right)^2 dx_1 \right\}^{-1}, \tag{4.30}$$

where $(\nu_j^{0,\pm}, u_j^{0,\pm})$ are eigenpairs of problem (4.9), the functions $\mathbf{B}(x_1), \mathbf{C}(x_1)$ are defined by (4.8) and (4.7), respectively; the function $F_1(x_1)$ and the constant $\overline{F}$ are given by (4.22) and (4.26) with $u^0 = u_j^{0,\pm}$ and $\nu^0 = \nu_j^{0,\pm}$.

The goal of this section is to prove the following result.

Theorem 4.3. *Let conditions* (H0)–(H3) *be fulfilled, and suppose that $\langle \rho(x_1, y)\rangle = 0$ for any $x_1 \in [-1,1]$. If $(\lambda_j^{\varepsilon,\pm}, u_j^{\varepsilon,\pm})$ are eigenpairs of problem* (2.1)*, and $(\nu_j^{0,\pm}, u_j^{0,\pm})$ are eigenpairs of the operator pencil* (4.9)*, then*

(i) *For any j, there exist ε_j and $C_j > 0$ such that*

$$\left|\lambda_j^{\varepsilon,\pm} - \left(\varepsilon^{-1}\nu_j^{0,\pm} + \nu_j^{1,\pm}\right)\right| \leqslant C_j \varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_j].$$

Here $\nu_j^{1,\pm}$ is defined in (4.30)*.*

(ii) *For any j,*

$$\left\| u_j^{\varepsilon,\pm} - U_j^{\varepsilon,\pm} \right\|_{H^1(G_\varepsilon)} \leqslant C_j \varepsilon \varepsilon^{\frac{d-1}{2}},$$

where $U_j^{\varepsilon,\pm}$ is defined by (4.29)*. Moreover, the "almost eigenfunctions" satisfy the almost orthogonality and normalization condition*

$$\left| \frac{\varepsilon^{-(d-1)}}{|Q|}\left(a^{\varepsilon}\nabla U_i^{\varepsilon,\pm}, \nabla U_j^{\varepsilon,\pm}\right)_{L^2(G_\varepsilon)} - \delta_{ij} \right| \leqslant C_j \varepsilon.$$

(iii) *For any $j \in \mathbb{N}$, $\lambda_j^{\varepsilon,\pm}$ are simple, for sufficiently small $\varepsilon > 0$.*

Proof of Theorem 3.2. As in Section 3, we make use of Lemma 3.2. Denote

$$\mathcal{U}_j^{\varepsilon,\pm} = \|U_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}^{-1} U_j^{\varepsilon,\pm}.$$

Lemma 4.1. *For any $j \in \mathbb{N}$ there is $\varepsilon_j > 0$ such that*

$$\big\|\mathcal{K}^\varepsilon \mathcal{U}_j^{\varepsilon,\pm} - \big(\varepsilon^{-1}\nu_j^{0,\pm} + \nu_j^{1,\pm}\big)^{-1}\mathcal{U}_j^{\varepsilon,\pm}\big\|_{\mathcal{H}^\varepsilon} \leqslant C_j\varepsilon^2, \quad \varepsilon < \varepsilon_j, \tag{4.31}$$

where the constant C_j depends only on j.

Proof. After straightforward rearrangements and integration by parts we have

$$\begin{aligned} I^\varepsilon &\equiv \big\|\mathcal{K}^\varepsilon \mathcal{U}_j^{\varepsilon,\pm} - \big(\varepsilon^{-1}\nu_j^{0,\pm} + \nu_j^{1,\pm}\big)^{-1}\mathcal{U}_j^{\varepsilon,\pm}\big\|_{\mathcal{H}^\varepsilon} \\ &= \frac{\|U_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}^{-1}}{|\varepsilon^{-1}\nu_j^{0,\pm} + \nu_j^{1,\pm}|} \sup_{\substack{w\in\mathcal{H}^\varepsilon \\ \|w\|_{\mathcal{H}^\varepsilon}=1}} \Big| \big(\mathcal{A}^\varepsilon U_j^{\varepsilon,\pm}, w\big)_{L^2(G_\varepsilon)} - \big(\varepsilon^{-1}\nu_j^{0,\pm} + \nu_j^{1,\pm}\big)\big(\rho^\varepsilon U_j^{\varepsilon,\pm}, w\big)_{L^2(G_\varepsilon)} \\ &\quad + \int\limits_{\Sigma_\varepsilon} \big(a^\varepsilon \nabla U_j^{\varepsilon,\pm}, n\big) w\, d\sigma\Big|. \end{aligned}$$

It is convenient to use the notation

$$U_j^{\varepsilon,\pm}(x) = u_j^{0,\pm}(x_1) + \varepsilon u_j^{1,\pm}(x_1, y)|_{y=x/\varepsilon} + \varepsilon u_{\mathrm{bl}}^\varepsilon(x).$$

Recall that $u_j^{0,\pm}(x_1) \in C^{2,\alpha}[-1,1]$ and $u_j^{1,\pm}(x_1, y) \in C^{1,\alpha}([-1,1]\times\overline{Y})$. In this way we obtain

$$\begin{aligned} I^\varepsilon = \frac{\|U_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}^{-1}}{|\varepsilon^{-1}\nu_j^{0,\pm} + \nu_j^{1,\pm}|} &\sup_{\substack{w\in\mathcal{H}^\varepsilon \\ \|w\|_{\mathcal{H}^\varepsilon}=1}} \Big| \int\limits_{G_\varepsilon} \mathcal{A}^\varepsilon\Big(u_j^{0,\pm}(x_1) + \varepsilon u_j^{1,\pm}\Big(x_1, \frac{x}{\varepsilon}\Big)\Big) w(x)\, dx \\ &- \big(\varepsilon^{-1}\nu_j^{0,\pm} + \nu_j^{1,\pm}\big) \int\limits_{G_\varepsilon} \rho^\varepsilon(x)\big(u_j^{0,\pm}(x_1) + \varepsilon u_j^{1,\pm}(x_1, y)\big|_{y=x/\varepsilon}\big) w(x)\, dx \\ &+ \varepsilon \int\limits_{\Sigma_\varepsilon} \big(a_{\cdot 1}^\varepsilon, n\big) \frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1, y)\big|_{y=x/\varepsilon} w\, d\sigma + \varepsilon\big(\mathcal{A}^\varepsilon u_{\mathrm{bl}}^\varepsilon, w\big)_{L^2(G_\varepsilon)} \\ &- \big(\nu_j^{0,\pm} + \varepsilon\nu_j^{1,\pm}\big)\big(\rho^\varepsilon u_{\mathrm{bl}}^\varepsilon, w\big)_{L^2(G_\varepsilon)} + \int\limits_{\Sigma_\varepsilon} \big(a_{\cdot 1}^\varepsilon, n\big) \nabla_y \tilde{u}_{\mathrm{bl}}^\varepsilon(y)\big|_{y=x/\varepsilon} w\, d\sigma\Big|. \end{aligned}$$

The last three terms containing $u_{\mathrm{bl}}^\varepsilon$ can be estimated exactly like in Lemma 3.1,

$$\begin{aligned} &\big|\varepsilon\big(\mathcal{A}^\varepsilon u_{\mathrm{bl}}^\varepsilon, w\big)_{L^2(G_\varepsilon)} + \varepsilon\big(a^\varepsilon \nabla u_{\mathrm{bl}}^\varepsilon w, n\big)_{L^2(\Sigma_\varepsilon)} - \big(\nu_j^{0,\pm} + \varepsilon\nu_j^{1,\pm}\big)\big(\rho^\varepsilon u_{\mathrm{bl}}^\varepsilon, w\big)_{L^2(G_\varepsilon)}\big| \\ &\quad \leqslant C\varepsilon\varepsilon^{(d-1)/2}\|w\|_{H^1(G_\varepsilon)}, \quad w \in \mathcal{H}^\varepsilon. \end{aligned} \tag{4.32}$$

Then

$$\int_{G_\varepsilon} \mathcal{A}^\varepsilon\left(u_j^{0,\pm}(x_1) + \varepsilon u_j^{1,\pm}\left(x_1, \frac{x}{\varepsilon}\right)\right) w(x)\,dx$$

$$- \left(\varepsilon^{-1}\nu_j^{0,\pm} + \nu_j^{1,\pm}\right) \int_{G_\varepsilon} \rho^\varepsilon(x)\left(u_j^{0,\pm}(x_1) + \varepsilon u_j^{1,\pm}(x_1, y)\big|_{y=x/\varepsilon}\right) w(x)\,dx$$

$$+ \varepsilon \int_{\Sigma_\varepsilon} \left(a_{\cdot 1}^\varepsilon, n\right) \frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1, y)\big|_{y=x/\varepsilon} w\,d\sigma$$

$$= \varepsilon^0 \left(I_0^\varepsilon, w\right)_{L^2(G_\varepsilon)} + \varepsilon^1 \left(I_1^\varepsilon, w\right)_{L^2(G_\varepsilon)} + \varepsilon \int_{\Sigma_\varepsilon} \left(a_{\cdot 1}^\varepsilon, n\right) \frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1, y)\big|_{y=x/\varepsilon} w\,d\sigma;$$

here

$$\begin{aligned} I_0^\varepsilon(x) &= I_0(x_1, y)|_{y=x/\varepsilon} \\ &= -\frac{\partial}{\partial x_1}\left(a_{1\cdot}(x_1, y)\nabla_y u_j^{1,\pm}(x_1, y)\right) - \frac{\partial}{\partial x_1}\left(a_{11}(x_1, y)\frac{du_j^{0,\pm}}{dx_1}(x_1)\right) \\ &\quad - \nu_j^{1,\pm}\rho(x_1, y)u_j^{0,\pm}(x_1) - \nu_j^{0,\pm}u_j^{1,\pm}(x_1, y)\rho(x_1, y)|_{y=x/\varepsilon}, \end{aligned}$$

$$I_1^\varepsilon(x) = -\left\{\operatorname{div}_x + \frac{1}{\varepsilon}\operatorname{div}_y\right\}\left(a_{\cdot 1}(x_1, y)\frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1, y)\right) - \nu_j^{1,\pm}\rho(x_1, y)u_j^{1,\pm}(x_1, y)|_{y=x/\varepsilon}.$$

By (4.9), the average of $I_0(x_1, y) \in C^{1,\alpha}([-1, 1]; C^\alpha(\overline{Y}))$ over Y is equal to zero, thus, by Lemma 3.4

$$\left|\left(I_0^\varepsilon, w\right)_{L^2(G_\varepsilon)}\right| \leqslant C\varepsilon\varepsilon^{\frac{d-1}{2}}\|w\|_{H^1(G_\varepsilon)}.$$

Integrating by parts and bearing in mind the regularity properties of $u_j^{1,\pm}$ and assumption (H0), one can see that

$$\begin{aligned} &\left|\left(I_1^\varepsilon, w\right)_{L^2(G_\varepsilon)} + \int_{\Sigma_\varepsilon}\left(a_{\cdot 1}^\varepsilon, n\right)\frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1, y)\big|_{y=x/\varepsilon} w\,d\sigma\right| \\ &= \left|\int_{G_\varepsilon}\left(a_{\cdot 1}^\varepsilon(x), \nabla w\right)\frac{\partial u_j^{1,\pm}}{\partial x_1}\bigg|_{y=x/\varepsilon} dx - \nu_j^{1,\pm}\int_{G_\varepsilon}\rho^\varepsilon(x)u_j^{1,\pm}\left(x_1, \frac{x}{\varepsilon}\right)w(x)\,dx\right| \\ &\leqslant C\varepsilon^{\frac{d-1}{2}}\|w\|_{H^1(G_\varepsilon)}. \end{aligned}$$

Thus,

$$I^\varepsilon \leqslant C\frac{\|U_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}^{-1}}{|\varepsilon^{-1}\nu_j^{0,\pm} + \nu_j^{1,\pm}|}\varepsilon\varepsilon^{\frac{d-1}{2}}. \tag{4.33}$$

Let us estimate $\|U_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}$. Rearranging the terms in the expression for $(U_i^{\varepsilon,\pm},U_j^{\varepsilon,\pm})_{\mathcal{H}^\varepsilon}$ yields

$$\big(U_i^{\varepsilon,\pm},U_j^{\varepsilon,\pm}\big)_{\mathcal{H}^\varepsilon}=J_{xx}^\varepsilon+J_{xy}^\varepsilon+J_{yx}^\varepsilon+J_{yy}^\varepsilon,$$

where

$$J_{xx}^\varepsilon=\int\limits_{G_\varepsilon}a_{11}^\varepsilon\frac{du_i^{0,\pm}}{dx_1}\frac{du_j^{0,\pm}}{dx_1}\,dx+\varepsilon\int\limits_{G_\varepsilon}a_{11}^\varepsilon\frac{du_i^{0,\pm}}{dx_1}\frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1,y)\,dx$$
$$+\varepsilon\int\limits_{G_\varepsilon}a_{11}^\varepsilon\frac{du_j^{0,\pm}}{dx_1}\frac{\partial u_i^{1,\pm}}{\partial x_1}(x_1,y)\,dx+\varepsilon^2\int\limits_{G_\varepsilon}a_{11}^\varepsilon\frac{\partial u_i^{1,\pm}}{\partial x_1}\frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1,y)\,dx,$$

$$J_{xy}^\varepsilon=\int\limits_{G_\varepsilon}\big(a_{\cdot 1}^\varepsilon,\nabla_yN^{1,1}\big)\big|_{y=x/\varepsilon}\frac{du_i^{0,\pm}}{dx_1}\frac{du_j^{0,\pm}}{dx_1}\,dx+\int\limits_{G_\varepsilon}\big(a_{\cdot 1}^\varepsilon,\nabla_y\tilde u_{\rm bl}^\varepsilon\big)\big|_{y=x/\varepsilon}\frac{du_i^{0,\pm}}{dx_1}\,dx$$
$$+\nu_j^{0,\pm}\int\limits_{G_\varepsilon}\big(a_{\cdot 1}^\varepsilon,\nabla_yN^{1,0}\big)\big|_{y=x/\varepsilon}\frac{du_i^{0,\pm}}{dx_1}u_j^{0,\pm}\,dx+\varepsilon\int\limits_{G_\varepsilon}\big(a_{\cdot 1}^\varepsilon,\nabla_yu_j^{1,\pm}\big)\big|_{y=x/\varepsilon}\frac{\partial u_i^{1,\pm}}{\partial x_1}(x_1,y)\,dx$$
$$+\varepsilon\int\limits_{G_\varepsilon}\big(a_{\cdot 1}^\varepsilon,\nabla_y\tilde u_{\rm bl}^\varepsilon\big)\big|_{y=x/\varepsilon}\frac{\partial u_i^{1,\pm}}{\partial x_1}(x_1,y)\,dx,$$

$$J_{yx}^\varepsilon=\int\limits_{G_\varepsilon}\big(a_{\cdot 1}^\varepsilon,\nabla_yN^{1,1}\big)\big|_{y=x/\varepsilon}\frac{du_i^{0,\pm}}{dx_1}\frac{du_j^{0,\pm}}{dx_1}\,dx+\int\limits_{G_\varepsilon}\big(a_{\cdot 1}^\varepsilon,\tilde u_{\rm bl}^\varepsilon\big)\big|_{y=x/\varepsilon}\frac{du_j^{0,\pm}}{dx_1}\,dx$$
$$+\nu_i^{0,\pm}\int\limits_{G_\varepsilon}\big(a_{\cdot 1}^\varepsilon,\nabla_yN^{1,0}\big)\big|_{y=x/\varepsilon}\frac{du_j^{0,\pm}}{dx_1}u_i^{0,\pm}\,dx++\varepsilon\int\limits_{G_\varepsilon}\big(a_{\cdot 1}^\varepsilon,\nabla_yu_i^{1,\pm}\big)\big|_{y=x/\varepsilon}\frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1,y)\,dx$$
$$+\varepsilon\int\limits_{G_\varepsilon}\big(a_{\cdot 1}^\varepsilon,\nabla_y\tilde u_{\rm bl}^\varepsilon\big)\big|_{y=x/\varepsilon}\frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1,y)\,dx,$$

$$J_{yy}^\varepsilon=\int\limits_{G_\varepsilon}\big(a^\varepsilon\nabla_yN^{1,1},\nabla_yN^{1,1}\big)\big|_{y=x/\varepsilon}\frac{du_i^{0,\pm}}{dx_1}\frac{du_j^{0,\pm}}{dx_1}\,dx$$
$$+\nu_i^{0,\pm}\int\limits_{G_\varepsilon}\big(a^\varepsilon\nabla_yN^{1,0},\nabla_yN^{1,1}\big)\big|_{y=x/\varepsilon}u_i^{0,\pm}\frac{du_j^{0,\pm}}{dx_1}\,dx$$
$$+\nu_j^{0,\pm}\int\limits_{G_\varepsilon}\big(a^\varepsilon\nabla_yN^{1,0},\nabla_yN^{1,1}\big)\big|_{y=x/\varepsilon}u_j^{0,\pm}\frac{du_i^{0,\pm}}{dx_1}\,dx$$
$$+\nu_i^{0,\pm}\nu_j^{0,\pm}\int\limits_{G_\varepsilon}\big(a^\varepsilon\nabla_yN^{1,0},\nabla_yN^{1,0}\big)\big|_{y=x/\varepsilon}u_i^{0,\pm}u_j^{0,\pm}\,dx.$$

There are several "typical" terms in the expressions for J^ε_{xx}, J^ε_{xy}, J^ε_{yx} and J^ε_{yy} to be estimated. For example, using the regularity properties of $a(x_1, y)$, $u_j^{0,\pm}$ and $u_j^{1,\pm}$ we get

$$\left|\varepsilon \int\limits_{G_\varepsilon} a^\varepsilon_{11} \frac{du_i^{0,\pm}}{dx_1} \frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1, y)\, dx\right| \leqslant C\varepsilon |G_\varepsilon| = C\varepsilon \varepsilon^{d-1}.$$

Then, taking into account the exponential decay of $\tilde{u}^\varepsilon_{\mathrm{bl}}$ one can see that

$$\left|\varepsilon \int\limits_{G_\varepsilon} \big(a_{\cdot 1}, \nabla_y \tilde{u}^\varepsilon_{\mathrm{bl}}\big)\big|_{y=x/\varepsilon} \frac{du_i^{0,\pm}}{dx_1}\, dx\right| \leqslant C\varepsilon^d \int\limits_{-1/\varepsilon}^{1/\varepsilon} dx_1 \int\limits_Q \big|\nabla_y \tilde{u}^\varepsilon_{\mathrm{bl}}\big|\, dy' \leqslant C\varepsilon\varepsilon^{d-1}.$$

In view of boundedness of $\partial u_j^{1,\pm}/\partial x_1$ and periodicity of $N^{1,1}$, $N^{1,0}$

$$\begin{aligned}
&\left|\varepsilon \int\limits_{G_\varepsilon} \big(a_{\cdot 1}, \nabla_y u_j^{1,\pm}\big)\big|_{y=x/\varepsilon} \frac{\partial u_i^{1,\pm}}{\partial x_1}(x_1, y)\, dx\right| \\
&\quad \leqslant C\varepsilon \int\limits_{G_\varepsilon} \big[\big|\nabla_y N^{1,1}(x_1, y)\big| + \big|\nabla_y N^{1,0}(x_1, y)\big|\big]\big|_{y=x/\varepsilon}\, dx \\
&\quad \leqslant C\varepsilon^d \max_{x_1\in[-1,1]} \left[\int\limits_Y \big|\nabla_y N^{1,1}(x_1, y)\big|\, dy + \int\limits_Y \big|\nabla_y N^{1,0}(x_1, y)\big|\, dy\right] \\
&\quad \leqslant C\varepsilon\varepsilon^{d-1}.
\end{aligned}$$

Notice that

$$\int\limits_Y \big\{\big(a_{1\cdot}(x_1, y), \nabla_y N^{1,0}(x_1, y)\big) + \big(a_{1\cdot}(x_1, y)\nabla_y N^{1,0}(x_1, y), \nabla_y N^{1,1}(x_1, y)\big)\big\}\, dy = 0,$$

and, thus, by Lemma 3.4

$$\begin{aligned}
&\bigg|\nu_i^{0,\pm} \int\limits_{G_\varepsilon} \big\{\big(a_{1\cdot}(x_1, y), \nabla_y N^{1,0}(x_1, y)\big) \\
&\qquad + \big(a_{1\cdot}(x_1, y)\nabla_y N^{1,0}(x_1, y), \nabla_y N^{1,1}(x_1, y)\big)\big\}\big|_{y=x/\varepsilon} u_i^{0,\pm}(x_1) \frac{du_j^{0,\pm}}{dx_1}(x_1)\, dx\bigg| \\
&\quad \leqslant C\varepsilon\varepsilon^{d-1}.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&\biggl| \nu_j^{0,\pm} \int\limits_{G_\varepsilon} \bigl\{ \bigl(a_{1\cdot}(x_1, y), \nabla_y N^{1,0}(x_1, y) \bigr) \\
&\quad + \bigl(a_{1\cdot}(x_1, y) \nabla_y N^{1,0}(x_1, y), \nabla_y N^{1,1}(x_1, y) \bigr) \bigr\} \big|_{y=x/\varepsilon} u_j^{0,\pm}(x_1) \frac{du_i^{0,\pm}}{dx_1}(x_1)\, dx \biggr| \\
&\leqslant C\varepsilon \varepsilon^{d-1}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\bigl(U_i^{\varepsilon,\pm}, U_j^{\varepsilon,\pm} \bigr)_{\mathcal{H}^\varepsilon} = {} & \int\limits_{G_\varepsilon} \bigl(a_{11}^\varepsilon + a_{\cdot 1}^\varepsilon \nabla_y N^{1,1}(x_1, y) \bigr) \big|_{y=x/\varepsilon} \frac{du_i^{0,\pm}}{dx_1} \frac{du_j^{0,\pm}}{dx_1}\, dx \\
& + \int\limits_{G_\varepsilon} \bigl\{ \bigl(a_{1\cdot}^\varepsilon, \nabla_y N^{1,1} \bigr) + \bigl(a^\varepsilon \nabla_y N^{1,1}, \nabla_y N^{1,1} \bigr) \bigr\} \big|_{y=x/\varepsilon} \frac{du_i^{0,\pm}}{dx_1} \frac{du_j^{0,\pm}}{dx_1}\, dx \\
& + \nu_i^{0,\pm} \nu_j^{0,\pm} \int\limits_{G_\varepsilon} \bigl(a^\varepsilon \nabla_y N^{1,0}, \nabla_y N^{1,0} \bigr) \big|_{y=x/\varepsilon} u_i^{0,\pm} u_j^{0,\pm}\, dx + \mathcal{R}_\varepsilon,
\end{aligned}
$$

where $|\mathcal{R}_\varepsilon| \leqslant C\varepsilon\varepsilon^{d-1}$.

Recalling the definition of the effective coefficient a^{eff} and of the function $\mathbf{C}(x_1)$ (see (3.3) and (4.7), respectively), by Lemma 3.4, we have

$$
\begin{aligned}
&\biggl| \bigl(U_i^{\varepsilon,\pm}, U_j^{\varepsilon,\pm} \bigr)_{\mathcal{H}^\varepsilon} - \varepsilon^{d-1} |Q| \int\limits_{-1}^{1} a^{\mathrm{eff}}(x_1) \frac{du_i^{0,\pm}}{dx_1} \frac{du_j^{0,\pm}}{dx_1}\, dx_1 \\
&\quad - \nu_i^{0,\pm} \nu_j^{0,\pm} \varepsilon^{d-1} |Q| \int\limits_{-1}^{1} \mathbf{C}(x_1) u_i^{0,\pm} u_j^{0,\pm}\, dx_1 \biggr| \\
&\leqslant C\varepsilon\varepsilon^{d-1}.
\end{aligned}
$$

In view of the normalization condition (4.11),

$$
\left| \frac{\varepsilon^{-(d-1)}}{|Q|} \bigl(U_i^{\varepsilon,\pm}, U_j^{\varepsilon,\pm} \bigr)_{\mathcal{H}^\varepsilon} - \delta_{ij} \right| \leqslant C\varepsilon. \tag{4.34}
$$

Estimate (4.34) implies the lower bound for the norm $\|U_i^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}$:

$$
\bigl\| U_i^{\varepsilon,\pm} \bigr\|_{\mathcal{H}^\varepsilon} \geqslant \frac{|Q|^{1/2}}{2} \varepsilon^{\frac{d-1}{2}}, \quad \varepsilon < \varepsilon_i. \tag{4.35}
$$

Combining (4.33) and (4.35) yields the desired estimate (4.31). Lemma 4.1 is proved. □

We turn back to the proof of Theorem 4.3. By Lemma 3.2, in view of estimate (4.31), for any j there exists an eigenvalue $\mu_q^{\varepsilon,\pm}$ of the operator $\mathcal{K}^\varepsilon$ such that

$$\left|\mu_q^{\varepsilon,\pm} - \left(\varepsilon^{-1}\nu_j^{0,\pm} + \nu_j^{1,\pm}\right)^{-1}\right| < C_j\varepsilon^2, \quad \varepsilon < \varepsilon_j.$$

Considering the relation $\lambda_j^{\varepsilon,\pm} = (\mu_j^{\varepsilon,\pm})^{-1}$, we get

$$\left|\lambda_q^{\varepsilon,\pm} - \left(\varepsilon^{-1}\nu_j^{0,\pm} + \nu_j^{1,\pm}\right)\right| < C_j\varepsilon, \quad \varepsilon < \varepsilon_j. \tag{4.36}$$

Our next goal is to prove that, for any j, there is a unique $\lambda_j^{\varepsilon,\pm}$ satisfying inequality (4.36). The proof consists of three steps presented below. Lemma 4.2 gives the lower and upper bounds for $\lambda_j^{\varepsilon,\pm}$. Lemma 4.3 claims that, up to a subsequence, $\varepsilon\lambda_q^{\varepsilon,\pm}$ converges to an eigenvalue of the operator pencil (4.9). Then we show that there exists a unique eigenvalue $\lambda_j^{\varepsilon,\pm}$ satisfying (4.36).

Lemma 4.2. *For any j, the estimate holds true*

$$0 < m \leqslant \varepsilon\left|\lambda_j^{\varepsilon,\pm}\right| \leqslant M_j \tag{4.37}$$

with some constants m and M_j.

Proof. By the definition of the operator $\mathcal{K}^\varepsilon$,

$$\left\|\mathcal{K}^\varepsilon\right\| = \sup_{(v,v)_{\mathcal{H}^\varepsilon}=1}\left(\mathcal{K}^\varepsilon v, v\right)_{\mathcal{H}^\varepsilon} = \sup_{(v,v)_{\mathcal{H}^\varepsilon}=1}\left(\rho^\varepsilon v, v\right)_{L^2(G_\varepsilon)}.$$

Arguments similar to those in Lemma 3.4 yield

$$\left|\int_{G_\varepsilon}\rho^\varepsilon(v)^2\,dx\right| \leqslant C\varepsilon\|v\|^2_{H^1(G_\varepsilon)}.$$

Thus,

$$\left\|\mathcal{K}^\varepsilon\right\| \leqslant C\varepsilon, \qquad \left|\mu_j^{\varepsilon,\pm}\right| \leqslant C\varepsilon, \quad \forall j.$$

Considering the equality $\lambda_j^{\varepsilon,\pm} = (\mu_j^{\varepsilon,\pm})^{-1}$, we obtain the lower bound in (4.37). The upper bound in (4.37) follows easily from estimate (4.36). Lemma 4.2 is proved. □

Lemma 4.3. *For any j, up to a subsequence, $\varepsilon\lambda_j^{\varepsilon,\pm}$ converges to an eigenvalue ν_* of problem* (4.9)*.*

Proof. In view of Lemma 4.2, $\varepsilon\lambda_j^{\varepsilon,\pm}$ converges to some $\nu_* \in \mathbb{R}\setminus\{0\}$. Let us show that ν_* is an eigenvalue of the operator pencil (4.9). The weak formulation of problem (2.1) has the form

$$\left(\mathcal{A}^\varepsilon u_i^{\varepsilon,\pm} - \lambda_i^{\varepsilon,\pm}\rho^\varepsilon u_i^{\varepsilon,\pm}, w\right)_{L^2(G_\varepsilon)} = 0, \quad w \in \mathcal{H}^\varepsilon.$$

Integrating by parts leads to the equality

$$\left(u_i^{\varepsilon,\pm}, \mathcal{A}^\varepsilon w - \lambda_i^{\varepsilon,\pm}\rho^\varepsilon w\right)_{L^2(G_\varepsilon)} + \int_{\Sigma^\varepsilon}\left(a^\varepsilon\nabla w, n\right)u_i^{\varepsilon,\pm}\,d\sigma = 0, \quad w \in \mathcal{H}^\varepsilon. \tag{4.38}$$

By the normalization condition (2.6), $u_i^{\varepsilon,\pm}(x) \in L^2(K_d, \mu_\varepsilon)$ converges strongly in the variable space $L^2(K_d, \mu_\varepsilon)$ to a function $u_*(x_1) \in L^2(K_d, \mu_*)$, $K_d = [-1, 1]^d$ (see Lemma 3.6 for the details). Thus, showing that $\mathcal{A}^\varepsilon w - \lambda_i^{\varepsilon,\pm}\rho^\varepsilon w$ converges weakly in $L^2(K_d, \mu_\varepsilon)$ will allow us to pass to the limit in (4.38). For this purpose we construct a test function

$$V^\varepsilon(x) = v(x_1) + \varepsilon N^{1,1}\left(x_1, \frac{x}{\varepsilon}\right)\frac{dv(x_1)}{dx_1} + \varepsilon^2\lambda_i^{\varepsilon,\pm}N^{1,0}\left(x_1, \frac{x}{\varepsilon}\right)v(x_1), \quad v \in C_0^\infty[-1, 1].$$

We would like to emphasize that, in contrast with ansatz (4.29), we do not add the boundary layer corrector here. The reason is that $v(x_1)$ is equal to zero at points ± 1 together with all its derivatives, that yields $V^\varepsilon(\pm 1, x') = 0$.

Simple transformations yield

$$\mathcal{A}^\varepsilon V^\varepsilon - \lambda_i^{\varepsilon,\pm}\rho^\varepsilon V^\varepsilon = J_1^\varepsilon(x_1, y) + J_2^\varepsilon(x_1, y)|_{y=x/\varepsilon},$$

where

$$\begin{aligned} J_1^\varepsilon(x_1, y) = &-\frac{\partial}{\partial x_1}\left(a_{1\cdot}(x_1, y)\nabla_y N^{1,1}(x_1, y)\frac{dv(x_1)}{dx_1}\right) \\ &- \frac{\partial}{\partial x_1}\left(a_{11}(x_1, y)\frac{dv(x_1)}{dx_1}\right) \\ &- \varepsilon\lambda_i^{\varepsilon,\pm}\frac{\partial}{\partial x_1}\big(a_{1\cdot}(x_1, y)\nabla_y N^{1,0}(x_1, y)v(x_1)\big) \\ &- \varepsilon\lambda_i^{\varepsilon,\pm}\rho(x_1, y)N^{1,1}(x_1, y)\frac{dv(x_1)}{dx_1} \\ &- \big(\varepsilon\lambda_i^{\varepsilon,\pm}\big)^2\rho(x_1, y)N^{1,0}(x_1, y)v(x_1), \end{aligned}$$

$$\begin{aligned} J_2^\varepsilon(x_1, y) = &-\varepsilon\left\{\operatorname{div}_x + \frac{1}{\varepsilon}\operatorname{div}_y\right\}\left[a_{\cdot 1}(x_1, y)\frac{\partial}{\partial x_1}\left(N^{1,1}(x_1, y)\frac{dv(x_1)}{dx_1}\right)\right] \\ &- \varepsilon^2\lambda_i^{\varepsilon,\pm}\left\{\operatorname{div}_x + \frac{1}{\varepsilon}\operatorname{div}_y\right\}\left[a_{\cdot 1}(x_1, y)\frac{\partial}{\partial x_1}\big(N^{1,0}(x_1, y)v(x_1)\big)\right]. \end{aligned}$$

In view of (3.3), (4.8) and (4.7),

$$\int_Y J_1^\varepsilon(x_1, y)\,dy = -\frac{\partial}{\partial x_1}\left(a^{\mathrm{eff}}(x_1)\frac{dv(x_1)}{dx_1}\right) + \varepsilon\lambda_i^{\varepsilon,\pm}\mathbf{B}(x_1)v(x_1) - \big(\varepsilon\lambda_i^{\varepsilon,\pm}\big)^2\mathbf{C}(x_1)v(x_1).$$

Using Lemma 3.4 and normalization condition (2.6), we obtain

$$\begin{aligned} &\left|\int_{G_\varepsilon} J_1^\varepsilon(x_1, y)\big|_{y=x/\varepsilon}u_i^{\varepsilon,\pm}(x)\,dx - \int_{G_\varepsilon}\int_Y J_1^\varepsilon(x_1, y)u_i^{\varepsilon,\pm}(x)\,dy\,dx\right| \\ &\leqslant C\varepsilon\varepsilon^{\frac{(d-1)}{2}}\big\|u_i^{\varepsilon,\pm}\big\|_{H^1(G_\varepsilon)} \\ &\leqslant C\varepsilon\varepsilon^{d-1}. \end{aligned}$$

Then, integrating by parts one gets

$$\int_{G_\varepsilon} J_2^\varepsilon(x_1,y)\big|_{y=x/\varepsilon} u_i^{\varepsilon,\pm}(x)\,dx + \int_{\Sigma_\varepsilon} \big(a^\varepsilon \nabla V^\varepsilon, n\big) u_i^{\varepsilon,\pm}\,d\sigma$$
$$= \varepsilon \int_{G_\varepsilon} a_{\cdot 1}(x_1,y)\frac{\partial}{\partial x_1}\left(N^{1,1}(x_1,y)\frac{dv(x_1)}{dx_1}\right)\bigg|_{y=x/\varepsilon} \nabla u_i^{\varepsilon,\pm}(x)\,dx$$
$$+ \varepsilon^2 \lambda_i^{\varepsilon,\pm} \int_{G_\varepsilon} a_{\cdot 1}(x_1,y)\frac{\partial}{\partial x_1}\big(N^{1,0}(x_1,y)v(x_1)\big)\big|_{y=x/\varepsilon} \nabla u_i^{\varepsilon,\pm}(x)\,dx.$$

Estimating the terms on the right-hand side of the last equality yields

$$\left| \int_{G_\varepsilon} J_2^\varepsilon(x_1,y)\big|_{y=x/\varepsilon} u_i^{\varepsilon,\pm}(x)\,dx + \int_{\Sigma_\varepsilon} \big(a^\varepsilon \nabla V^\varepsilon, n\big) u_i^{\varepsilon,\pm}\,d\sigma \right|$$
$$\leqslant C\varepsilon |G_\varepsilon|^{1/2} \big\|\nabla u_i^{\varepsilon,\pm}\big\|_{L^2(G_\varepsilon)}$$
$$\leqslant C\varepsilon\varepsilon^{d-1}.$$

Consequently,

$$0 = \big(u_i^{\varepsilon,\pm}, \mathcal{A}^\varepsilon w - \lambda_i^{\varepsilon,\pm}\rho^\varepsilon w\big)_{L^2(G_\varepsilon)} + \int_{\Sigma^\varepsilon} \big(a^\varepsilon \nabla w, n\big) u_i^{\varepsilon,\pm}\,d\sigma$$
$$= \big(u_i^{\varepsilon,\pm}, \boldsymbol{\Pi}\big(\varepsilon\lambda_i^{\varepsilon,\pm}\big)v\big)_{L^2(G_\varepsilon)} + r^\varepsilon, \quad \big|r^\varepsilon\big| \leqslant C\varepsilon\varepsilon^{d-1}.$$

By definition of the measure μ_ε (see Section 3)

$$\int_{K_d} u_i^{\varepsilon,\pm}(x)\boldsymbol{\Pi}\big(\varepsilon\lambda_i^{\varepsilon,\pm}\big)v(x_1)\,d\mu_\varepsilon + \frac{r^\varepsilon}{\varepsilon^{d-1}|Q|} = 0.$$

Passing to the limit in the last equality, taking into account the strong convergence of $u_i^{\varepsilon,\pm}$ in $L^2(K_d,\mu_\varepsilon)$, yields

$$\int_{K_d} u_*(x_1)\boldsymbol{\Pi}(\nu_*)v(x_1)\,d\mu_*(x) = 0.$$

Integration by parts gives

$$\int_{K_d} v(x_1)\boldsymbol{\Pi}(\nu_*)u_*(x_1)\,d\mu_*(x) = 0, \quad v \in C_0^\infty[-1,1].$$

Thus, u_* satisfies the equation

$$\boldsymbol{\Pi}(\nu_*)u_*(x_1) = -\frac{d}{dx_1}\left(a^{\mathrm{eff}}\frac{du_*}{dx_1}\right) + \nu_*\mathbf{B}u_* - (\nu_*)^2\mathbf{C}u_* = 0. \tag{4.39}$$

By the definition of $u_i^{\varepsilon,\pm}$ and $\lambda_i^{\varepsilon,\pm}$ we have

$$\left\|u_i^{\varepsilon,\pm}\right\|_{\mathcal{H}^\varepsilon}^2 = \lambda_i^{\varepsilon,\pm}\left(\rho^\varepsilon u_i^{\varepsilon,\pm}, u_i^{\varepsilon,\pm}\right)_{L^2(G_\varepsilon)}.$$

Since $\langle \rho(x_1,\cdot)\rangle = 0$, then

$$\left|\int\limits_{G_\varepsilon} \rho^\varepsilon \left(u_i^{\varepsilon,\pm}\right)^2 dx\right| \leqslant C\varepsilon \left\|u_i^{\varepsilon,\pm}\right\|_{L^2(G_\varepsilon)} \left\|u_i^{\varepsilon,\pm}\right\|_{H^1(G_\varepsilon)},$$

and, consequently,

$$\left\|u_i^{\varepsilon,\pm}\right\|_{\mathcal{H}^\varepsilon}^2 \leqslant C\varepsilon\lambda_i^{\varepsilon,\pm} \left\|u_i^{\varepsilon,\pm}\right\|_{L^2(G_\varepsilon)} \left\|u_i^{\varepsilon,\pm}\right\|_{H^1(G_\varepsilon)}.$$

Taking into account estimate (4.37) and the definition of the measure μ_ε, we have

$$\left\|u_i^{\varepsilon,\pm}\right\|_{L^2(K_d,\mu_\varepsilon)} \geqslant c > 0.$$

Considering the strong convergence of $u_i^{\varepsilon,\pm}$ in $L^2(K_d,\mu_\varepsilon)$ leads to the inequality

$$\|u_*\|_{L^2(-1,1)} \geqslant c > 0,$$

which means, together with (4.39), that (ν_*, u_*) is an eigenpair of the operator pencil (4.9). Lemma 4.3 is proved. □

Assume that

$$\varepsilon\lambda_i^{\varepsilon,\pm} \to \nu_j^{0,\pm}, \quad \varepsilon \to 0,$$
$$\varepsilon\lambda_k^{\varepsilon,\pm} \to \nu_j^{0,\pm}, \quad \varepsilon \to 0.$$

Then necessarily $i = k$. Indeed, by Lemma 4.3 the eigenfunctions $u_i^{\varepsilon,\pm}$ and $u_k^{\varepsilon,\pm}$ converge to the eigenfunctions $u_1^{*,\pm}$ and $u_2^{*,\pm}$ of (4.9) corresponding to $\nu_j^{0,\pm}$, and, as was proved above, $u_1^{*,\pm} \neq 0$ and $u_2^{*,\pm} \neq 0$. Since the eigenvalue $\nu_j^{0,\pm}$ is simple, we have

$$u_1^{*,\pm} + \overline{c_1} u_2^{*,\pm} = 0,$$

for some $\bar{c}_1 \neq 0$. Assume that $i \neq k$, and consider the expression

$$T^\varepsilon = \frac{1}{\varepsilon}\left(\rho^\varepsilon\left(u_i^{\varepsilon,\pm} + \bar{c}_1 u_k^{\varepsilon,\pm}\right), \left(u_i^{\varepsilon,\pm} + \bar{c}_1 u_k^{\varepsilon,\pm}\right)\right)_{L^2(K_d,\mu^\varepsilon)}.$$

Considering (2.6), (4.37) and (3.26), we obtain

$$\begin{aligned} T^\varepsilon &= \frac{1}{\varepsilon\lambda_i^{\varepsilon,\pm}} \frac{(u_i^{\varepsilon,\pm}, u_i^{\varepsilon,\pm})_{\mathcal{H}^\varepsilon}}{\varepsilon^{d-1}|Q|} + \frac{c_1^2}{\varepsilon\lambda_k^{\varepsilon,\pm}} \frac{(u_k^{\varepsilon,\pm}, u_k^{\varepsilon,\pm})_{\mathcal{H}^\varepsilon}}{\varepsilon^{d-1}|Q|} \\ &= \frac{1}{\varepsilon\lambda_i^{\varepsilon,\pm}} + \frac{c_1^2}{\varepsilon\lambda_k^{\varepsilon,\pm}} \quad \longrightarrow \quad \frac{1}{\nu_j^{0,\pm}} + \frac{c_1^2}{\nu_j^{0,\pm}} = \frac{1+c_1^2}{\nu_j^{0,\pm}} \neq 0. \end{aligned} \tag{4.40}$$

It was shown in the proof of Lemma 3.6 that $u_i^{\varepsilon,\pm}$ and $u_k^{\varepsilon,\pm}$ converges strongly in $L^2(K_d,\mu^\varepsilon)$, therefore,

$$\|u_i^{\varepsilon,\pm}+c_1u_k^{\varepsilon,\pm}\|_{L^2(K_d,\mu^\varepsilon)}\to 0,$$

as $\varepsilon\to 0$. Denote by $S(x,y)$ a solution to the following problem

$$\begin{cases} -\Delta_y S(x_1,y)=\rho(x_1,y), & y\in G^\pm,\\ \nabla_y S(x_1,y)\cdot n(y)=0, & y\in\Sigma^\pm,\\ S(x_1,y)\ \text{is 1-periodic} & \text{in } y_1.\end{cases}$$

Since $\langle\rho(x_1,\cdot)\rangle=0$, this problem is solvable. Setting $R(x_1,y)=\nabla_y S(x_1,y)$ we have

$$\frac{1}{\varepsilon}\rho\left(x_1,\frac{x}{\varepsilon}\right)=\operatorname{div}R\left(x_1,\frac{x}{\varepsilon}\right)-\frac{\partial}{\partial x_1}R(x_1,y)\big|_{y=x/\varepsilon}.$$

Denoting $R_1^\varepsilon(x)=\frac{\partial}{\partial x_1}R(x_1,y)|_{y=x/\varepsilon}$ and $R^\varepsilon(x)=R(x_1,\frac{x}{\varepsilon})$, we rewrite T^ε as follows

$$\begin{aligned} T^\varepsilon &= \big(\operatorname{div}R^\varepsilon\big(u_i^{\varepsilon,\pm}+c_1u_k^{\varepsilon,\pm}\big),\big(u_i^{\varepsilon,\pm}+c_1u_k^{\varepsilon,\pm}\big)\big)_{L^2(K_d,\mu^\varepsilon)}\\ &\quad-\big(R_1^\varepsilon\big(u_i^{\varepsilon,\pm}+c_1u_k^{\varepsilon,\pm}\big),\big(u_i^{\varepsilon,\pm}+c_1u_k^{\varepsilon,\pm}\big)\big)_{L^2(K_d,\mu^\varepsilon)}.\end{aligned}$$

Clearly, R_1^ε is uniformly in ε bounded. Therefore, the second term on the right-hand side tends to zero, as $\varepsilon\to 0$. Integration by parts in the first term yields

$$\begin{aligned} &\big(\operatorname{div}R^\varepsilon\big(u_i^{\varepsilon,\pm}+c_1u_k^{\varepsilon,\pm}\big),\big(u_i^{\varepsilon,\pm}+c_1u_k^{\varepsilon,\pm}\big)\big)_{L^2(K_d,\mu^\varepsilon)}\\ &\quad=-2\big(R^\varepsilon\big(u_i^{\varepsilon,\pm}+c_1u_k^{\varepsilon,\pm}\big),\nabla\big(u_i^{\varepsilon,\pm}+c_1u_k^{\varepsilon,\pm}\big)\big)_{L^2(K_d,\mu^\varepsilon)}.\end{aligned}$$

Since $\|\nabla u_i^{\varepsilon,\pm}\|_{L^2(K_d,\mu^\varepsilon)}$ and $\|\nabla u_i^{\varepsilon,\pm}\|_{L^2(K_d,\mu^\varepsilon)}$ are uniformly in ε bounded, the first term also tends to zero, as $\varepsilon\to 0$, which implies that $\lim_{\varepsilon\to 0}T^\varepsilon=0$. This contradicts (4.40). We conclude that $i=k$.

Finally, we conclude that for any j there is only one $\lambda_j^{\varepsilon,\pm}$ satisfying inequality (4.36), and thus, it is simple for sufficiently small ε. In view of the geometric simplicity of $\nu_j^{0,\pm}$ and Lemma 3.2, the corresponding eigenfunction $u_j^{\varepsilon,\pm}$ can be approximated by the “almost eigenfunction” $U_j^{\varepsilon,\pm}$:

$$\|u_j^{\varepsilon,\pm}-U_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}\leqslant c_j\varepsilon,\quad \varepsilon<\varepsilon_j.$$

The proof of Theorem 4.3 is complete. □

5. The case of sign-changing $\langle\rho(x_1,\cdot)\rangle$

In the case of sign-changing $\langle\rho(x_1,\cdot)\rangle$ the limit spectral problem takes the form

$$\begin{cases} \mathcal{A}^0u^0(x_1)\equiv-\dfrac{d}{dx_1}\left(a^{\mathrm{eff}}(x_1)\dfrac{du^0(x_1)}{dx_1}\right)\\ \qquad\qquad=\lambda^0\big\langle\rho(x_1,\cdot)\big\rangle u^0(x_1),\quad x_1\in(-1,1),\\ u^0(\pm 1)=0.\end{cases}\tag{5.1}$$

Here the effective coefficient a^{eff} is defined by (3.3). By Lemma 3.1 the coefficient $a^{\mathrm{eff}}(\cdot)$ is a $C^{1,\alpha}[-1,1]$ function such that $a^{\mathrm{eff}}(x_1) > 0$ for all $x_1 \in [-1,1]$.

Since $\langle \rho(x_1,\cdot)\rangle$ changes sign, one can see in the same way as in Theorem 2.1 that the spectrum of problem (5.1) is discrete and consists of two infinite sequences

$$0 < \lambda_1^{0,+} < \lambda_2^{0,+} < \cdots < \lambda_j^{0,+} \cdots \to +\infty,$$
$$0 > \lambda_1^{0,-} > \lambda_2^{0,-} > \cdots > \lambda_j^{0,-} \cdots \to -\infty.$$

Moreover, since problem (5.1) is one-dimensional, all the eigenvalues $\lambda_j^{0,\pm}$ are simple. The corresponding eigenfunctions $u_i^{0,\pm} \in C^{2,\alpha}[-1,1]$ of problem (5.1) can be normalized by

$$\int_{-1}^{1} a^{\mathrm{eff}}(x_1)\frac{du_i^{0,\pm}}{dx_1}\frac{du_j^{0,\pm}}{dx_1}\,dx_1 = \delta_{ij}. \tag{5.2}$$

For any $j \in \mathbb{N}$ we denote

$$\begin{aligned} U_j^{\varepsilon,\pm}(x) = u_j^{0,\pm}(x_1) + \varepsilon N^{1,1}(x_1,y)\frac{du_j^{0,\pm}(x_1)}{dx_1}\bigg|_{y=x/\varepsilon} \\ + \varepsilon v_j^{1,\pm}(x_1) + \varepsilon\big(u_{\mathrm{bl}}^{\varepsilon,+}(x) + u_{\mathrm{bl}}^{\varepsilon,-}(x)\big), \end{aligned} \tag{5.3}$$

where $u_j^{0,\pm}$, $N^{1,1}$ and $v_j^{1,\pm}$ solve problems (5.1), (3.2) and (3.15), respectively, with $u^0 = u_j^{0,\pm}$ and $\lambda^0 = \lambda_j^{0,\pm}$. The boundary layer functions $u_{\mathrm{bl}}^{\varepsilon,\pm}$ are defined by (3.17) and (3.13) with $u^0 = u_j^{0,\pm}$.

Theorem 5.1. *Let conditions* (H0)–(H3) *be fulfilled, and suppose that $\langle \rho(x_1,\cdot)\rangle$ changes its sign on* $[-1,1]$. *If $(\lambda_j^{\varepsilon,\pm}, u_j^{\varepsilon,\pm})$ are eigenpairs of problem* (2.1), *and $(\lambda_j^{0,\pm}, u_j^{0,\pm})$ are those of problem* (5.1), *then the following statements hold*:

(i) *For any $j \in \mathbb{N}$, there exist ε_j and $C_j > 0$ such that*

$$\big|\lambda_j^{\varepsilon,\pm} - \lambda_j^{0,\pm}\big| \leqslant C_j\varepsilon, \quad \forall\varepsilon \in (0,\varepsilon_j].$$

(ii) *For any $j \in \mathbb{N}$,*

$$\big\|u_j^{\varepsilon,\pm} - U_j^{\varepsilon,\pm}\big\|_{H^1(G_\varepsilon)} \leqslant C_j\varepsilon\varepsilon^{\frac{d-1}{2}}$$

where $U_j^{\varepsilon,\pm}$ is defined by (5.3). *Moreover, the "almost eigenfunctions" satisfy the almost orthogonality and normalization condition*

$$\left|\frac{\varepsilon^{-(d-1)}}{|Q|}\big(a^\varepsilon\nabla U_i^{\varepsilon,\pm}, \nabla U_j^{\varepsilon,\pm}\big)_{L^2(G_\varepsilon)} - \delta_{ij}\right| \leqslant C_j\varepsilon.$$

(iii) *For $j \in \mathbb{N}$, $\lambda_j^{\varepsilon,\pm}$ are simple, for sufficiently small $\varepsilon > 0$.*

Proof. Since the proof of Theorem 5.1 is similar to that of Theorem 3.2, we give here just a sketch of this proof.

First, we construct a formal asymptotic expansion for a solution $(\lambda^\varepsilon, u^\varepsilon)$ of problem (2.1). In the case under consideration it takes the same form as in the case $\langle\rho(x_1,\cdot)\rangle > 0$ (see (3.1)). Namely,

$$u^\varepsilon(x) = u^0(x_1) + \varepsilon u^1(x_1, y) + \varepsilon^2 u^2(x_1, y) + \varepsilon^3 u^3(x_1, y) + \cdots,$$
$$\lambda^\varepsilon = \lambda^0 + \varepsilon\lambda^1 + \cdots, \qquad y = \frac{x}{\varepsilon}, \tag{5.4}$$

where unknown functions $u^k(x_1, y)$ are 1-periodic in y_1. We substitute these ansatz for u^ε and λ^ε in (2.1), collect power-like terms, and repeat the computations of Section 3.1. At the first step we obtain that

$$u^1(x_1, y) = N^{1,1}(x_1, y)\frac{du^0(x_1)}{dx_1} + v^1(x_1)$$

with $N^{1,1}$ defined in (3.2). At the second step this yields problem (3.4), that is the pair (λ^0, u^0) solves problem (5.1).

Notice that, since 0 does not belong to the spectrum of (5.1), for each $u^0 \neq 0$ we have

$$\lambda^0 \neq 0, \qquad \int_{-1}^{1} \langle\rho(x_1,\cdot)\rangle\big(u^0(x_1)\big)^2\, dx_1 \neq 0. \tag{5.5}$$

In order to determine the function $v^1(x_1)$ we set, like in (3.6),

$$u^2(x_1, y) = N^{2,2}(x_1, y)\frac{d^2u^0(x_1)}{dx_1^2} + N^{2,1}(x_1, y)\frac{du^0(x_1)}{dx_1} + N^{2,0}(x_1, y)u^0(x_1)$$
$$+ N^{1,1}(x_1, y)\frac{dv^1(x_1)}{dx_1} + v^2(x_1),$$

where $N^{2,2}$, $N^{2,1}$ and $N^{2,0}$ are y_1-periodic functions defined in (3.7)–(3.9). Recalling the definition of the boundary layer functions $w^\pm(y_1, y')$ (see (3.13)) and the corresponding constants $\hat{w}^\pm$, and repeating once again the computations of Section 3.1, we arrive at problem (3.15) that reads

$$\begin{cases} -\dfrac{d}{dx_1}\left(a^{\mathrm{eff}}(x_1)\dfrac{dv^1}{dx_1}\right) - \lambda^0\langle\rho(x_1,\cdot)\rangle v^1(x_1) \\ \quad = F(x_1) + \lambda^1\langle\rho(x_1,\cdot)\rangle u^0, \quad x_1 \in (-1, 1), \\ v^1(\pm 1) = \hat{w}^\pm \end{cases} \tag{5.6}$$

with $F(x_1)$ defined by (3.11).

In view of (5.5), normalization condition (5.2), and by the Fredholm theorem, the solvability condition of the last problem reads

$$\lambda^1 = -\lambda^0 \int_{-1}^{1} F(x_1)u^0(x_1)\, dx_1 + \lambda^0\left(a^{\mathrm{eff}}(1)\frac{du^0}{dx_1}(1)\hat{w}^+ - a^{\mathrm{eff}}(-1)\frac{du^0}{dx_1}(-1)\hat{w}^-\right). \tag{5.7}$$

Imposing the normalization condition

$$\int_{-1}^{1} v^1(x_1)u^0(x_1)\,dx_1 = 0$$

and letting

$$u_{\rm bl}^{\varepsilon,\pm}(x) = w^{\pm}\left(\frac{x_1 \mp 1}{\varepsilon}, \frac{x'}{\varepsilon}\right) - \hat{w}^{\pm}, \tag{5.8}$$

we finally obtain a formal asymptotic expansion of u^ε:

$$U^\varepsilon(x) = u^0(x_1) + \varepsilon N^{1,1}\left(x_1, \frac{x'}{\varepsilon}\right)\frac{du^0(x_1)}{dx_1} + \varepsilon v^1(x_1) + \varepsilon\big(u_{\rm bl}^{\varepsilon,+}(x) + u_{\rm bl}^{\varepsilon,-}(x)\big). \tag{5.9}$$

Let $\lambda_j^{0,+}$ ($\lambda_j^{0,-}$) be the jth positive (negative) eigenvalue of problem (3.4). We substitute the corresponding eigenfunction $u_j^{0,+}$ ($u_j^{0,-}$) for u^0 in (5.9) and denote

$$\begin{aligned} U_j^{\varepsilon,\pm}(x) = {} & u_j^{0,\pm}(x_1) + \varepsilon N^{1,1}(x_1, y)\frac{du_j^{0,\pm}(x_1)}{dx_1}\bigg|_{y=x/\varepsilon} + \varepsilon v_j^{1,\pm}(x_1) \\ & + \varepsilon\big(u_{\rm bl}^{\varepsilon,+}(x) + u_{\rm bl}^{\varepsilon,-}(x)\big), \end{aligned} \tag{5.10}$$

where $u_j^{0,\pm}$, $N^{1,1}$ and $v_j^{1,\pm}$ solve problems (5.1), (3.2) and (5.6), respectively, with $u^0 = u_j^{0,\pm}$ and $\lambda^0 = \lambda_j^{0,\pm}$. The boundary layer functions $u_{\rm bl}^{\varepsilon,\pm}$ are defined by (3.17) and (3.13) again with $u^0 = u_j^{0,\pm}$.

Notice that by construction the function $U_j^{\varepsilon,\pm}$ are elements of the space $\mathcal{H}^\varepsilon$.

Consider the normalized ansatz (5.10)

$$\mathcal{U}_j^{\varepsilon,\pm} = \big(\|U_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}\big)^{-1} U_j^{\varepsilon,\pm}$$

and the numbers $(\lambda_j^{0,\pm} + \varepsilon\lambda_j^{1,\pm})^{-1}$ with $\lambda_j^{1,\pm}$ defined by formula (5.7) with $u^0 = u_j^{0,\pm}$ and $\lambda^0 = \lambda_j^{0,\pm}$.

The statement of Lemma 3.3 remains valid in the case under consideration both for positive and negative parts of the spectrum.

Lemma 5.1. *For any $j \in \mathbb{N}$ there are $\varepsilon_j > 0$ and $C_j > 0$ that only depend on j, such that*

$$\big\|\mathcal{K}^\varepsilon \mathcal{U}_j^{\varepsilon,\pm} - \big(\lambda_j^{0,\pm} + \varepsilon\lambda_j^{1,\pm}\big)^{-1}\mathcal{U}_j^{\varepsilon,\pm}\big\|_{\mathcal{H}^\varepsilon} \leqslant C_j\varepsilon \quad \textit{for all } \varepsilon < \varepsilon_j. \tag{5.11}$$

Proof. As in the proof of Lemma 3.3 we set

$$I_j^{\varepsilon,\pm} \equiv \big\|\mathcal{K}^\varepsilon \mathcal{U}_j^{\varepsilon,\pm} - \big(\lambda_j^{0,\pm} + \varepsilon\lambda_j^{1,\pm}\big)^{-1}\mathcal{U}_j^{\varepsilon,\pm}\big\|_{\mathcal{H}^\varepsilon},$$

and after straightforward rearrangements get

$$I_j^{\varepsilon,\pm} = \sup_{\substack{w\in\mathcal{H}^\varepsilon \\ \|w\|_{\mathcal{H}^\varepsilon}=1}} \big|\big(\mathcal{K}^\varepsilon \mathcal{U}_j^{\varepsilon,\pm} - \big(\lambda_j^{0,\pm} + \varepsilon\lambda_j^{1,\pm}\big)^{-1}\mathcal{U}_j^{\varepsilon,\pm}, w\big)_{\mathcal{H}^\varepsilon}\big|$$

$$= \frac{\|U_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}^{-1}}{|\lambda_j^{0,\pm}+\varepsilon\lambda_j^{1,\pm}|} \sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}} \left|\left(\left(\lambda_j^{0,\pm}+\varepsilon\lambda_j^{1,\pm}\right)\mathcal{K}^\varepsilon U_j^{\varepsilon,\pm} - U_j^{\varepsilon,\pm}, w\right)_{\mathcal{H}^\varepsilon}\right|$$

$$= \frac{\|U_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}^{-1}}{|\lambda_j^{0,\pm}+\varepsilon\lambda_j^{1,\pm}|} \sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}} \left|\left(\lambda_j^{0,\pm}+\varepsilon\lambda_j^{1,\pm}\right)\left(\rho^\varepsilon U_j^{\varepsilon,\pm}, w\right)_{L^2(G_\varepsilon)} - \left(a^\varepsilon\nabla U_j^{\varepsilon,\pm}, \nabla w\right)_{L^2(G_\varepsilon)}\right|.$$

Estimate (4.35) justified in the proof of Lemma 3.3 did not rely on the positiveness of $\langle\rho(x_1,\cdot)\rangle$. Thus it also holds in the case of sign-changing $\langle\rho(x_1,\cdot)\rangle$. Namely, for all sufficiently small $\varepsilon>0$ we have

$$\left\|U_i^{\varepsilon,\pm}\right\|_{\mathcal{H}^\varepsilon} \geqslant \frac{|Q|^{1/2}}{2}\varepsilon^{\frac{(d-1)}{2}}. \tag{5.12}$$

Analogously, in the same way as in the proof of Lemma 3.3, we obtain

$$\sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}} \left|\left(\lambda_j^{0,\pm}+\varepsilon\lambda_j^{1,\pm}\right)\left(\rho^\varepsilon U_j^{\varepsilon,\pm}, w\right)_{L^2(G_\varepsilon)} - \left(a^\varepsilon\nabla U_j^{\varepsilon,\pm}, \nabla w\right)_{L^2(G_\varepsilon)}\right| \leqslant C\varepsilon\varepsilon^{\frac{(d-1)}{2}}. \tag{5.13}$$

Since $\lambda_j^{0,\pm}\neq 0$, then for sufficiently small $\varepsilon>0$ we have $|\lambda_j^{0,\pm}+\varepsilon\lambda_j^{1,\pm}|\geqslant C$ with some $C>0$. Combining this estimate with (5.12) and (5.13) yields (5.11). □

From Lemma 5.1 and Lemma 3.2 it follows that for any $j\in\mathbb{N}$ there are $\varepsilon_j>0$ and $q^\pm$ such that

$$\left|\lambda_{q^\pm}^{\varepsilon,\pm}-\lambda_j^{0,\pm}\right| \leqslant c_j\varepsilon, \quad \varepsilon<\varepsilon_j. \tag{5.14}$$

By the same arguments as in Lemmata 3.5 and 3.6 it is easy to deduce that for any $q\in\mathbb{N}$,

$$0<m\leqslant\left|\lambda_q^{\varepsilon,\pm}\right|\leqslant M_q,$$

and that any limit point λ_* of a sequence $\{\lambda_j^{\varepsilon,+}\}$ or $\{\lambda_j^{\varepsilon,-}\}$ is an eigenvalue of problem (5.1).

In the same way as in the proof of Theorem 3.2 this readily implies all the statements of Theorem 5.1.

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