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ON THE BEHAVIOUR AT INFINITY OF SOLUTIONS TO STATIONARY CONVECTION-DIFFUSION EQUATION IN A CYLINDER

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Abstract. The work focuses on the behaviour at infinity of solutions to second order elliptic equation with first order terms in a semi-infinite cylinder. Neumann's boundary condition is imposed on the lateral boundary of the cylinder and Dirichlet condition on its base. Under the assumption that the coefficients stabilize to a periodic regime, we prove the existence of a bounded solution, its stabilization to a constant, and provide necessary and sufficient condition for the uniqueness.

1. **Introduction.** This work deals with the behaviour at infinity of solutions to stationary convection-diffusion equations defined in a semi-infinite cylinder. We assume that Neumann boundary condition is imposed on the lateral boundary of the cylinder, and that the coefficients of the equation are periodic along the cylinder axis or stabilize at the exponential rate to a periodic regime for asymptotically large axial distance. Under these assumptions we study the existence and uniqueness of a bounded solution, and its stabilization to a constant at infinity.

The question of validity of the Saint-Venant and Phragmén–Lindelöf principles, as well as other questions related to the behaviour at infinity of solutions to elliptic equations and systems of equations, received a lot of attention of mechanicians and mathematicians starting from the beginning of 20th century.

A number of rigorous mathematical works are devoted to this subject. Dirichlet and Neumann boundary value problems in a cylindrical domains for second order linear elliptic equations in divergence form were studied by many authors. Early contributions include [10], [6] and [7] which contain results like Saint-Venant's principle for special classes of Neumann problems. As to the later works on this topic, we mention just some of them closely related to the present paper.

In [14] an equation in divergence form in a half-cylinder with periodic coefficients on all variables except for one was considered, the exponential stabilization

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to a constant was proved. The periodic boundary conditions were imposed on the lateral boundary of the cylinder. The technique used in this work relies on specific geometrical methods. As was communicated to the authors ([22]), the question of the existence of a solution that converges exponentially to a constant with large axial distance was studied in 1976 for an equation in divergence form under some natural assumptions on the right-hand side. The method relied on a variation of the Lax-Milgram lemma. This result has partially been written in [15]. Another proof of the exponential decay of the solution to the same equation was given in [1]and it is valid also for non-flat base of the cylinder.

A boundary value problem for a second order elliptic equation with first order terms on a half-cylinder with periodic boundary conditions on the lateral boundary of the cylinder was studied in [19] and [20]. In these works, under the assumption of C^2 regularity and periodicity of the coefficients, the existence of a bounded solution and its exponential stabilization to a constant at infinity was proved by means of diffusion processes techniques. Moreover, the necessary and sufficient conditions for the uniqueness of a bounded solution were given. In [20] the obtained results were applied to a homogenization problem for singularly perturbed operators defined in a layer. Also, without the assumption on periodicity in axial variable the following conditional result was obtained in [20]: if the adjoint problem has a bounded uniformly positive solution, then the effective axial drift can be defined and the results proved in periodic case, remain valid.

In the present paper we study operators with measurable coefficients and assume only Lipschitz continuity of the boundary of the cross-section. In this case the usage of probabilistic techniques is getting embarrassing and sometimes impossible, especially if the boundary condition is not homogeneous. Our approach relies on the various results from local qualitative elliptic theory, such as Harnack's inequality, Nash and De Giorgi estimates, the maximum principle, positive operators theory and a number of non-trivial a priori estimates which include as a weight the function forming the kernel of the adjoint periodic operator. We consider here not only operators with periodic coefficients, but also with coefficients which stabilize to periodic regime at infinity. Another issue addressed in the paper is the generalization of the existence, uniqueness and stabilization results to the case of nonhomogeneous equations with H^{-1} function on the right-hand side. It should be noted that obtaining the a priori estimates in this case is getting more complicated than in the case of data from L^2.

Also we pay special attention to the asymptotic behaviour of the solutions defined in a growing family of finite cylinders. This gives a clear picture of how solutions defined in finite cylinders approximate the limit bounded solution. This analysis allows us, in particular, to distinguish the special case when the so-called effective axial drift is equal to zero.

In [3], [4] and [8] specific classes of semi-linear elliptic equations in a half-cylinder were considered. It was proved that a global solution, when it exists, decays at least exponentially with large axial distance. The technique involves the derivation of a first order differential inequality for the energy flux across a cross-section of the cylinder. With the help of this technique spatial behaviour of solutions to elliptic systems, in particular those of linearized and linear elasticity, was studied also (see, for example, [9]).

A priori estimates similar to Saint-Venant's principle in elasticity theory were discussed in [17] under some dissipativity type assumptions on the coefficients. Also in this work interesting uniqueness results in proper classes of growing functions were obtained for Dirichlet and Neumann problems for second order linear elliptic equations in unbounded domains .

In [11] the authors investigated elliptic systems with complex constant coefficients, assuming that a weighted Dirichlet integral is bounded. The paper deals with finite energy solutions for the system of linear elasticity.

The asymptotic behaviour at infinity of solutions to symmetric elliptic systems were treated in [16]. This work focused on the existence of solutions in weighted spaces with various exponentially growing or decaying weights.

In [13] the behaviour of solutions to nonlinear elliptic equations with a dissipative nonlinear zero order terms was studied by means of the barrier functions techniques.

The goal of this work is to study the behaviour at infinity of solutions to a linear stationary convection-diffusion equation in a semi-infinite cylinder. We impose Dirichlet boundary condition on the base of the cylinder and Neumann condition on the lateral boundary. Under the assumptions that the coefficients of the equation stabilize exponentially to a periodic regime, and the functions on right-hand side of the equation and of the boundary operator decays sufficiently fast at infinity, we prove the existence of a bounded solution and its stabilization to a constant at the exponential rate. Also we provide a necessary and sufficient condition for the uniqueness of a bounded solution. It should be noted that, in contrast with the divergence form operators, for the operators with first-order terms the question of uniqueness of a bounded solution is getting more complicated. We show that whether a solution is unique or not depends on the sign of some constant called effective axial drift (or flux), which can be determined in terms of a solution to auxiliary periodic problem for the formally adjoint operator.

The problems of this type appear while constructing the asymptotic expansions of solutions to equations describing different phenomena in highly inhomogeneous medium. For instance, such results allow one to construct boundary layer functions in various homogenization problems. Moreover, these results are of independent interest in mechanics and other applied fields and, of course, in mathematics.

The paper is organized as follows. Sections 1–6 focus on the homogeneous problem with periodic coefficients. In these sections we start with the problem setup and auxiliary results, and then proceed with the existence, the uniqueness and the stabilization to a constant of a bounded solution to the problem under consideration. In Sections 7–8 we obtain similar results for inhomogeneous problems and equations with coefficients stabilizing to a periodic regime.

2. **Problem statement.** Let $G = (0,\infty)\times Q$ be a semi-infinite cylinder in $\mathbb{R}^d$ with the axis directed along x_1, where Q is a bounded domain in $\mathbb{R}^{d-1}$ with a Lipschitz boundary ∂Q. The lateral boundary of G is denoted by $\Sigma = (0,+\infty)\times\partial Q$. We study the following boundary-value problem:

$$\begin{cases} -\mathrm{div}\ (a(x)\,\nabla\, u(x)) - (b(x),\nabla\, u(x)) = 0, & x\in G,\\ \dfrac{\partial u}{\partial n_a} = 0, & x\in\Sigma,\\ u(0,x') = \varphi(x'), & x'\in Q. \end{cases} \tag{1}$$

Here $a(x)$ is a $d\times d$ matrix and $b(x)$ is a vector in $\mathbb{R}^d$, $x=(x_1,x')$, $\varphi(x')\in H^{1/2}(Q)$; $(\cdot,\cdot)$ stands for the standard scalar product in $\mathbb{R}^d$; $\partial u/\partial n_a = \sum_{i,j=1}^d a_{ij}(x)n_i\,\partial_j u$ is the conormal derivative, n is the external unit normal. The matrix-valued function

$a(x)$ and the vector field $b(x)$ are supposed to be measurable and bounded, that is $a_{ij}(x) \in L^\infty(G), b_i(x) \in L^\infty(G)$, and periodic in x_1 functions. Without loss of generality we assume that the period is equal to 1. For the sake of simplicity the matrix $a(x)$ is supposed to be symmetric. Moreover, we assume that $a(x)$ satisfies the uniform ellipticity condition, that is there exists a positive constant Λ such that, for almost all $x \in \mathbb{R}^d$,

$$\Lambda\,|\xi|^2 \le \sum_{i,j} a_{ij}(x)\xi_i\xi_j, \quad \forall \xi \in \mathbb{R}^d, \tag{2}$$

The first goal of this work is to study the behavior of bounded (in a proper sense) solutions of problem (1).

3. **Auxiliary function** $p(x)$**.** Consider the following periodic problem:

$$\left\{\begin{array}{ll} -\mathrm{div}(a(x)\nabla\, u) - (b(x), \nabla\, u) = f(x), & x \in G_0^1 = (0,1)\times Q,\\ \dfrac{\partial u}{\partial n_a} = 0, & x \in \Sigma_0^1 = (0,1)\times \partial Q,\\ u -\ x_1 - \text{periodic}. & \end{array}\right. \tag{3}$$

This problem has a unique up to an additive constant solution $u(x)$. We denote by A an unbounded operator in $L^2(G_0^1)$ which maps $u(x)$ into $f(x) \in L^2(G_0^1)$. In view of x_1-periodicity we can identify functions defined on G_0^1 with the corresponding functions defined on the set $Y = \mathfrak{S}_1 \times Q$, where $\mathfrak{S}_1$ is a 1-dimensional circle. Then problem (3) reads

$$\left\{\begin{array}{ll} -\mathrm{div}(a(x)\nabla\, u) - (b(x), \nabla\, u) = f(x), & x \in Y,\\ \dfrac{\partial u}{\partial n_a} = 0, & x \in \partial Y. \end{array}\right. \tag{4}$$

The operator A is an unbounded operator from $L^2(Y)$ into itself with dense domain $D(A)$, that consists of the functions $u(x) \in L^2(Y)$ such that there exists $f(x) \in L^2(Y)$: $A\,u = f$ and $\partial u/\partial n_a = 0$ for $x \in \partial Y$. For large $\lambda > 0$ the inverse operator $(A+\lambda I)^{-1}$ exists and it is compact. Moreover, using the De Giorgi–Nash estimates (see, for example, [5]) it is easy show that $(A+\lambda I)^{-1}$ is a compact operator in $C(Y)$.

The formally adjoint problem takes the following form:

$$\left\{\begin{array}{ll} -\mathrm{div}(a\nabla\, v) + \mathrm{div}\,(b\, v) = f, & x \in Y,\\ \dfrac{\partial v}{\partial n_a} - (b,n)\, v = 0, & x \in \partial Y. \end{array}\right.$$

In the sequel we will need an auxiliary function $p(x)$ which belongs to the null space of the adjoint operator:

$$\left\{\begin{array}{ll} -\mathrm{div}(a\nabla\, p) + \mathrm{div}\,(b\, p) = 0, & x \in Y,\\ \dfrac{\partial p}{\partial n_a} - (b,n)\, p = 0, & x \in \partial Y. \end{array}\right. \tag{5}$$

The goal of this section is to show that such function exists and is positive.

Definition 3.1. We say that the operator B from $L^2(Y)$ $(C(Y))$ into itself is positive if from the inequality $u \ge 0$ it follows that $Bu \ge 0$.

The linear positive operator B is called v-bounded, for some $v \in C(Y)$, $v > 0$, if for every positive function $u \in C(Y)$ there exists two constants $\alpha = \alpha(u)$ and $\beta = \beta(u)$ such that

$$0 < \alpha(u)\, v \le B\, u \le \beta(u)\, v.$$

First let us show that the operator $(A + \lambda I)^{-1}$ is positive. By the maximum principle, if $f > 0$, then u cannot have a negative minimum in the interior of the domain Y. The assumption that a negative minimum is attained on the boundary ∂Y, will also contradict the maximum principle in view of the positiveness of f. Indeed, since ∂Q is Lipschitz then for every point $\tilde{x} \in \partial Q$ there exists a neighborhood $U(\tilde{x}) \subset \mathbb{R}^d$ such that the surface $\partial Q \cap U(\tilde{x})$ is represented by the equality

$$x_1 = \mathcal{F}(x_2, ..., x_d),$$

where $\mathcal{F}$ is a Lipschitz function. Let us make a change of variables straightening the boundary ∂Q, so that the piece of the boundary $\partial Q \cap U(\tilde{x})$ is mapped into the piece of the plane $\xi_1 = 0$ and $Y \cap U(\tilde{x})$ into some domain where $\xi_1 > 0$:

$$\begin{cases} \xi_1 = x_1 - \mathcal{F}(x_2, ..., x_d), \\ \xi_k = x_k, \quad k = 2, ..., d. \end{cases}$$

One more change of the variables transfers the co–normal derivative to the normal derivative:

$$\begin{cases} \eta_1 = \xi_1, \\ \eta_k = \xi_k - \dfrac{a_{1k}}{a_{11}}\, \xi_1, \quad k = 2, ..., d. \end{cases}$$

By construction, in the vicinity of the point $\tilde{\eta} = \mathcal{F}(\tilde{x})$ the solution u is only defined for $\eta_1 \ge 0$. We define an extension of u (keeping the notation u for the extended function) by setting $u(\eta_1, \eta') = u(-\eta_1, \eta')$ for negative η_1. One can check that after inverse changing the variables, due to the homogeneous Neumann boundary condition on ∂Q, the extended function $u(x)$ remains a solution of some convection-diffusion equation with a positive right hand side in a neighbourhood of $\tilde{x}$. Thus, in view of the maximum principle, $u(x)$ cannot attend a negative minimum at $\tilde{x}$.

Let us show that the operator $(A + \lambda I)^{-1}$ is 1-bounded in $C(Y)$. First, we note that, in view of the boundedness of the coefficients, the following estimate takes place (see, for example, [5]):

$$\|u\|_{C(Y)} \le C \|f\|_{C(Y)}$$

for some constant C independent of f. Thus,

$$(A + \lambda I)^{-1} f = u \le C.$$

It remains to show that $(A+\lambda I)^{-1}\, f > 0$ if $f(x) > 0$, $x \in Y$. Let us suppose that $\min\limits_{x \in Y} u(x) = 0$. In the interior of Y the function $u(x)$ cannot attain a nonpositive minimum unless u is equal to zero. If we assume that u achieves zero minimum on the boundary $\mathfrak{S}^1 \times \partial Q$, then, in the same way as above, we can extend u to a larger domain so that the extended function remains a solution of some elliptic convection-diffusion type equation with a positive right-hand side and, consequently, u cannot achieve its zero minimum on the boundary. Hence, we conclude that $u(x) \ge c(f) > 0$ if $f(x) > 0$.

Now we apply the Krein-Rutman theorem (see, for example, [12]) to compact, positive, 1-bounded operator $(A+\lambda I)^{-1}$ in $C(Y)$. According to this theorem, there exists a simple positive eigenvalue λ_0 of the operator $(A + \lambda I)^{-1}$ with a positive eigenvector, and there is no others eigenvalues with positive eigenvectors. Moreover, if we consider $(A+\lambda I)^{-1}$ as an operator in $L^2(Y)$, then there exists a nonnegative periodic in x_1 eigenvector $p(x) \in L^2(Y)$ of the adjoint operator $(A^*+\lambda I)^{-1}$, which corresponds to the same eigenvalue λ_0. Let us note that the operator A has a positive eigenvector (which is equal to 1) corresponding to zero eigenvalue. In view of the uniqueness of eigenvalue with positive eigenfunction $\left(\frac{1}{\lambda_0} - \lambda\right) = 0$, and, therefore, $p(x)$ belongs to the kernel of the adjoint operator A^*.

In order to prove the positiveness and boundedness of the function $p(x)$ up to the boundary $\partial Y = \partial Q$ we extend it to some bigger domain containing $\tilde{G}$.

Since ∂Q is Lipschitz, for any point $\bar{x} \in \partial Q$ there exists a neighborhood $U(\bar{x})$ such that $\Gamma = \partial Q \cap U(\bar{x}) = \{x : x_1 = f(x')\}$, with Lipschitz function $f(x')$. Let us make a change of variables which straightens Γ:

$$\begin{cases} \xi_1 = x_1 - f(x'), \\ \xi_k = x_k, \quad k = 2, ..., d, \end{cases}$$

such that the domain $\Omega^+ = Y \cap U(\bar{x})$ is mapped into $\tilde{\Omega}^+$, where $\xi_1 > 0$, and Γ is mapped into $\tilde{\Gamma} = \{\xi : \ \xi_1 = 0\}$.

We define the "extended" coefficients $\tilde{a}_{ij}(\xi)$ and $\tilde{b}_j(\xi)$ in the domain

$$\tilde{\Omega}^- = \Big\{\xi = (\xi_1, \xi') : \ \xi_1 < 0, \ (-\xi_1, \xi') \in \tilde{\Omega}^+\Big\}$$

as follows:

$$\tilde{a}(\xi_1, \xi') = S\, a(-\xi_1, \xi')\, S^*,$$
$$\tilde{b}(\xi_1, \xi') = S\, b(-\xi_1, \xi'),$$

where the matrix S is given by the expression

$$S = \begin{pmatrix} -1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 1 \end{pmatrix}.$$

If we define the extended function $\tilde{p}$ as

$$\tilde{p}(\xi) = \begin{cases} p(\xi_1, \xi'), & \xi_1 > 0, \\ p(-\xi_1, \xi'), & \xi_1 < 0, \end{cases}$$

then it can be checked that $\tilde{p}$ is a solution of the equation

$$-\operatorname{div}(\tilde{a}(\xi)\, \nabla\, \tilde{p}(\xi)) + \operatorname{div}(\tilde{b}(\xi)\, \tilde{p}(\xi)) = 0, \quad \tilde{\Omega} = \tilde{\Omega}^+ \cup \tilde{\Omega}^-. \tag{6}$$

Indeed, by definition

$$\frac{\partial \tilde{p}}{\partial n_{\tilde{a}}} - (\tilde{b}, n)\, \tilde{p} = 0, \quad \xi \in \tilde{\Gamma},$$

where n is an external normal to $\tilde{\Omega}^+$. Since by construction

$$\frac{\partial \tilde{p}^+}{\partial n_{\tilde{a}}} - (\tilde{b}, n)\, \tilde{p}^+ = -\left(\frac{\partial \tilde{p}^-}{\partial n_{\tilde{a}}} - (\tilde{b}, n)\, \tilde{p}^-\right), \quad \xi \in \tilde{\Gamma},$$

where $\tilde{p}^{\pm}$ are limit values of the function $\tilde{p}$ on the surface $\tilde{\Gamma}$ from different sides of it, $\tilde{\Omega}^+$ and $\tilde{\Omega}^-$ respectively. Then

$$\int\limits_{\tilde{\Gamma}} \left(\frac{\partial \tilde{p}^+}{\partial n_{\tilde{a}}} - (\tilde{b}, n)\, \tilde{p}^+\right) \varphi(\xi)\, d\xi' = \int\limits_{\tilde{\Gamma}} \left(\frac{\partial \tilde{p}^-}{\partial n_{\tilde{a}}} - (\tilde{b}, n)\, \tilde{p}^-\right) \varphi(\xi)\, d\xi' = 0,$$

for any function $\varphi(\xi) \in C_0^\infty(\tilde{\Omega})$. Keeping in mind the last equality, one can easily show that $\tilde{p}$ is a solution of (6).

By construction the obtained function $\tilde{p}(\xi)$ is nonnegative. In view of the Harnack inequality in any compact subset $\tilde{\Omega}'$ of $\tilde{\Omega}$, $\tilde{p}$ is bounded from below by some positive constant δ which depends on the point $\bar{x}$ and the choice of a compact subset (otherwise it is equal to zero which contradicts the definition of p). Moreover, $\tilde{p}$ is a Holder continuous function in $\tilde{\Omega}'$ (see, e.g. [21], [5]), where again the upper bound for $\tilde{p}$ depends on the point $\bar{x} \in \tilde{\Gamma}$ and the choice of a compact subset.

Making inverse change of variables we conclude that $p(x)$ is positive and bounded in some neighborhood $U'(\bar{x})$ of any point $\bar{x} \in \partial Q$ up to the boundary ∂Q. Let us take a covering of ∂Q which consists of these neighborhoods $U'(\bar{x})$. Since ∂Q is a compact set, there exist a finite subcovering of it. By means of the standard compactness arguments, one can prove that $p(x)$ is a positive continuous function in the closed set $\overline{Y}$.

4. **Existence of bounded solutions.** In what follows G_α^β is a finite cylinder $(\alpha,\beta)\times Q$, $\Sigma_\alpha^\beta = (\alpha, \beta \times \partial Q)$ is its lateral boundary and $S_\alpha = \{x = (x_1, x') : x_1 = \alpha,\, x' \in Q\}$.

Definition 4.1. We say that a weak solution of problem (1) is bounded if for any $N > 0$ the following inequality holds:

$$\|u\|_{L^2(G_N^{N+1})} \le C,$$

where C does not depend on N.

Lemma 4.2. *A bounded solution $u(x)$ of problem (1) in terms of Definition 4.1 exists. Moreover,*

$$\|\nabla u\|_{L^2(G)} < \infty, \quad \|u\|_{L^\infty(G_1^\infty)} < \infty. \tag{7}$$

Proof. First we consider the following boundary value problem in a finite cylinder

$$\begin{cases} -\mathrm{div}(a(x)\nabla u^k) - (b(x), \nabla u^k) = 0, & x \in G_0^k, \\ \dfrac{\partial u^k}{\partial n_a} = 0, & x \in \Sigma_0^k, \\ u^k(0,x') = \varphi(x'), \quad u^k(k,x') = 0, & x' \in Q. \end{cases} \tag{8}$$

It is known that the solution to problem (8) exists and for any $k > 0$ has finite H^1 and L^∞ norms. Obviously, in view of the maximum principle since $u^k(k,x') = 0$

$$\|u^k\|_{L^\infty(S_1)} \le \|u^k\|_{L^\infty(S_{1/2})}.$$

Let us consider in the cylinder G_0^1 the following auxiliary problem

$$\begin{cases} -\mathrm{div}(a(x)\nabla z^k) - (b(x), \nabla z^k) = 0, & x \in G_0^1, \\ \dfrac{\partial z^k}{\partial n_a} = 0, & x \in \Sigma_0^1, \\ z^k(0,x') = \varphi(x'), \quad z^k(1,x') = u^k(1,x'), & x' \in Q. \end{cases}$$

Since the last problem is linear, we can represent z^k as a sum $z_1 + z_2^k$, where z_1 and z_2^k satisfy the homogeneous equation and lateral boundary conditions, $z_1(0,x') = \varphi(x')$, $z_1(1,x') = z_2^k(0,x') = 0$, $z_2^k(1,x') = u^k(1,x')$. It is known that for the function $z_1(x)$ as a solution of elliptic problem in a fixed domain the following estimate holds:

$$\|z_1^k\|_{L^\infty(S_{1/2})} \le C\,\|\varphi\|_{H^{1/2}(Q)}.$$

The $L^\infty(S_{1/2})$ norm of the function z_2^k can be estimated in terms of $L^\infty(S_1)$ norm of u^k as follows

$$\|z_2^k\|_{L^\infty(S_{1/2})} \le \alpha \, \|u^k\|_{L^\infty(S_1)}, \tag{9}$$

where $0 < \alpha < 1$, α does not depend on k. Indeed, $|z_2^k| \le v^k$ in G_0^1, where v^k satisfies the same equation and boundary conditions as z_2^k, except for the boundary conditions on S_1, which reads $v^k(1,x') = \|u^k\|_{L^\infty(S_1)}$. Due to the strong maximum principle, $v^k \le \alpha \|u^k\|_{L^\infty(S_1)}$ with $0<\alpha<1$, that yields (9). In this way we obtain

$$\begin{aligned}\|u^k\|_{L^\infty(S_1)} &\le \|u^k\|_{L^\infty(S_{1/2})} \le \|z_1\|_{L^\infty(S_{1/2})} + \|z_2^k\|_{L^\infty(S_{1/2})}\\ &\le C\|\varphi\|_{H^{1/2}(Q)} + \alpha\, \|u^k\|_{L^\infty(S_1)}, \quad \alpha<1,\end{aligned}$$

and, finally

$$\|u^k\|_{L^\infty(S_1)} \le \frac{C}{1-\alpha}\, \|\varphi\|_{H^{1/2}(Q)}, \quad 0<\alpha<1.$$

Moreover, the $L^2(G_0^1)$ norm of z_1 is bounded

$$\|z_1\|_{L^2(G_0^1)} \le C_1 \, \|\varphi\|_{H^{1/2}(Q)},$$

and, since $u^k(1,x') \in L^\infty(S_1)$ then

$$\|z_2^k\|_{L^2(G_0^1)} \le C_2 \, \|\varphi\|_{H^{1/2}(Q)},$$

where C_1 and C_2 do not depend on k. Also, in view of the maximum principle,

$$\|u^k\|_{L^\infty(G_1^k)} \le C \, \|\varphi\|_{H^{1/2}(Q)}, \tag{10}$$

with C independent on k. Obviously, it follows from the last estimates that

$$\|u^k\|_{L^2(G_N^{N+1})} \le C, \quad N \ge 0. \tag{11}$$

Let us note that the estimate (10) is valid in $L^\infty(G_\delta^k)$, for any $\delta>0$:

$$\|u^k\|_{L^\infty(G_\delta^k)} \le C(\delta)\|\varphi\|_{H^{1/2}(Q)}, \quad \forall \delta>0, \tag{12}$$

with $C(\delta)$ independent on k.

In order to estimate the L^2-norm of the gradient of u^k in G_0^k, notice first that by the standard elliptic estimates in the cylinder G_0^2 we get

$$\|\nabla u^k\|_{L^2(G_0^2)} \le C\|\varphi\|_{H^{1/2}(Q)}. \tag{13}$$

Notice also, that $p\, u^k$ is $H^1(G_1^2)$ function because both $p(x)$ and $u^k(x)$ are elements of $H^1(G_1^k)\cap L^\infty(G_1^k)$. Moreover, the estimate holds true

$$\|pu^k\|_{H^1(G_1^2)} \le C\|\varphi\|_{H^{1/2}(Q)}. \tag{14}$$

Since $\mathrm{div}\big(a\nabla u^k\big) \in L^2(G_0^k)$ and $\mathrm{div}\big(a\nabla p - bp\big) = 0$, then the normal components of $\big(a\nabla u^k\big)$ and $\big(a\nabla p - bp\big)$ on S_1 are well-defined elements of $H^{-1/2}(Q)$ (see [2]), and the inequality holds

$$\|a_{1j}\partial_{x_j} u^k\|_{H^{-1/2}(Q)} \le C\|\varphi\|_{H^{1/2}(Q)}, \quad \|a_{1j}\partial_{x_j} p - b_1 p\|_{H^{-1/2}(Q)} \le C. \tag{15}$$

If we multiply the equation in (8) by pu^k and integrate the resulting relation over the cylinder G_1^k, then considering (5) and integrating several times by parts, we obtain

$$\int\limits_{G_1^k} (a\nabla\, u^k, \nabla\, u^k)\, p\, dx = \int\limits_{S_1} u^k p\, a_{1j} \frac{\partial u^k}{\partial_{x_j}}\, dx' - \frac{1}{2}\int\limits_{S_1} (u^k)^2 \, (a_{1j}\frac{\partial p}{\partial_{x_j}} - bp)\, dx.$$

Taking into account (10) and (13)-(15), we estimate the integral on the left-hand side as follows

$$\int\limits_{G_1^k} (a\nabla\, u^k, \nabla\, u^k)\, p\, dx \le C\|\varphi\|^2_{H^{1/2}(Q)}.$$

This estimate and (13) imply the desired bound

$$\|\nabla\, u^k\|_{L^2(G_0^k)} \le C\|\varphi\|_{H^{1/2}(Q)}, \tag{16}$$

where C does not depend on k.

Finally, using (11), (16) and compactness arguments, we conclude that $u^k(x)$ converges weakly in $H^1_{\rm loc}(G)$ to a function $u(x)$ which is a solution of problem (1) such that (7) holds true. Let us note that in view of the Nash–De Giorgi estimates (see [21]), for any $\delta > 0$ a solution of problem (1) is a Hölder-continuous function in $\overline{G_\delta^\infty}$ up to the lateral boundary of the cylinder. □

Remark 1. Let us note that we did not use the x_1-periodicity of the coefficients $a_{ij}(x)$ and $b_j(x)$ to prove the estimates (10) and (11). The proof is valid for the case of arbitrary measurable bounded coefficients $a_{ij}(x)$ and $b_j(x)$ and uniformly elliptic matrix $a(x)$.

5. **Stabilization of solutions.** In this section we are going to show that every bounded solution of problem (1) stabilizes to a constant at the exponential rate. To this end let us consider two functions of the variable x_1:

$$M(x_1) = \max_{x'\in Q} u(x_1, x') \quad \text{and} \quad m(x_1) = \min_{x'\in Q} u(x_1, x').$$

By the maximum principle the function $M(x_1)$ does not assume a local maximum point in the open interval $x_1 \in (0, +\infty)$). This implies that $M(x_1)$ has at most one minimum point on $[0, +\infty)$), and that, starting form this minimum point, $M(x_1)$ is monotonous. If $M(x_1)$ does not have minimum point, then it is monotonous on the whole interval $[0, +\infty)$. Similarly, $m(x_1)$ is monotonous, possibly starting from some point.

Therefore, we have only three possibilities for the behavior of the functions $M(x_1)$ and $m(x_1)$:

- $M(x_1)$ monotonously decreases and $m(x_1)$ monotonously increases;
- $M(x_1)$ and $m(x_1)$ monotonously increase (maybe starting from some point);
- $M(x_1)$ and $m(x_1)$ monotonously decrease (maybe starting from some point).

5.1. $M(x_1)$ **monotonously decreases and** $m(x_1)$ **monotonously increases.** Denote $G_N^{N+2} = (N, N+2) \times Q$, $N \ge 0$. Our aim is to estimate the oscillation of $u(x)$ over the cross-section S_{N+1} in terms of the oscillation of $u(x)$ over S_N. Since problem (1) is linear, then we can assume without loss of generality that $m(N) = 0$. Then in G_N^{N+2} the function $u(x)$ is nonnegative. As was shown above, the solution $u(x)$ can be extended to a larger domain $(N, N+2)\times\tilde{Q}$, $\overline{Q} \subset \tilde{Q}$, in such a way that the extended function satisfies a convection-diffusion equation in $(N, N+2) \times \tilde{Q}$, and the maximum and the minimum of the extended function over cross-section $\{x_1 = k,\, x' \in \tilde{Q}\}$ coincide with $M(k)$ and $m(k)$, respectively.

Thus, the Harnack inequality holds:

$$m(k) \ge \alpha M(k), \quad \forall k \ge 1,$$

where a constant α depends only on Λ, d and Q. Then

$$M(N+1) - m(N+1) \le (1-\alpha)\, M(N+1) \le (1-\alpha)\, M(N). \tag{17}$$

Taking into account (17) and the assumption $m(N) = 0$, we obtain

$$\operatorname*{osc}_{x_1=N+1} u(x) = M(N+1) - m(N+1) \le (1-\alpha) \operatorname*{osc}_{x_1=N} u(x),\ 0<\alpha<1,\ N>0.$$

The last inequality implies that $u(x)$ stabilizes to a constant exponentially. Indeed, since this inequality holds for all $N > 0$, then

$$\operatorname*{osc}_{x_1=N} u(x) \le (1-\alpha)^{N-1} \operatorname*{osc}_{x_1=1} u(x).$$

Finally, taking into account the boundedness of the function $u(x)$ (see (7)) and denoting by C_∞ the limit of $m(x_1)$ as $x_1 \to \infty$, we obtain

$$|u(x) - C_\infty| \le C_0\, e^{-\gamma_0\, x_1}, \quad \gamma_0 = -\log(1-\alpha) > 0. \tag{18}$$

Remark 2. One can see that the constant C_0 in (18) in this case takes the form

$$C_0 \le C_1\, \|\varphi\|_{H^{1/2}(Q)} + C_2\, C_\infty \le C_1 \|\varphi\|_{H^{1/2}(Q)} + C_2 C \|\varphi\|_{H^{1/2}(Q)}.$$

Indeed, taking into account the linearity of the problem and estimate (10), we have

$$|\operatorname*{osc}_{x_1=1} u| \le 2\, \|u\|_{L^\infty(S_1)} \le C\, \|\varphi\|_{H^{1/2}(Q)}.$$

Let us emphasize also that the constant γ_0 depends only on the ellipticity constant Λ, the space dimension d and the domain Q.

Lemma 5.1. *There always exists a unique solution $u_0(x)$ of problem (1) for which $M(x_1)$ decreases and $m(x_1)$ increases.*

The function $u_0(x)$ stabilizes to a constant C_φ^∞ exponentially, as $x_1 \to \infty$:

$$|u_0 - C_\varphi^\infty| \le C_0\, \|\varphi\|_{H^{1/2}(Q)}\, e^{-\gamma_0 x_1}, \quad x_1 > 1.$$

If $\varphi(x') \in L^\infty(Q)$ then for $u_0(x)$ the maximum principle is valid, that is

$$\min_{x' \in Q} \varphi(x') \le u_0(x) \le \max_{x' \in Q} \varphi(x').$$

Proof. Indeed, such a solution can be constructed with the help of the following auxiliary problems:

$$\begin{cases} -\mathrm{div}(a(x)\nabla\, u^k) - (b(x), \nabla\, u^k) = 0, & x \in G_0^k, \\ \dfrac{\partial u^k}{\partial n_a} = 0, & x \in \Sigma_0^k, \\ u^k(0,x') = \varphi(x'), \quad \dfrac{\partial u^k}{\partial n_a}(k,x') = 0, & x' \in Q. \end{cases}$$

By the maximum principle, $M^k(x_1) = \max\limits_{x' \in Q} u^k(x_1, x')$ is decreasing and $m^k(x_1) = \min\limits_{x' \in Q} u^k(x')$ is increasing function, for any k. If $\varphi \in L^\infty(Q)$ then

$$\min_{x' \in Q} \varphi(x') \le u^k(x) \le \max_{x' \in Q} \varphi(x'), \quad \forall k > 0.$$

Passing to the limit as $k \to \infty$ completes the proof. Due to the maximum principle the obtained solution u_0 to problem (1) is unique. In view of Remark 2 the rate of exponential stabilization of u_0 to C_φ^∞ depends only on Λ, d and Q. ∎

5.2. $M(x_1)$ **and** $m(x_1)$ **are monotonously decreasing (increasing) functions.** If $M(x_1)$ and $m(x_1)$ decrease for sufficiently large x_1, then $M(x_1)$ monotonously decreases on the whole half-line $[0, +\infty)$, while $m(x)$ might have at most one maximum point. One can take N_0 large enough so that on the interval $[N_0, \infty)$ both functions are monotonous. Obviously, it is sufficient to prove the stabilization in the case of monotonously decreasing at infinity functions $M(x_1)$ and $m(x_1)$: the case when $M(x_1)$ and $m(x_1)$ are monotonously increasing functions can be considered in a similar way. As before we assume that $m(N+2) = 0$.

First of all, due to monotonicity and boundedness of $M(x_1)$ and $m(x_1)$ (we consider only bounded solutions) the following limits exist:

$$\lim_{x_1\to\infty} M(x_1) = M, \quad \lim_{x_1\to\infty} m(x_1) = m.$$

For arbitrary $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, let $N > 0$ be such that

$$M(N) - M(N+2) < \varepsilon_1, \quad m(N) - m(N+2) = m(N) < \varepsilon_2.$$

Then, by the Harnack inequality, in the domain $S_{N+1} = \{N+1\} \times Q$ the estimate holds

$$\varepsilon_2 > m(N+1) \geq \alpha\, M(N+1) \geq \alpha M(N+2) > \alpha M(N) - \alpha\, \varepsilon_1.$$

Thus, we have that

$$\operatorname*{osc}_{x_1=N} \tilde{u}(x) \to 0, \quad N \to \infty.$$

The last equality shows that the functions $M(x_1)$ and $m(x_1)$ converge to the same constant, that is $u(x)$ stabilizes to the constant.

Now we are going to prove that $u(x)$ stabilizes to the constant exponentially. Without loss of generality we can assume that $u(x)$ stabilizes to zero. Instead of the original function $u(x)$ we consider shifted function $\tilde{u}(x) = u(x_1 + N_0, x')$ for $N_0 \geq 0$. Due to the periodicity of the coefficients, $\tilde{u}$ remains a solution of the same problem but with different boundary function at S_0, which we denote by $\psi(x') = \tilde{u}(0, x') = u(N, x')$. Clearly, $\psi(x')$ is a positive continuous function. Let us define a function $v(x)$ as a solution of problem (1) with $v(0, x') = 1$ and $v \to 0$ as $x_1 \to \infty$. The existence of such a solution can be justified as follows. As in (8), one can construct approximations $\tilde{u}^k$ and v^k for the functions $\tilde{u}(x)$ and $v(x)$. By the maximum principle $\tilde{u}^k(x) \geq \big(\min\limits_{x'\in Q} \psi\big) v^k(x)$. Passing to the limit, as $k \to \infty$, in this inequality, we obtain:

$$v \leq \frac{\tilde{u}(x)}{\min\limits_{x'\in Q} \psi} \to 0, \quad x_1 \to \infty.$$

Thus, the required solution v exists. By the maximum principle

$$\frac{\tilde{u}(x)}{\max\limits_{x'\in Q} \psi} = \frac{\tilde{u}(x)}{M(N)} \leq v(x_1, x'),$$

so, setting $x_1 = 1$ and taking maximum over $x \in Q$ of both sides of the last inequality, we obtain

$$\frac{M(N+1)}{M(N)} \leq \max_{x'\in Q} v(1, x') \leq \beta < 1.$$

Consequently we have:

$$M(N+1) \leq \beta\, M(N), \quad \forall N \geq 0,$$

or

$$M(N) \le e^{(N-1)\,\ln\beta} \max_{x'\in Q} u(1,x'), \quad N \ge 1.$$

Denoting by γ the positive constant $-\log\beta$ and using estimate (10), we obtain

$$M(N) \le C_1 \, \|\varphi\|_{H^{1/2}(Q)} \, e^{-\gamma N},$$

or, in other words,

$$|u| \le C_0 \, e^{-\gamma x_1}, \quad x_1 > 1.$$

In the general case, when a bounded solution $u(x)$ to problem (1) stabilizes to a nonzero constant C_∞, one can see that C_0 in the last inequality takes the form

$$C_0 = C_1 \, \|\varphi\|_{H^{1/2}(Q)} + C_2 \, C_\infty,$$

with constants C_1 and C_2 which depend only on Λ, d and Q. In this way we have proved the following

Lemma 5.2. *Under our standing assumptions on $a_{ij}(x)$ and $b_j(x)$, $i,j = 1,...,d$, every bounded solution of problem (1) stabilizes to a constant at the exponential rate, as $x_1 \to \infty$.*

Remark 3. It should be noted that if we replace in (1) the homogeneous Neumann boundary condition on Σ with zero-flux condition

$$\frac{\partial u}{\partial n_a} - (b,n)u = 0,$$

then the corresponding periodic cell problem need not have a nontrivial kernel; in particular, a constant need not be an eigenfunction. In this case the problem

$$\begin{cases} -\mathrm{div}\,\left(a(x)\,\nabla\, u(x)\right) - (b(x), \nabla\, u(x)) = 0, & x \in G, \\ \dfrac{\partial u}{\partial n_a} - (b,n)u = 0, & x \in \Sigma, \\ u(0,x') = \varphi(x'), & x' \in Q \end{cases} \tag{19}$$

might have a bounded solution which does not stabilize to a constant at infinity. For example, a function $u(x_1,x_2) = \sin(\sqrt{2}x_1)e^{x_2}$ satisfies the problem

$$\begin{cases} -\dfrac{\partial^2 u}{\partial x_1^2} - \dfrac{\partial^2 u}{\partial x_2^2} - \dfrac{\partial u}{\partial x_1} = 0, & x \in (0,+\infty)\times(0,1), \\ \big(\dfrac{\partial u}{\partial x_2} - u\big)(x_1,0) = \big(\dfrac{\partial u}{\partial x_2} - u\big)(x_1,1) = 0, & x_1 \in (0,+\infty), \\ u(0,x_2) = 0, & x_2 \in (0,1), \end{cases}$$

clearly, this solution is bounded, but does not converge to a constant, as $x_1 \to \infty$.

The detail analysis of problem (19) requires quite delicate arguments of spectral theory and is out of the scope of the present paper.

6. **Main result.** In order to formulate the main result we introduce the notation

$$\bar{b}_1 = \int\limits_{G_0^1} \left(a_{1j}(x) \frac{\partial p(x)}{\partial x_j} - b_1(x)p(x) \right) dx, \tag{20}$$

where the auxiliary function $p(x)$ was introduced in Section 3. Let us notice that in view of the periodicity of the coefficients, the integral on the right-hand side of (20) can be taken over G_k^{k+1} for any $k > 0$. Moreover, this integral can be taken over arbitrary cross section $S_\xi = \{\xi\} \times Q$. Indeed, integrating (5) over G_ξ^η we obtain the following equality

$$\int\limits_{S_\xi} \left(-a_{1j}(x) \frac{\partial p(x)}{\partial x_j} + b_1(x) p(x) \right) dx' = \int\limits_{S_\eta} \left(-a_{1j}(x) \frac{\partial p(x)}{\partial x_j} + b_1(x) p(x) \right) dx',$$

for any positive ξ and η. Thus, for any $\xi > 0$

$$\bar{b}_1 = \int\limits_{S_\xi} \left(a_{1j}(x) \frac{\partial p(x)}{\partial x_j} - b_1(x) p(x) \right) dx' = \text{Const.}$$

Theorem 6.1. *Let $a_{ij}(x) \in L^\infty(G)$, $b_j(x) \in L^\infty(G)$ be x_1-periodic functions, and suppose that the condition (2) is fulfilled. Then the following statements hold:*

1. *Every bounded (in terms of Definition 4.1) solution $u(x)$ of problem (1) stabilizes to a constant at the exponential rate as $x_1 \to \infty$, that is*
$$|u(x) - C_\infty| \le C\, e^{-\gamma x_1}, \quad C,\ \gamma > 0,\ x_1 > 1,$$
where the convergence rate γ does not depend on φ and C_∞.
2. *$\bar{b}_1 < 0$ if and only if for any $\varphi(x') \in H^{1/2}(Q)$ and for any $l \in \mathbb{R}$, there exists a bounded solution $u(x)$ of problem (1) that converges to the constant l, as $x_1 \to \infty$;*
3. *$\bar{b}_1 \ge 0$ if and only if there exists a unique bounded solution $u(x)$ of problem (1) and it converges to a constant $m = m(\varphi)$, as $x_1 \to \infty$.*

Remark 4. In the case $\bar{b}_1 \ge 0$ for a solution $u(x)$ of problem (1), the function $M(x_1)$ is decreasing and the function $m(x_1)$ is increasing.

Indeed, by virtue of Lemma 5.1, there exists a solution to problem (1) such that the corresponding $M(x_1)$ monotonously decreases and $m(x_1)$ monotonously increases. Since $\bar{b}_1 \ge 0$, the mentioned solution is unique, and the required statement follows.

Although in the case $\bar{b}_1 < 0$ a bounded solution is not unique, the solution for which $M(x_1)$ decreases and $m(x_1)$ increases remains unique. Such a solution depends continuously on the boundary data $\varphi(x')$ and defines uniquely the constant C_φ^∞, to which it converges. This constant will play an important role in the sequel.

Remark 5. Let us note that in the case when $M(x_1)$ and $m(x_1)$ are both decreasing or increasing functions, the stabilization rate may depend on $\bar{b}_1$ (cf. Remark 2). In general, γ may tend to zero, as $\bar{b}_1$ goes to zero. Indeed, let us consider the following problem with constant coefficients:

$$\begin{cases} \Delta u + b_1\, \partial_{x_1} u = 0, \quad x \in G, \\ u(0, x') = 1. \end{cases} \tag{21}$$

It is easy to see that in this case $p(x) = \text{Const}$, $\bar{b}_1 = -b_1$ and all the solutions of problem (21) depend only on x_1; furthermore,

$$u(x) = C_1 + C_2\, e^{-b_1 x_1}, \tag{22}$$

with some constants C_1 and C_2. Obviously, if $\bar{b}_1 \ge 0$ then a solution to problem (21) is unique and equal to 1; if $\bar{b}_1 < 0$ then every bounded solution stabilizes to a constant at the exponential rate. As follows from (22), the stabilization rate goes to zero as $\bar{b}_1 = -b_1 \to 0$.

Proof of Theorem 6.1. 1. Stabilization of every bounded solution had been proved above in Section 5.

2. Assume that for any $\varphi(x')$ and for every constant k there exists a solution that converges to this constant. We are going to prove that in this case $\bar{b}_1 < 0$. To this end we denote by $\tilde{u}(x)$ the solution of problem (1) with $\varphi(x') \equiv 1$ such that $\tilde{u}(x) \to 0$, as $x_1 \to \infty$. Letting $u(x) = 1 - \tilde{u}$, we obtain a solution of problem (1) with $u(0, x') = 0$. If we multiply the equation in (1) by $p(x)u(x)$ and integrate the resulting relation over $G_0^\xi = (0, \xi) \times Q$, then we obtain

$$\int\limits_{G_0^\xi} (a\nabla u, \nabla u)\, p\, dx + \frac{1}{2} \int\limits_{S_\xi} \left(\frac{\partial p}{\partial n_a} - (b, n)p \right) u^2\, dx' + \int\limits_{S_\xi} a_{1j} \frac{\partial u}{\partial x_j} u\, p\, dx' = 0.$$

Integrating on ξ from N to $N+1$, for some $N > 0$, gives:

$$\int\limits_N^{N+1} \int\limits_{G_0^\xi} (a\nabla u, \nabla u)\, p(x)\, dx\, d\xi + \frac{1}{2} \int\limits_{G_N^{N+1}} \left(\frac{\partial p}{\partial n_a} - (b, n)p \right) u^2\, dx \\ + \int\limits_{G_N^{N+1}} a_{1j} \frac{\partial u}{\partial x_j} u(x)\, p(x)\, dx = 0.$$

Now, we use the facts that the integral $\int_{G_0^\xi} (a\nabla u, \nabla u)\, p(x)\, dx$ is an increasing function of ξ, $p(x) > 0$ is bounded and, due to our assumption, $u(x)$ stabilizes to 1 at the exponential rate, as $x_1 \to \infty$. Then for sufficiently large N the following inequality holds:

$$\int\limits_{G_0^N} (a\nabla u, \nabla u)\, p\, dx + \frac{1}{2} \bar{b}_1 \le C\, \|\nabla u\|_{L^2(G_N^{N+1})} \|u\|_{L^2(G_N^{N+1})}$$

Combining standard elliptic estimates for $\tilde{u}(x) = 1 - u(x)$ (extended as in Section 3 to a bigger domain) with the assumption on $\tilde{u}(x)$, one can see that

$$\|\nabla u\|_{L^2(G_N^{N+1})} = \|\nabla \tilde{u}\|_{L^2(G_N^{N+1})} \le C \|\tilde{u}\|_{L^2(G_{N-1}^{N+2})} \le C\, e^{-\gamma N}, \quad \gamma > 0,$$

and, therefore,

$$\int\limits_{G_0^N} (a\nabla u, \nabla u)\, p\, dx + \frac{1}{2} \bar{b}_1 \le C\, e^{-\gamma N}.$$

Passing to the limit as $N \to \infty$ implies that $\bar{b}_1 < 0$. The inverse implication will follow from Lemma 6.2 below.

3. We consider the following sequence of auxiliary boundary value problems:

$$\begin{cases} -\mathrm{div}\,(a(x)\nabla u^k) - (b, \nabla u^k) = 0, & x \in G_0^k, \\ \dfrac{\partial u^k}{\partial n_a} = 0, & x \in \Sigma_0^k, \\ u^k(0, x') = 1, \quad u^k(k, x') = 0. \end{cases} \tag{23}$$

First we show that if the sequence $u^k(x)$ of solutions of the auxiliary problems (23) converges uniformly to 1 on every compact set as $k \to \infty$, then $\bar{b}_1 \ge 0$.

Let us multiply the first equation in (23) by $p(x)u^k(x)$ and integrate the resulting relation over G_0^k. Integrating by parts and taking into account the boundary conditions $u^k(k,x')=0$, we obtain

$$\int\limits_{G_0^k}(a\,\nabla\,u^k,\nabla\,u^k)\,p\,dx-\frac{1}{2}\int\limits_{S_0}\left(a_{1j}\frac{\partial p}{\partial x_j}-b_1\,p\right)\,dx'-\int\limits_{S_0}a_{11}\frac{\partial u^k}{\partial x_1}\,p\,dx'=0.$$

In view of the maximum principle u^k cannot attend its maximum in the interior of the domain G_0^k, so

$$\bar{b}_1=2\int\limits_{G_0^k}(a\,\nabla\,u^k,\nabla\,u^k)\,p\,dx+\int\limits_{S_0}a_{11}\frac{\partial u^k}{\partial x_1}\,p\,dx'\geq 0.$$

Next we prove the following

Lemma 6.2. *The following two conditions are equivalent:*

(i) For every boundary condition $\varphi(x')$ there exists a unique bounded solution of problem (1) and this solution converges to a constant $m=m(\varphi)$, as $x_1\to\infty$;

(ii) Solutions $u^k(x)$ of problem (20) with the boundary condition $\varphi(x')=1$ converge uniformly on every compact set $K\Subset G_0^k$ to 1, as $k\to\infty$.

Proof of Lemma 6.2. Let condition (i) be fulfilled. Then, obviously, $u(x)\equiv 1$ if $\varphi=1$. Since $u^k\to u$, as $k\to\infty$ in the space $H^1_{\rm loc}(G)$, then in view of De Giorgi estimates

$$u^k\rightrightarrows u=1,\quad k\to\infty$$

on every compact set in G.

Let (ii) hold true. Suppose that there exist $\varphi(x')$, two constants C^1_∞ and C^2_∞ and two bounded solutions u_1 and u_2 of problem (1) such that

$$u_1\to C^1_\infty,\quad u_2\to C^2_\infty,\quad x_1\to\infty.$$

Then the function $v=1-(u_1-u_2)/(C^1_\infty-C^2_\infty)$, which stabilizes to zero as $x_1\to\infty$, solves the following problem:

$$\begin{cases}-\mathrm{div}\ (a(x)\,\nabla\,v)-(b(x),\nabla\,v)=0, & x\in G,\\ \dfrac{\partial v}{\partial n_a}=0, & x\in\Sigma,\\ v(0,x')=1, & x'\in Q.\end{cases}$$

On the other hand, by the maximum principle $v(x)\geq u^k(x)$ where u^k is a solution of problem (23). According to (ii), u^k converges to 1 uniformly on every compact set in G_0^k, as $k\to\infty$. Thus $v(x)\geq 1$, $x\in G$. We arrive at contradiction. Lemma 6.2 is proved. □

4. It remains to prove that there are only two possible options for the behaviour of u^k: either $u^k(x)$ decays at the exponential rate, or $u^k(x)$ converges to 1 uniformly on every compact set, as $k\to\infty$.

Obviously, in view of the maximum principle, $\{u^k(x)\}$, for any $x\in G$, is a monotonously increasing sequence and $0\leq u^k(x)\leq 1$, for all $x\in G_0^k$. Thus $u^k(x)$ converges uniformly on every compact subset of G to a function $u(x)$, $0\leq u\leq 1$, which is a solution of problem (1) with $\varphi(x')=1$. In view of the maximum principle if $u(x)=1$ in some interior point of G, then $u(x)\equiv 1$,

$x \in G$. Hence, either u^k converges uniformly to 1 on every compact subset of G or

$$\lim_{k\to\infty} \max_{x'\in Q} u^k(x_1, x') < 1, \qquad \forall x_1 > 0.$$

Suppose that the latter case takes place, and denote $\lim\limits_{k\to\infty} \max\limits_{x'\in Q} u^{k+1}(1, x') = \beta < 1$. If we introduce

$$v_1^k(x_1, x') = \frac{u^{k+1}(x_1 + 1, x')}{\max\limits_{x'\in Q} u^{k+1}(1, x')},$$

then $v_1^k(0, x') \le 1$ and, due to the maximum principle, $v_1^k(x_1, x') \le u^k(x_1, x')$. This yields

$$u^{k+1}(x_1 + 1, x') \le u^k(x_1, x') \max_{x'\in Q} u^{k+1}(1, x') \le u^{k+1}(x_1, x') \max_{x'\in Q} u^{k+1}(1, x');$$

thus,

$$u^{k+1}(2, x') \le \beta\, u^{k+1}(1, x') \le \beta^2.$$

Similarly, we can construct

$$v_2^k(x_1, x') = \frac{u^{k+2}(x_1 + 2, x')}{\max\limits_{x'\in Q} u^{k+2}(2, x')}$$

and show that $\lim\limits_{k\to\infty} u^k(3, x') \le \beta^3$. Repeating this procedure, we obtain for any $N > 0$ the inequality $u^k(N, x') \le \beta^N$ which implies the exponential decay for $u^k(x)$, as $x_1 \to \infty$.

Theorem 6.1 is proved. □

Although in the statement of Theorem 6.1 one does not see any difference between the cases $\bar b_1 = 0$ and $\bar b_1 > 0$, the behaviour of the approximations u^k is rather distinct in these two cases. The lemmata below specify the difference.

Lemma 6.3. *Let $\bar b_1 > 0$. Then the solution u^k to problem (23) satisfies the estimate*

$$|u^k - 1| \le C\, e^{-\gamma(k-x_1)}, \quad x \in G_0^k, \tag{24}$$

where the constant C depends on Λ, d and Q; γ is a positive parameter which may depend on $\bar b_1$.

Proof. Making change of variables $z_1 = k - x_1$, $z' = x'$ in (23) and denoting

$$\tilde a(z) = S\, a(k - z_1, z')\, S^* = S\, a(-z_1, z')\, S^*, \quad \tilde b(z) = s\, b(k - z_1, z') = S\, b(-z_1, z'),$$

with

$$S = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{pmatrix},$$

we transform problem (23) to the form

$$\begin{cases} -\mathrm{div}\,(\tilde a(z)\nabla\, \tilde u^k) - (\tilde b, \nabla\, \tilde u^k) = 0, & z \in G_0^k, \\ \dfrac{\partial \tilde u^k}{\partial n_{\tilde a}} = 0, & z \in \Sigma_0^k, \\ \tilde u^k(0, z') = 1, \quad \tilde u^k(k, z') = 0. \end{cases}$$

It is easy to see that for the obtained problem the effective drift is negative. As was shown in the proof of Theorem 6.1, the function $(1-\tilde{u}^k)$ tends to zero exponentially, that is

$$|1-\tilde{u}^k(z)| \le C\, e^{-\gamma z_1}, \quad z \in G_0^k.$$

Making the inverse change of variables and taking into account that $\tilde{u}^k(z) = u^k(k-z_1, z')$, we obtain (24). □

Lemma 6.4. $\bar{b}_1 = 0$ *if and only if a solution* u^k *of the auxiliary problem (23) is close to the linear function on every compact set* $K \Subset G$*, that is if we denote*

$$l^k(x) = \begin{cases} 1 - \dfrac{x_1}{k}, & x_1 \le k, \\ 0, & x_1 > k \end{cases}$$

then

$$\|u^k(x) - l^k(x)\|_{L^\infty(G_0^k)} \to 0, \quad k \to \infty.$$

Proof. The method we use is borrowed from the homogenization theory (see, for example, [18]). Let us denote $\varepsilon = 1/k$ in 23) and make the change of variables

$$x_1 \mapsto \varepsilon\, x_1, \qquad x_j \mapsto x_j,\ j = 2, ..., d.$$

If we introduce the notation

$$a^\varepsilon(x_1, x') = a\left(\frac{x_1}{\varepsilon}, x'\right), \quad v^\varepsilon(x_1, x') = u\left(\frac{x_1}{\varepsilon}, x'\right),$$

then in the new variables equation (23) reads

$$\begin{aligned} &\varepsilon^2\, \partial_{x_1}(a_{11}^\varepsilon\, \partial_{x_1} v^\varepsilon) + \varepsilon \sum_{i\ne 1} \partial_{x_i}(a_{i1}^\varepsilon\, \partial_{x_i} v^\varepsilon) + \varepsilon \sum_{k \ne 1} \partial_{x_k}(a_{1k}^\varepsilon\, \partial_{x_1} v^\varepsilon) \\ &+ \sum_{i,k\ne 1} \partial_{x_k}(a_{ik}^\varepsilon\, \partial_{x_i} v^\varepsilon) + \varepsilon\, b_1^\varepsilon\, \partial_{x_1} v^\varepsilon + \sum_{k\ne 1} b_k^\varepsilon\, \partial_{x_k} v^\varepsilon = 0, \quad x \in G_0^1. \end{aligned} \tag{25}$$

The periodicity of the coefficients suggests the following ansatz

$$\tilde{v}^\varepsilon(x, y_1) = v^0(x, y_1) + \varepsilon\, v^1(x, y_1), \quad y_1 = \frac{x_1}{\varepsilon},$$

where the functions v^0 and v^1 are 1-periodic in y_1. Substituting this expression into the equality (25), collecting power–like terms related to ε^0 and taking into account the periodicity of all the functions in y_1, we obtain an equation for the function v^0:

$$\begin{aligned} \partial_{y_1}(a_{11}(y_1, x')\, \partial_{y_1} v^0(x, y_1)) &+ \sum_{i\ne 1} \partial_{y_1}(a_{i1}\, \partial_{x_i} v^0) + \sum_{k\ne 1} \partial_{x_k}(a_{1k}^\varepsilon\, \partial_{y_1} v^0) \\ &+ \sum_{i,k\ne 1} \partial_{x_k}(a_{ik}^\varepsilon\, \partial_{x_i} v^0) + b_1(y_1, x')\, \partial_{y_1} v^0 + \sum_{k\ne 1} b_k\, \partial_{x_k} v^0 = 0, \\ &\qquad x \in G_0^1,\ y_1 \in (0,1). \end{aligned} \tag{26}$$

Since $\partial u/\partial n_a = 0$ for $x \in \Sigma_0^1$, then making simple rearrangements we obtain the boundary condition for the function v^0:

$$\sum_j a_{1j}(y_1, x')\, n_j\, \partial_{y_1} v^0 + \sum_{j;i\ne 1} a_{ij}(y_1, x')\, n_j\, \partial_{x_i} v^0 = 0, \quad x' \in \partial Q,\ y_1 \in (0,1). \tag{27}$$

The solution of the boundary value problem (26)–(27) does not depend on the variables y_1 and x', that is:

$$v^0(x, y_1) = v^0(x_1).$$

Following the ideas of the homogenization theory, we represent $\tilde{v}^\varepsilon(x, y_1)$ as follows:

$$\tilde{v}^\varepsilon(x, y_1) = v^0(x_1) + \varepsilon\, \chi_1(y_1, x')\, \partial_{x_1} v^0(x_1) + \varepsilon^2\, \psi(x_1, y_1, x'), \tag{28}$$

where the periodic in y_1 scalar functions χ_1 and ψ are to be found. For convenience let us denote the "fast" variables (y_1, x') by $z = (z_1, z')$. Then, collecting the terms of order ε, one can obtain the following equation for the function $\chi_1(z)$:

$$-\mathrm{div}_z\,(a(z)\,\nabla\,\chi_1) - (b(z), \nabla_z\,\chi_1) = \sum_{k=1}^{d} \partial_{z_k}\, a_{1k} + b_1, \quad z \in Y = (0,1)\times Q. \tag{29}$$

The boundary conditions for the function χ_1 on the lateral boundary of the cylinder take the form

$$(a(z)\nabla_z\,\chi_1, n) = -\sum_i a_{i1} n_i, \quad z \in \partial Y = (0,1)\times\partial Q. \tag{30}$$

Due to the Fredholm Alternative, problem (29) – (30) is solvable if and only if the following equality holds:

$$\int\limits_Y \left(\sum_{k=1}^{d} \partial_{z_k}\, a_{1k} + b_1\right) p(z)\, dz - \int\limits_{\partial Y} \sum_{i=1}^{d} a_{i1} n_i p\, d\sigma = 0, \tag{31}$$

where the function $p(z)$ is a solution of the following problem:

$$\begin{cases} -\mathrm{div}(a(z)\nabla\, p) + \mathrm{div}\,(b(z)\, p) = 0, & z \in Y,\\ (a(z)\nabla p, n) - (b(z), n)\, p = 0, & z \in \partial Y, \end{cases} \tag{32}$$

Since we assume that $\bar{b}_1$ is equal to zero, then the condition (31) holds. Indeed, integrating by parts and making simple rearrangements, it is easy to see that the left hand side of (31) coincides with $\bar{b}_1$.

Finally, collecting the terms of order ε^2, we obtain the following problem for the function $\psi(x_1, z)$:

$$\begin{aligned} -\mathrm{div}(a(z), \nabla_z\psi) - (b(z), \nabla_z\psi) \quad = \quad & \Big[a_{11} + \sum_{k=1}^{d} \partial_{z_k}(a_{1k}\chi_1) + \sum_{i=1}^{d} a_{i1}\partial_{z_i}\chi_1 \\ & + a_{11}\partial_{z_1}\chi_1 + b_1\chi_1\Big]\, \partial_{x_1x_1} v^0,\ \ z \in Y, \end{aligned} \tag{33}$$

$$(a(z)\nabla_z\psi, n) = -\sum_i a_{i1} n_i \chi_1 \partial_{x_1x_1} v^0, \quad z \in \partial Y. \tag{34}$$

Using one more time the Fredholm Alternative for problem (33) – (34) we get:

$$\bar{a}_{11}\, \partial_{x_1x_1} v^0(x_1) = 0, \quad x_1 \in (0,1),$$

where the constant $\bar{a}_{11}$ is given by

$$\begin{aligned} \bar{a}_{11} = \int\limits_Y \left(a_{11} + \sum_{k=1}^{d} \partial_{z_k}\,(a_{1k}\,\chi_1) + \sum_{i=1}^{d} a_{i1}\,\partial_{z_i}\,\chi_1 + a_{11}\partial_{z_1}\,\chi_1 + b_1\chi_1\right) p\, dz \\ - \int\limits_{\partial Y} \sum_i a_{i1} n_i \chi_1 p\, d\sigma. \end{aligned}$$

Integrating by parts we obtain the following expression for the constant $\bar{a}_{11}$:

$$\bar{a}_{11} = \int\limits_Y \left(a_{11} - \sum_{k=1}^{d} a_{1k}\chi_1\partial_{z_k}\, p + \sum_{i=1}^{d} a_{i1}\,\partial_{z_i}\,\chi_1\, p + b_1\,\chi_1\, p\right) dz. \tag{35}$$

Let us show that $\bar{a}_{11} > 0$. Then $\partial_{x_1x_1} v^0(x_1) = 0$ and, as a consequence, $v^0(x_1)$ is a linear function on x_1. The scheme of the proof is as follows:

1. We construct the matrix $\bar{A}$ such that $\bar{A}_{11} = \bar{a}_{11}$. Namely, we set

$$\bar{A}_{ij} = \int\limits_{Y} a_{ik}(\delta_{kj} + \partial_{z_k}\chi_j)\, p\, dz - \int\limits_{Y} \chi_i a_{mj}\partial_{z_m} p\, dz + \int\limits_{Y} \chi_i \tilde{b}_j p\, dz,$$

where χ_1 is defined in (29) and the functions χ_k for $k \neq 1$ are defined by the equations:

$$\begin{cases} -\mathrm{div}\,(a(z)\nabla\chi_k) - (b(z), \nabla\chi_k) = -\partial_{z_k} a_{k1} - b_k - \bar{b}_k, & z \in Y, \\ \dfrac{\partial \chi_k}{\partial n_a} = -a_{ki}\, n_i, \quad z \in \partial Y, \end{cases}$$

and $\bar{b}_k$ are given by the formula:

$$\bar{b}_k = \int\limits_{Y} (a_{ki}\,\partial_{z_i} p - b_k\, p)\, dz.$$

2. On the second step we prove that $\bar{A}$ is nonnegative definite matrix. For this purpose we show that this matrix can be represented in the form

$$\bar{A} = \int\limits_{Y} (I + \nabla\chi)^T a(z)(I + \nabla\chi)\, p(z)\, dz, \tag{36}$$

where B^T denotes the adjoint of B.
3. Then we show that $(I + \nabla\chi)\, e_1 \neq 0$.
4. For an arbitrary nonnegative definite matrix $C = \{c_{ij}\}$ we state that if $c_{11} = 0$, then $c_{1k} = 0$, $k = \overline{2, d}$. Thus $C\, e_1 = 0$. We then show that $\bar{A}e_1 \neq 0$ in our case. Therefore $\bar{a}_{11}$ cannot be equal to zero.

Now we proceed with the detail proof. The fact that $\bar{A}_{11} = \bar{a}_{11}$ readily follows from from the definition of the matrix $\bar{A}$. In order to prove (36) let us re-arrange the expression on the right hand side:

$$\begin{aligned} \int\limits_{Y} (\delta_{im} + \partial_{z_m}\chi_i)\, a_{mk}\, (\delta_{kj} + \partial_{z_k}\chi_j)\, p\, dz &= \int\limits_{Y} \delta_{im}\, a_{mk}\, (\delta_{kj} + \partial_{z_k}\chi_j)\, p\, dz \\ &\quad + \int\limits_{Y} \partial_{z_m}\chi_i\, a_{mk}\, (\delta_{kj} + \partial_{z_k}\chi_j)\, p\, dz. \end{aligned}$$

Integrating the second term by parts gives

$$\begin{aligned} \bar{A}_{ij} = \int\limits_{Y} a_{ik}\, (\delta_{kj} + \partial_{z_k}\chi_j)\, p\, dz - \frac{1}{2}\int\limits_{Y} \chi_i\chi_j \partial_{z_k}(b_k\, p)\, dz + \frac{1}{2}\int\limits_{\partial Y} \chi_i\chi_j\, b_k\, n_k\, p\, dz \\ + \int\limits_{Y} \chi_i\tilde{b}_j\, p\, dz - \int\limits_{Y} \chi_i\, a_{mj}\, \partial_{z_m} p\, dz + \frac{1}{2}\int\limits_{Y} \chi_i\, \chi_j\, \partial_{z_k}(a_{mk}\, \partial_{z_m} p)\, dz \\ - \frac{1}{2}\int\limits_{\partial Y} \chi_i\, \chi_j\, a_{mk}\, n_k\, \partial_{z_m} p\, d\sigma + \int\limits_{\partial Y} \chi_i\, a_{mk}\, (\delta_{kj} + \partial_{z_k}\chi_j)\, n_m\, p\, d\sigma. \end{aligned}$$

Finally, the last equality and (31) lead to (36).

Let us show that $(I+\nabla\chi)\,e_1\neq 0$. Suppose that $(I+\nabla\chi)\,e_1=0$. Then $\partial_{z_1}\chi_1=-1$, or

$$\int\limits_Y \partial_{z_1}\chi_1\,dz_1=-1,$$

which contradicts the periodicity of χ.

Consider a nonnegative definite symmetric matrix $C=\{c_{ij}\}_{i,j=1}^d$, and suppose that $c_{11}=0$. Evaluating the quadratic form $c_{ij}\xi_i\xi_j$ at the vector

$$\xi=\{N,1,0,...,0\},\quad N>0,$$

we get

$$c_{ij}\xi_i\xi_j=c_{12}N+c_{21}N+c_{22}=2c_{12}N+c_{22}.$$

If $c_{12}\neq 0$, then for large N (positive if $c_{12}<0$, and negative, if $c_{12}>0$) we obtain $c_{ij}\xi_i\xi_j<0$, which contradicts the non-negativeness of the matrix C. Thus, $c_{12}=c_{21}$ is equal to zero. Similarly we can show that $c_{1k}=0$, $k=2,...,d$. Therefore, $C\,e_1=0$.

In our case $\bar{A}e_1=(I+\nabla\chi)\,e_1\neq 0$, and we conclude that $\bar{a}_{11}=\bar{A}_{11}>0$. Consequently, v^0 is a linear function on x_1. Thus, $\psi(x_1,z)$ satisfies the homogeneous problem with respect to the variable z and $\psi=\psi(x_1)$.

Let us return to problem (23). We have shown that the expansion (28) takes the form:

$$\tilde{v}^\varepsilon=v^0(x_1)+\varepsilon\,\chi_1\left(\frac{x_1}{\varepsilon},x'\right)\partial_{x_1}v^0+\varepsilon^2\psi(x_1)$$

with $v^0(x_1)=1-x_1$ and $\chi(z)$ solving problem (29)-(30). Denote

$$v_1^\varepsilon=v^0(x_1)+\varepsilon\,\chi_1\left(\frac{x_1}{\varepsilon},x'\right)\partial_{x_1}v^0.$$

One can easily check that the difference $(v^\varepsilon-v_1^\varepsilon)$ satisfies the following problem:

$$\begin{cases} -\mathrm{div}(a^\varepsilon(x)\nabla(v^\varepsilon-v_1^\varepsilon))-(b^\varepsilon,\nabla(v^\varepsilon-v_1^\varepsilon))=0, & x\in G_0^1,\\ (v^\varepsilon-v_1^\varepsilon)(0,x')=\varepsilon\chi_1(0,x'), & x'\in Q,\\ (v^\varepsilon-v_1^\varepsilon)(1,x')=-\varepsilon\chi_1(1/\varepsilon,x'), & x'\in Q,\\ \dfrac{\partial(v^\varepsilon-v_1^\varepsilon)}{\partial n_{a^\varepsilon}}=0, & x\in\Sigma_0^1.\end{cases}$$

We can rewrite problem for the function χ_1 in the form

$$\begin{cases} \sum\limits_{i,k}\partial_{z_k}\left(a_{ik}\partial_{z_i}(\chi_1+z_1)\right)+\sum\limits_k b_k\,\partial_{z_k}(\chi_1+z_1)=0, & z\in Y,\\ \sum\limits_{i,k}a_{ik}\partial_{z_k}(\chi_1+z_1)\,n_i=0, & z\in\partial Y.\end{cases}\tag{37}$$

By means of the extension techniques in the same way as above, one can show that (χ_1+z_1) is a Hölder-continuous function in $[0,1]\times\overline{Q}$. Consequently,

$$\|v^\varepsilon-v_1^\varepsilon\|_{L^\infty(S_0)}\le C\varepsilon$$

and

$$\|v^\varepsilon-v_1^\varepsilon\|_{L^\infty(S_1)}\le C\varepsilon,$$

and by the maximum principle

$$\|v^\varepsilon-v_1^\varepsilon\|_{L^\infty(G_0^1)}\le C\varepsilon.$$

□

The following statement characterizes in more general situation the asymptotic behaviour of solutions of auxiliary problems in finite cylinders in the three cases $\bar{b}_1 \lessgtr 0$ and $\bar{b}_1 = 0$. Namely, we consider the following boundary value problem in the finite cylinder G_0^k:

$$\begin{cases} -\mathrm{div}\,\left(a(x)\,\nabla\, v^k\right) - (b(x), \nabla\, v^k) = 0, & x \in G_0^k, \\ \dfrac{\partial v^k}{\partial n_a} = 0, & x \in \Sigma_0^k, \\ v^k(0, x') = \varphi(x'), \quad v^k(k, x') = M & x' \in Q, \end{cases} \tag{38}$$

where $\varphi(x') \in L^\infty(Q)$, M is a constant.

Theorem 6.5. *Let the assumptions of Theorem 6.1 be fulfilled. Then for the solution v^k of problem (38) the following statements hold:*

1. If $\bar{b}_1 > 0$ then

$$|v^k - C_\varphi^\infty| \le C_0\,\|\varphi\|_{L^\infty(Q)}\,\left(e^{-\gamma_0 x_1} + e^{-\gamma(k-x_1)}\right) + C\,M\,e^{-\gamma(k-x_1)}; \tag{39}$$

2. If $\bar{b}_1 < 0$ then

$$|v^k - M| \le C_0\,\|\varphi\|_{L^\infty(Q)}\,e^{-\gamma x_1} + C\,M\,e^{-\gamma x_1}; \tag{40}$$

3. If $\bar{b}_1 = 0$ then in G_0^k the function v^k is close to a linear function:

$$\left| v^k - \frac{C_\varphi^\infty(k - x_1) + M x_1}{k} \right| \le C_0\,\|\varphi\|_{L^\infty(Q)}\,e^{-\gamma_0 x_1} + \frac{C}{k}\,\left(\|\varphi\|_{L^\infty(Q)} + M\right). \tag{41}$$

The constant C_φ^∞ is uniquely determined by Lemma 5.1.

Proof. Due to Lemma 5.1, if $\varphi \in L^\infty(Q)$ then there always exists a solution $u_0(x)$ of problem (1) satisfying the maximum principle. Moreover, such a solution is unique and stabilizes to a constant C_φ^∞ exponentially:

$$|u_0 - C_\varphi^\infty| \le C_0\,\|\varphi\|_{L^\infty(Q)}\,e^{-\gamma_0 x_1}, \quad \gamma_0 > 0,\ x_1 > 1.$$

Recall that γ_0 depends only on Λ, d, Q and does not depend on $\bar{b}_1$.

We represent the solution v^k of problem (38) as a sum $v_1^k + v_2^k$, where v_1^k and v_2^k solve the following problems:

$$\begin{cases} -\mathrm{div}\,\left(a(x)\,\nabla\, v_1^k\right) - (b(x), \nabla\, v_1^k) = 0, & x \in G_0^k, \\ \dfrac{\partial v_1^k}{\partial n_a} = 0, & x \in \Sigma_0^k, \\ v_1^k(0, x') = \varphi(x'), \quad v_1^k(k, x') = C_\varphi^\infty & x' \in Q; \end{cases} \tag{42}$$

$$\begin{cases} -\mathrm{div}\,\left(a(x)\,\nabla\, v_2^k\right) - (b(x), \nabla\, v_2^k) = 0, & x \in G_0^k, \\ \dfrac{\partial v_2^k}{\partial n_a} = 0, & x \in \Sigma_0^k, \\ v_2^k(0, x') = 0, \quad v_2^k(k, x') = -C_\varphi^\infty + M & x' \in Q. \end{cases} \tag{43}$$

One can see that, due to the maximum principle, the difference $(u_0 - v_1^k)$ is of order $e^{-\gamma_0 k}$ everywhere in G_0^k, and, consequently,

$$|v_1^k - C_\varphi^\infty| \le |v_1^k - u_0| + |u_0 - C_\varphi^\infty| \le C_0 \|\varphi\|_{L^\infty(Q)}\,\left(e^{-\gamma_0 k} + e^{-\gamma_0 x_1}\right),\ x \in G_0^k. \tag{44}$$

- Assume that $\bar{b}_1 > 0$. By Lemma 6.3 v_2^k satisfies the estimate
$$|v_2^k| \le C_0(C_\varphi^\infty + |M|)\, e^{-\gamma(k-x_1)}, \quad x \in G_0^k.$$
Combining the last estimate and (44) and taking into account the bound $C_\varphi^\infty \le \|\varphi\|_{L^\infty(Q)}$, we obtain (39).
- If $\bar{b}_1 < 0$ then the solution u^k of problem (23) decays exponentially, which leads to the estimate for $v_2^k(x)$
$$|v_2^k - (M - C_\varphi^\infty)| \le C_0 \left(\|\varphi\|_{L^\infty(Q)} + |M|\right) e^{-\gamma_0 x_1}, \quad x \in G_0^k,$$
which proves (40).
- In the case $\bar{b}_1 = 0$ to estimate $v_2^k(x)$ we make use of Lemma 6.4. Namely, v_2^k is close to a linear function in this case:
$$\left| v_2^k - \frac{M - C_\varphi^\infty}{k} x_1 \right| \le \frac{C}{k} \left(\|\varphi\|_{L^\infty(Q)} + |M|\right).$$
The last estimate and (44) implies (41). Theorem 6.5 is proved.

□

7. Equivalent definitions of a bounded solution.

Lemma 7.1. *For a solution of problem (1) the following conditions are equivalent:*

(i)
$$\|u\|_{L^2(G_N^{N+1})}\, dx \le C, \quad \forall N \ge 0,$$
where C does not depend on $N \in [0, \infty)$;

(ii)
$$\|u\|_{L^\infty(G_1^\infty)} < \infty;$$

(iii)
$$\|\nabla\, u\|_{L^2(G)} < \infty.$$

Proof.
- $(i) \to (ii)$ Under assumptions of uniform ellipticity of matrix $a(x)$ (2) and boundedness of the coefficients, for any compact set G' in $(N, N+1) \times \overline{Q}$ the generalized solution of problem (1) satisfies the following estimate:
$$\|u\|_{C^\alpha(G')} \le C \|u\|_{L^2(G_N^{N+1})},$$
for some constants C and $\alpha > 0$ independent of N and, consequently
$$\|u\|_{C^\alpha(G')} \le C,$$
for any compact set G' in G_N^{N+1} with C independent of N. Thus,
$$\|u\|_{L^\infty(G_\delta^\infty)} \le C(\delta) < \infty, \quad \delta > 0.$$
- $(i) \to (iii)$ In Section 5 we proved that any bounded solution $u(x)$ stabilizes to a constant C_∞ at the exponential rate with large axial distance. Then the function $(u(x) - C_\infty)$ solves problem (1) with boundary condition $(u - C_\infty)(0, x') = (\varphi(x') - C_\infty)$ and vanishes at infinity at the exponential rate. Extending $u(x)$ to a larger domain (as in Section 3) and applying standard elliptic estimates to $(u(x) - C_\infty)$ one deduces that
$$\|\nabla(u - C_\infty)\|_{L^2(G_N^{N+1})} \le C \|u - C_\infty\|_{L^2(G_{N-1}^{N+2})} \le C e^{-\gamma N},$$

where the constant C does not depend on N. Thus $\nabla u(x)$ stabilizes to zero at the exponential rate, as $x_1 \to \infty$, and

$$\int\limits_{G} |\nabla u|^2\, dx = \sum_{N=0}^{\infty} \int\limits_{G_N^{N+1}} |\nabla u|^2\, dx \le C_0 \sum_{N=0}^{\infty} e^{-\gamma N} \le C.$$

- $(iii) \to (i)$ Let $u(x)$ be a solution of problem (1) such that
$$\|\nabla u\|_{L^2(G)} \le C. \tag{45}$$
The Friedrichs inequality gives an estimate for the L^2-norm of $u(x)$ in the finite cylinder G_N^{N+1}:
$$\|u\|^2_{L^2(G_N^{N+1})} \le C_1 + C_2 N$$
with constants $C_1 = C_1(\varphi)$ and C_2 independent on N. Note that if $0 \le N \le 1$ then
$$\|u\|_{L^2(G_N^{N+1})} \le C;$$
below we suppose that $N \ge 1$.

Let $v(x)$ be a bounded solution of problem (1) (it exists by Lemma 4.2). Notice that the difference $(u - v)$ satisfies the estimates
$$\|u - v\|^2_{L^2(G_N^{N+1})} \le C_1 + C_2\, N, \quad \|\nabla (u - v)\|^2_{L^2(G)} \le C.$$
If we denote
$$w_N = \frac{1}{\sqrt{N}}(u - v), \tag{46}$$
then
$$\|w_N\|^2_{L^2(G_N^{N+1})} \le \frac{C_1}{N} + C_2, \quad \|\nabla w_N\|^2_{L^2(G)} \le \frac{C}{N}.$$
Since w_N is a solution of problem (1) with zero boundary condition on the base of G, then the last estimates imply
$$\|w_N\|_{L^\infty(G_0^{N+1})} \le \bar{w},$$
with $\bar{w}$ independent of N. By the maximum principle $|w_N|$ does not exceed the solution v_N of the following problem:
$$\begin{cases} -\mathrm{div}\,(a\nabla v_N) - (b, \nabla v_N) = 0, \quad x \in G_0^{N+1}, \\ \dfrac{\partial v_N}{\partial n_a} = 0, \quad x \in \Sigma_0^{N+1}, \\ v_N(0, x') = 0, \quad v_N(N+1, x') = \bar{w}, \quad x' \in Q. \end{cases} \tag{47}$$
We will consider separately the cases $\bar{b}_1 > 0$, $\bar{b}_1 = 0$ and $\bar{b}_1 < 0$.

Let first $\bar{b}_1 > 0$. From Theorem 6.5 for N large enough we conclude that the function w_N is close to zero for $x_1 < N/2$:
$$|w_N| \le v_N \le C_0\, e^{-\gamma N}, \quad x_1 < N/2.$$
Therefore, considering the definition of w_N (see (46)) one obtain the following estimate for the difference $(u - v)$:
$$|u - v| \le C_0 \sqrt{N} e^{-\gamma N} \to 0, \quad N \to \infty,$$
which implies that $u = v$ and thus
$$\|u\|_{L^2(G_N^{N+1})} \le C.$$

Consider the case $\bar{b}_1 = 0$. As was proved in Lemma 6.3, in this case a solution of problem (47) is close in the cylinder G_0^{N+1} to the linear function, namely

$$\|v_N - x_1 \frac{\bar{w}}{(N+1)}\|_{L^\infty(G_0^{N+1})} \le \frac{C}{N+1}, \quad x \in G_0^{N+1}.$$

Consequently

$$|u - v| \le \frac{C\sqrt{N}}{N+1} + x_1 \frac{\bar{w}}{(N+1)} \sqrt{N}, \quad \forall N > 0.$$

For $x_1 < N^\alpha$, $\alpha < 1/2$, we obtain that

$$|u - v| \to 0, \quad N \to \infty,$$

thus $u(x) = v(x)$ is a bounded solution.

Finally, let us consider the case $\bar{b}_1 < 0$. As was discussed in Section 5, due to the maximum principle, either $m(x_1)$ increases or $M(x_1)$ decreases in the neighbourhood of infinity. Suppose that

$$\min_{x' \in Q} u \to \infty, \quad x_1 \to \infty,$$

the case of decreasing $M(x_1)$ can be studied in a similar way. Subtracting from $u(x)$ a bounded solution $v(x)$ of problem (1) with $v(0,x') = u(0,x')$, one can assume without loss of generality that $u(0,x') = 0$. Then $u(N,x')/m(N)$ will be greater than or equal to 1. Let us introduce a function v_N as a solution to the problem

$$\left\{ \begin{array}{l} -\mathrm{div}\,(a\nabla\, v_N) - (b, \nabla\, v_N) = 0, \quad x \in G_0^N, \\ \dfrac{\partial v_N}{\partial n_a} = 0, \quad x \in \Sigma_0^N, \\ v_N(0,x') = 0, \quad v_N(N,x') = 1, \quad x' \in Q. \end{array} \right. \tag{48}$$

By the maximum principle $u(x)/m(N) \ge v_N$. As was shown in Theorem 6.1, v_N satisfies the estimate

$$|v_N - 1| \le C_0\, e^{-\gamma x_1}, \quad x_1 > 1.$$

Thus

$$u(x) - m(N) \ge -C_0 m(N) e^{-\gamma x_1}.$$

Let $\bar{x}_1 = \frac{1}{\gamma}\ln(2C_0)$, then for any $x_1 > \bar{x}_1$ the following estimate holds:

$$C_0 e^{-\gamma x_1} < \frac{1}{2}.$$

Then $v_N > 1/2$ and, consequently,

$$u(\bar{x}_1, x') \ge \frac{1}{2} m(N), \quad x' \in Q.$$

From the last estimate, using Friedrichs inequality, we obtain

$$\|\nabla\, u\|_{L^2(G_0^{\bar{x}_1})} \ge \frac{1}{4\bar{x}_1}\, m^2(N) \to \infty, \quad N \to \infty,$$

that contradicts (iii).

Lemma 7.1 is proved. $\square$

8. **Inhomogeneous problem with periodic coefficients.** We proceed with studying the existence and the stabilization to a constant of a solution to the following boundary value problem:

$$\begin{cases} -\mathrm{div}\,(a(x)\nabla\, u) - (b(x),\, \nabla\, u) = f(x) + \mathrm{div}\, F, & x \in G = (0,\infty)\times Q, \\ \dfrac{\partial u}{\partial n_a} = g(x) - (F, n), & x \in \Sigma = (0,\infty)\times \partial Q, \\ u(0,x') = 0, & x' \in Q. \end{cases} \tag{49}$$

Here the assumptions on the coefficients $a_{ij}(x)$, $b_j(x)$ and the cylinder G are the same as in the previous sections. Concerning the functions f, F and g we suppose that $f(x) \in L^2(G)$, $F \in (L^2(G))^d$, $g(x) \in L^2(\Sigma)$, and that these functions decay exponentially as x_1 goes to infinity, i.e.

$$\begin{gathered} \|f\|_{L^2(G_N^{N+1})} \le C_1\, e^{-\gamma_1 N}, \quad \|F\|_{L^2(G_N^{N+1})} \le C_1\, e^{-\gamma_1 N}, \\ \|g\|_{L^2(\Sigma_N^{N+1})} \le C_2\, e^{-\gamma_1 N} \end{gathered} \tag{50}$$

for some positive γ_1.

Definition 8.1. We say that $u(x) \in H^1_{loc}(G)$ is a weak solution to problem (49) if the following integral equality holds for any $\psi(x) \in C_0^\infty((0,\infty); C^\infty(\bar{Q}))$:

$$\int\limits_G (a(x)\nabla u, \nabla\psi)\, dx - \int\limits_\Sigma g(x)\psi(x)\, d\sigma - \int\limits_G (b,\nabla u)\,\psi(x)\, dx = -\int\limits_G (F,\nabla\psi)\, dx.$$

We begin with the case $F = 0$. In this case we can use the integration by parts technic in the weighted space with the weight $p(x)$, as we did in the proofs of the previous statements. It turns out that this technic fails to work if F is not equal to zero. That is why we consider the case of nonzero F separately and reduce it to the case $F = 0$.

Lemma 8.2. *Let $F = 0$. Then there exists a solution $u(x)$ of problem (49), which stabilizes to a constant at the exponential rate, as $x_1 \to \infty$, and satisfies the estimates*

$$\|\nabla\, u\|_{L^2(G)} \le C(\|(1+\sqrt{x_1})\, f\|_{L^2(G)} + \|(1+\sqrt{x_1})\, g\|_{L^2(\Sigma)}); \tag{51}$$

$$\|u\|_{L^2(G_N^{N+1})} \le C(\|(1+\sqrt{x_1})\, f\|_{L^2(G)} + \|(1+\sqrt{x_1})\, g\|_{L^2(\Sigma)}),\ \forall N \ge 0. \tag{52}$$

Proof. Let us consider the sequence of auxiliary problems

$$\begin{cases} -\mathrm{div}\,(a(x)\nabla\, u_m^k) - (b(x),\, \nabla\, u_m^k) = f_m(x), & x \in G_0^k, \\ \dfrac{\partial u_m^k}{\partial n_a} = g_m(x), & x \in \Sigma_0^k, \\ u_m^k(0,x') = 0, \quad u_m^k(k,x') = 0, & x' \in Q. \end{cases} \tag{53}$$

Here $f_m(x) = f(x)\chi(G_m^{m+1})$ and $g_m(x) = g(x)\chi(G_m^{m+1})$, $\chi(G_\alpha^\beta)$ is a characteristic function of G_α^β. Multiplying the first equation of (53) by the product $p(x)u_m^k(x)$, integrating by parts over G_0^k and using boundary conditions for u_m^k, we obtain

$$\int\limits_{G_0^k} (a(x)\nabla\, u_m^k, \nabla\, u_m^k)\, p\, dx - \int\limits_{\Sigma_m^{m+1}} g_m(x)\, u_m^k\, p\, d\sigma = \int\limits_{G_m^{m+1}} f_m(x) u_m^k\, p\, dx. \tag{54}$$

Let us estimate the integral on the right-hand side. Using the boundedness of $p(x)$ and Schwartz inequality one has

$$\left|\int\limits_{G_0^k} f_m(x)\, u_m^k\, p\, dx\right| \le C\|f_m\|_{L^2(G_m^{m+1})}\, \|u_m^k\|_{L^2(G_m^{m+1})}.$$

The Friedrichs inequality yields

$$\int\limits_{G_m^{m+1}} (u_m^k)^2\, dx \le (m+1) \int\limits_{G_0^k} |\nabla\, u_m^k|^2\, dx, \tag{55}$$

and, finally,

$$\left|\int\limits_{G_0^k} f_m(x)\, u_m^k\, p\, dx\right| \le C\, \|\nabla u_m^k\|_{L^2(G_0^k)}(1+\sqrt{m})\, \|f_m\|_{L^2(G_m^{m+1})}.$$

Using analogous arguments one can estimate the integral over the lateral boundary of the cylinder:

$$\left|\int\limits_{\Sigma_m^{m+1}} g_m(x)\, u_m^k\, p(x)\, d\sigma\right| \le C\,(1+\sqrt{m})\, \|g_m\|_{L^2(\Sigma_m^{m+1})}\, \|\nabla\, u_m^k\|_{L^2(G_0^k)}.$$

Combining the above bounds with the integral identity (54), we conclude that

$$\|\nabla u_m^k\|_{L^2(G_0^k)} \le C\,(1+\sqrt{m}) \left(\|f_m\|_{L^2(G_m^{m+1})} + \|g_m\|_{L^2(\Sigma_m^{m+1})}\right), \tag{56}$$

where the constant C does not depend on m, k. Estimate (55) implies that the L^2-norm of the function u_m^k is uniformly in k bounded on each G_N^{N+1} for all $N \le m$:

$$\|u_m^k\|_{L^2(G_N^{N+1})} \le C\,(1+m) \left(\|f_m\|_{L^2(G_m^{m+1})} + \|g_m\|_{L^2(\Sigma_m^{m+1})}\right).$$

In the cylinder G_{m+1}^k the function u_m^k satisfies homogeneous equation and

$$\begin{aligned} \|u_m^k\|_{H^{1/2}(S_{m+1})} &\le C\|u_m^k\|_{H^1(G_m^{m+1})} \\ &\le C(1+m) \left(\|f_m\|_{L^2(G_m^{m+1})} + \|g_m\|_{L^2(\Sigma_m^{m+1})}\right). \end{aligned}$$

For $u_m^k(x)$ estimate (10) obtained while proving Theorem 6.1 takes the form

$$\|u_m^k\|_{L^\infty(G_{m+1}^k)} \le \|u_m^k\|_{H^{1/2}(S_{m+1})}.$$

Consequently, the following inequality holds

$$\|u_m^k\|_{L^2(G_N^{N+1})} \le C\,(1+m) \left(\|f_m\|_{L^2(G_m^{m+1})} + \|g_m\|_{L^2(\Sigma_m^{m+1})}\right) \tag{57}$$

for $N \ge 0$ with the constant C independent of k and m.

Since $f(x) = \sum_0^{k-1} f_m$ and $g(x) = \sum_0^{k-1} g_m$, then $u^k = \sum_0^{k-1} u_m^k$ is a solution of problem

$$\begin{cases} -\mathrm{div}\,(a(x)\nabla\, u^k) - (b(x),\, \nabla\, u^k) = f(x), & x \in G_0^k, \\ \dfrac{\partial u^k}{\partial n_a} = g(x), & x \in \Sigma_0^k, \\ u^k(0,x') = 0, \quad u^k(k,x') = 0, & x' \in Q, \end{cases}$$

and, in view of (56) – (57), satisfies the following estimates:

$$\|u^k\|_{L^2(G_N^{N+1})} \le C\left(\|(1+\sqrt{x_1})f\|_{L^2(G)} + \|(1+\sqrt{x_1})g\|_{L^2(\Sigma)}\right), \tag{58}$$

$$\|\nabla u^k\|_{L^2(G_0^k)} \le C\left(\|(1+\sqrt{x_1})f\|_{L^2(G)} + \|(1+\sqrt{x_1})g\|_{L^2(\Sigma)}\right) \tag{59}$$

with C independent of k. Hence, up to a subsequence, u^k converges weakly in the space $H^1_{loc}(G)$, as $k \to \infty$, to a function $u(x)$ which satisfies (51) and (52). We will prove the exponential stabilization to a constant only in the case $f, g = 0$, $F \ne 0$. This proof can be extended to the case of nontrivial f and g with minor modifications. We leave it to the reader. □

It remains to consider problem (49) with a non-trivial F and with $f = g = 0$.

Lemma 8.3. *There exists a solution of problem (49) with $f = g = 0$, which satisfies the estimates*

$$\begin{aligned} \|\nabla u\|_{L^2(G)} &\le C\,\|(1+\sqrt{x_1})F\|_{L^2(G)},\\ \|u\|_{L^2(G_N^{N+1})} &\le C\,\|(1+\sqrt{x_1})F\|_{L^2(G)}, \quad \forall N \ge 0. \end{aligned} \tag{60}$$

This solution stabilizes to a constant at the exponential rate, as $x_1 \to \infty$.

Proof. Consider the sequence of auxiliary problems:

$$\begin{cases} -\mathrm{div}\,(a(x)\nabla u^k) - (b(x),\, \nabla u^k) = \mathrm{div} F, & x \in G_0^k,\\ \dfrac{\partial u^k}{\partial n_a} = -(F, n), & x \in \Sigma_0^k,\\ u^k(0, x') = u^k(k, x') = 0, & x' \in Q, \end{cases} \tag{61}$$

Let us represent the function F in G_0^k as follows:

$$F(x) = \sum_{m=0}^{M} \chi(G_{m\tau}^{(m+1)\tau})\, F(x) \equiv \sum_{m=0}^{M} F^m(x),$$

where $\chi(G_{m\tau}^{(m+1)\tau})(x)$ is a characteristic function of the domain $G_{m\tau}^{(m+1)\tau}$,

$$\tilde\tau = \Lambda^4,\ \ M = \left[\frac{k}{\tilde\tau}\right],\ \ \tau = \frac{k}{M+1}.$$

Clearly, $\mathrm{supp} F^m \subset G_{m\tau}^{(m+1)\tau}$. Due to the linearity of the studied problem, we can represent a solution $u^k(x)$ of (61) as the sum $\sum_{m=0}^{M} u_m^k(x)$:

$$\begin{cases} -\mathrm{div}\,(a(x)\nabla u_m^k) - (b(x),\, \nabla u_m^k) = \mathrm{div} F^m, & x \in G_0^k,\\ \dfrac{\partial u_m^k}{\partial n_a} = -(F^m, n), & x \in \Sigma_0^k,\\ u_m^k(0, x') = u_m^k(k, x') = 0, & x' \in Q. \end{cases} \tag{62}$$

We will first assume that the coefficients a_{ij}, b_j are smooth functions.

Our analysis is based on the properties of the Green function $G^k(x, y)$ of problem (61):

$$\begin{cases} -\mathrm{div}_x\,(a(x)\nabla_x G^k(x,y)) - (b(x),\, \nabla_x G^k(x,y)) = \delta(x-y), & x \in G_0^k,\\ \dfrac{\partial G^k(x,y)}{\partial n_a} = 0, & x \in \Sigma_0^k,\\ G^k(0, x', y) = G^k(k, x', y) = 0, & x' \in Q. \end{cases}$$

Due to our assumptions on $a(x)$ and $b(x)$, the Green function $G^k(x,y)$ is well-defined. If we denote by v^k a solution of (61) with the function $\chi(G_{m\tau}^{(m+1)\tau})$ on the right-hand side

$$\begin{cases} -\mathrm{div}\,(a(x)\nabla\, v^k) - (b(x),\, \nabla\, u^k) = \chi(G_{m\tau}^{(m+1)\tau}), & x \in G_0^k, \\ \dfrac{\partial v^k}{\partial n_a} = 0, & x \in \Sigma_0^k, \\ v^k(0,x') = v^k(k,x') = 0, & x' \in Q, \end{cases}$$

then

$$v^k(x) = \int\limits_{G_{m\tau}^{(m+1)\tau}} G^k(x,y)\,dy.$$

As was shown in the proof of Lemma 8.2, the functions $v^k(x)$ satisfy the estimate:

$$\|v^k\|_{L^\infty(G_{(m-1)\tau}^{(m+2)\tau})} \le C(m\tau+1),$$

with the constant C which depends only on Λ, d and Q, but does not depend of k. In particular for all $x \in S_{(m-1)\tau} \cup S_{(m+2)\tau}$, since $y \in G_{m\tau}^{(m+1)\tau}$,

$$\int\limits_{G_{m\tau}^{(m+1)\tau}} G^k(x,y)\,dy \le C(1+m\tau).$$

Recalling the fact that $G^k(y,x)$ is the Green function of the adjoint problem, using the mean value theorem and the Harnack inequality for $G^k(x,\cdot)$, one can easily get the following inequality:

$$|Q|\,G^k(x,y) \le \alpha\,|Q|\,G^k(x,y_0) = \alpha \int\limits_{G_{m\tau}^{(m+1)\tau}} G^k(x,y)\,dy \le C(1+m\tau),$$

for all $x \in S_{(m-1)\tau}$, for all $y \in G_{m\tau}^{(m+1)\tau}$ and some $y_0 \in G_{m\tau}^{(m+1)\tau}$. Here $\alpha > 0$ depends only on the ellipticity constant Λ, the dimension d and the domain Q. Similar inequality is also valid for all $x \in S_{(m+2)\tau}$. Then the standard elliptic estimates read

$$\begin{aligned} \|\nabla_y G^k(x,\cdot)\|_{L^2(G_{m\tau}^{(m+1)\tau})} &\le C\|G^k(x,\cdot)\|_{L^2(G_{(m-1/2)\tau}^{(m+3/2)\tau})} \\ &\le C(1+m\tau), \quad x \in S_{(m-1)\tau} \cup S_{(m+2)\tau}. \end{aligned} \tag{63}$$

Let us emphasize that the constant C in (63) depends on Λ, d, Q and does not depend on k.

Now we turn back to problem (62). Considering the representation of u_m^k in terms of the Green function, one can see that

$$u_m^k(x) = -\int\limits_{G_{m\tau}^{(m+1)\tau}} (\nabla_y G^k(x,y), F^m(y))\,dy,$$

and, consequently, in view of (63),

$$\begin{aligned} \|u_m^k\|_{L^\infty(S_{(m-1)\tau})} &\le \|\nabla_y G^k(x,\cdot)\|_{L^2(G_{m\tau}^{(m+1)\tau})}\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})} \\ &\le C(1+m\tau)\,\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}. \end{aligned}$$

Similar estimate is valid for $x \in S_{(m+2)\tau}$:

$$\|u_m^k\|_{L^\infty(S_{(m+2)\tau})} \le C(1+m\tau)\,\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}.$$

By virtue of the maximum principle, since $u_m^k(0,x') = u_m^k(k,x') = 0$, we have

$$\|u_m^k\|_{L^\infty(G_0^{(m-1)\tau})} \le C(1+m\tau)\,\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}; \tag{64}$$

$$\|u_m^k\|_{L^\infty(G_{(m+2)\tau}^k)} \le C(1+m\tau)\,\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}. \tag{65}$$

In order to estimate the L^2-norms of u_m^k and ∇u_m^k in $G_{(m-1)\tau}^{(m+2)\tau}$, we represent u_m^k as a sum $v_m^k + w_m^k$, where

v_m^k is a solution of homogeneous equation, $v_m^k((m-1)\tau, x') = u_m^k((m-1)\tau, x')$, $v_m^k((m+2)\tau, x') = u_m^k((m+2)\tau, x')$,

w_m^k satisfies the nonhomogeneous equation and zero Dirichlet boundary conditions on $S_{(m-1)\tau}$ and $S_{(m+2)\tau}$.

In view of (64), (65) and the maximum principle we have

$$\|v_m^k\|_{L^\infty(G_{(m-1)\tau}^{(m+2)\tau})} \le C(1+m\tau)\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}.$$

Combining the last estimate with the standard elliptic H^1-estimates in the domain $G_{(m-1)\tau}^{(m+2)\tau}$, and taking into account the fact that the shape of the domain does not depend on m, we conclude that

$$\|v_m^k\|_{H^1(G_{(m-1)\tau}^{(m+2)\tau})} \le C(1+m\tau)\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}.$$

To estimate $w_m^k(x)$ let us multiply the equation by w_m^k and integrate over $G_{(m-1)\tau}^{(m+2)\tau}$. Then exploiting the Friedrichs inequality and taking into account the specific choice of τ, one can see that

$$\|\nabla v_m^k\|_{L^\infty(G_{(m-1)\tau}^{(m+2)\tau})} \le \frac{\Lambda}{2}\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})};$$
$$\|v_m^k\|_{L^\infty(G_{(m-1)\tau}^{(m+2)\tau})} \le \frac{\Lambda\,\tau}{2}\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}.$$

Consequently, one has

$$\|u_m^k\|_{H^1(G_{(m-1)\tau}^{(m+2)\tau})} \le C(1+m\tau)\,\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})},$$

where C depends only on Λ, d and Q.

Elliptic estimates for u_m^k in $G_0^{(m-1)\tau}$ yield the bound

$$\|\nabla u_m^k\|_{L^2(G_0^{(m-1)\tau})} \le C(1+m\tau)\,\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}.$$

Since in $G_{(m+2)\tau}^k$ the function u_m^k satisfies the homogeneous equation and homogeneous Neumann boundary conditions on the lateral boundary $\Sigma_{(m+2)\tau}^k$, then inequality (16) takes the form

$$\|\nabla u_m^k\|_{L^2(G_{(m+2)\tau})} \le C\|u_m^k\|_{H^{1/2}(S_{(m+2)\tau})} \le C(1+m\tau)\,\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}.$$

Thus,

$$\begin{gathered}\|u_m^k\|_{L^2(G_N^{N+1})} \le C(1+m\tau)\,\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})},\ \forall N \ge 0;\\ \|\nabla u_m^k\|_{L^2(G_0^k)} \le C(1+m\tau)\,\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}.\end{gathered} \tag{66}$$

And, consequently,

$$\begin{aligned} \|u^k\|_{L^2(G_N^{N+1})} &\le C\,\|(1+\sqrt{x_1})F\|_{L^2(G)}, \quad \forall N\ge 0,\\ \|\nabla u^k\|_{L^2(G_0^k)} &\le C\,\|(1+\sqrt{x_1})F\|_{L^2(G)}, \end{aligned} \tag{67}$$

where the constant C depends on Λ, d and Q.

Using the compactness arguments, we conclude that, along a subsequence, $u^k(x)$ converges weakly in $H^1_{\rm loc}(G)$, as $k\to\infty$, to a function $u(x)$ which solves problem (49) and (60) hold. This completes the proof of the existence of a bounded solution in the case of smooth coefficients.

In the general case of measurable bounded coefficients a_{ij} and b_j define

$$\begin{aligned} a^\delta_{ij}(x) &= \int\limits_{\mathbb{R}^d} a_{ij}(y)\psi^\delta(x-y)\,dy,\\ b^\delta_j(x) &= \int\limits_{\mathbb{R}^d} b_j(y)\psi^\delta(x-y)\,dy, \end{aligned}$$

where $\psi^\delta(\xi)\in C_0^\infty(\mathbb{R}^d)$ is such that $\psi^\delta(\xi)\ge 0$, $\psi^\delta(-\xi)=\psi^\delta(\xi)$ and $\int_{\mathbb{R}^d}\psi^\delta(\xi)d\xi=1$. In order to define $a^\delta(x)$ and $b^\delta(x)$ we should extend $a(x)$ and $b(x)$ outside $\mathbb{R}\times Q$ (on $\mathbb{R}_-\times Q$ the coefficients are extended by periodicity). For example, we can set $a(x)=\Lambda I$, I is a unit matrix, $b(x)=\{0,...0\}$ for $x'\notin Q$. Clearly, the obtained $a(x)$ and $b(x)$ satisfy the same uniform ellipticity and boundedness conditions as before. By construction, a^δ_{ij} converges to a_{ij}, and b^δ_j converges to b_j, as $\delta\to 0$, in $L^p(G_0^k)$, for any $k>0$ and $p\ge 1$. For the solution $u^k_\delta(x)$ of problem (61) with smoothed coefficients a^δ_{ij}, b^δ_j the following bounds are valid:

$$\begin{aligned} \|u^k_\delta\|_{L^2(G_N^{N+1})} &\le C\,\|(1+\sqrt{x_1})F\|_{L^2(G)}, \quad \forall N\ge 0,\\ \|\nabla u^k_\delta\|_{L^2(G_0^k)} &\le C\,\|(1+\sqrt{x_1})F\|_{L^2(G)}, \end{aligned}$$

with C independent of δ. Thus, up to a subsequence, $u^k_\delta\to u^k$ in $L^2(G_N^{N+1})$, $\nabla u^k_\delta \rightharpoonup \nabla u^k$ in $L^2(G_0^k)$, as $\delta\to 0$, where u^k solves problem (61) with measurable bounded coefficients. Clearly, $u^k(x)$ satisfies the estimates

$$\begin{aligned} \|u^k\|_{L^2(G_N^{N+1})} &= \lim_{\delta\to 0}\|u^k_\delta\|_{L^2(G_N^{N+1})} \le C\|(1+\sqrt{x_1})F\|_{L^2(G)},\\ \|\nabla u^k\|_{L^2(G)} &\le \liminf_{\delta\to 0}\|\nabla u^k_\delta\|_{L^2(G)} \le C\|(1+\sqrt{x_1})F\|_{L^2(G)}. \end{aligned}$$

Using the compactness arguments, we conclude that, along a subsequence, u^k converges weakly in $H^1_{loc}(G)$, as $k\to\infty$, to a function $u(x)$ which solves (49) with $f=g=0$, and estimates (60) are valid.

It is left to prove the stabilization of $u(x)$ at the exponential rate to a constant. It can be easily seen that along a subsequence the functions $\{u^k_m\}$ constructed above converge weakly in $H^1_{\rm loc}(G)$, as $k\to\infty$, to a function $u_m(x)$ which is a solution to the problem

$$\left\{\begin{array}{ll} -{\rm div}\,(a(x)\nabla\, u_m) - (b(x),\,\nabla\, u_m) = {\rm div}F^m, & x\in G,\\ \dfrac{\partial u_m}{\partial n_a} = -(F^m,n), & x\in\Sigma,\\ u_m(0,x')=0, & x'\in Q. \end{array}\right. \tag{68}$$

It is clear that $u(x)=\sum_{m=0}^\infty u_m(x)$. With regard to Theorem 6.1, one can see that there exists a constant C^∞_m such that for a solution $u_m(x)$ of problem (68) the following estimate holds:

$$|u_m - C^\infty_m| \le C_0\|u_m\|_{H^{1/2}(S_{(m+2)\tau})}\,e^{-\gamma(x_1-(m+2)\tau)}, \quad x_1>(m+2)\tau.$$

Notice that by construction, since $u_m^k(k,x')=0$, $|C_m^\infty|\le \|u_m\|_{H^{1/2}(S_{(m+2)\tau})}$. As was shown above,

$$\|u_m\|_{H^{1/2}(S_{(m+2)\tau})}\le C(1+m\tau)\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}.$$

Thus,

$$|u_m-C_m^\infty|\le C(1+m\tau)\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}\,e^{-\gamma(x_1-(m+2)\tau)},\quad x_1>(m+2)\tau. \tag{69}$$

Let us check that $u(x)$ converges to $C^\infty=\sum_{m=0}^\infty C_m^\infty$. To this end we estimate the $L^2(G_{N\tau}^{(N+1)\tau})$-norm of the difference $(u-C^\infty)$:

$$\|u-C^\infty\|_{L^2(G_{N\tau}^{(N+1)\tau})}\le \sum_{m=0}^\infty \|u_m-C_m^\infty\|_{L^2(G_{N\tau}^{(N+1)\tau})}.$$

Splitting the sum into two parts and taking into account (50), estimates (60) and (69), we have

$$\begin{aligned}
&\|u-C^\infty\|_{L^2(G_{N\tau}^{(N+1)\tau})}\le\left(\sum_{m=0}^{N-3}+\sum_{m=N-2}^{\infty}\right)\|u_m-C_m^\infty\|_{L^2(G_{N\tau}^{(N+1)\tau})}\\
&\le C\tau\sum_{m=0}^{N-3}(1+m\tau)e^{-\gamma m\tau}\,e^{-\gamma(N\tau-(m+2)\tau)}+C\sum_{m=N-2}^{\infty}\Big(\|u_m\|_{L^2(G_{N\tau}^{(N+1)\tau})}+|C_m^\infty|\Big)\\
&\le C\,N^2\,e^{-\gamma N\tau}+C\,e^{-\gamma(N-2)\tau/2}\sum_{m=N-2}^{\infty}(1+m\tau)\,e^{-\gamma m\tau/2}\le \tilde C\,e^{-\tilde\gamma N\tau},\ N\ge 0.
\end{aligned}$$

The case of nontrivial f and g in (49) can be considered analogously. It suffices to use estimates (51) – (52) instead of (60) and notice that bound (69) remains valid if we replace $\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}$ with $\|f_m\|_{L^2(G_{m\tau}^{(m+1)\tau})}+\|g_m\|_{L^2(\Sigma_{m\tau}^{(m+1)\tau})}$. The rest of the proof is exactly the same as above. Lemma 8.3 is proved. □

As in Section 7 we can define a bounded solution of problem (49).

Definition 8.4. We say that a weak solution $u(x)$ of problem (49) is bounded if one of the following conditions is fulfilled:

(i) $\|u\|_{L^2(G_N^{N+1})}\le C,\quad \forall N\ge 0,$

(ii) $\|\nabla u\|_{L^2(G)}\le C.$

Lemma 8.5. *The conditions (i) and (ii) are equivalent.*

Proof. In view of Lemma 8.2 there exists a solution $v(x)$ of problem (49) such that the conditions (i) and (ii) hold. Let us consider the difference $(u(x)-v(x))$. It satisfies the homogeneous problem (1) with $\varphi=0$. But for a solution to problem (1) conditions (i)–(ii) are equivalent. Lemma 8.5 is proved. □

The rest of this section is devoted to studying the uniqueness of solution to problem (49). The result similar to that of Theorem 6.1 takes place. As before, we denote

$$\bar b_1=\int\limits_{G_0^1}\left(a_{1j}(x)\frac{\partial p(x)}{\partial x_j}-b_1(x)p(x)\right)dx,$$

where the function $p(x)$ was introduced in Section 3.

Theorem 8.6. *1. Any bounded solution $u(x)$ of problem (49) stabilizes to a constant at the exponential rate as $x_1 \to \infty$, that is*

$$\|u(x) - C_\infty\|_{L^2(G_n^\infty)} \leq C_0\, e^{-\gamma\, n}, \quad \forall n \geq 0,$$

for some $C_0 > 0$ and $\gamma > 0$;

2. *$\bar{b}_1 < 0$ if and only if for any $\varphi(x') \in H^{1/2}(Q)$ and for any constant $l \in \mathbb{R}$, there exists a bounded solution $u(x)$ of problem (49) that converges to the constant l, as $x_1 \to \infty$;*
3. *$\bar{b}_1 \geq 0$ if and only if for every boundary condition $\varphi(x')$ there exists a unique constant $m(\varphi)$ such that a bounded solution of problem (49) converges to this constant as $x_1 \to \infty$.*

Proof. The existence of a bounded solution that stabilizes to a constant at the exponential rate was proved in Lemma 8.2. Denote this solution by $u_0(x)$. If $u(x)$ is an arbitrary boundary solution of problem (49), then Theorem 6.1 applies to the difference $(u(x) - u_0(x))$ and implies the first statement of Theorem 8.6. In order to obtain the second and the third statements, it suffices to observe that the uniqueness of a bounded solution to problem (49) is equivalent to that of problem (1). Indeed, if there are two distinct bounded solutions, say u_1 and u_2, of problem (49), then the difference $(u_1 - u_2) \neq 0$ is a bounded solution of the homogeneous problem, and thus a bounded solution of homogeneous problem is not unique.

Conversely, if we assume that problem (1) with $\varphi = 0$ has two distinct bounded solutions, say v_1 and v_2, then $(u_0 + v_1)$ and $(u_0 + v_2)$ are bounded solutions of (49). Theorem is proved. ∎

9. **Non-periodic coefficients.** The goal of this section is to generalize the results of Section 6 to the case of the coefficients which stabilize exponentially to a periodic regime. We will consider the following boundary value problem:

$$\begin{cases} -\mathrm{div}\ (\hat{a}(x)\,\nabla\, u(x)) - (\hat{b}(x), \nabla\, u(x)) = 0, & x \in G, \\ \dfrac{\partial u}{\partial n_{\hat{a}}} = 0, & x \in \Sigma, \\ u(0, x') = \varphi(x'), & x' \in Q, \end{cases} \tag{70}$$

where Q is a bounded domain in $\mathbb{R}^{d-1}$ with a sufficiently smooth boundary ∂Q. We suppose that the matrix $\hat{a}(x)$ and vector $\hat{b}(x)$ admit the representations

$$\hat{a}(x) = a(x) + a^\circ(x), \quad \hat{b}(x) = b(x) + b^\circ(x),$$

where $a(x)$ and $b(x)$ are x_1-periodic, while a°_{ij} and b°_j decay exponentially, that is for almost all $x \in G$

$$|a^\circ_{ij}| \leq C_1\, e^{-\gamma_1\, x_1}, \quad |b^\circ_j| \leq C_2\, e^{-\gamma_1\, x_1}, \quad \gamma_1 > 0. \tag{71}$$

Moreover, as in the previous sections, we assume that $\hat{a}(x)$ is a symmetric uniformly elliptic matrix, i.e. there exists a positive constant Λ_1 such that for almost all $x \in \mathbb{R}^d$ the following estimate holds:

$$\Lambda_1\, |\xi|^2 \leq \hat{a}_{ij}(x)\, \xi_i\, \xi_j, \quad \xi \in \mathbb{R}^d,$$

and $\hat{a}_{ij}(x), \hat{b}_j \in L^\infty(G)$.

Lemma 9.1. *Let the above conditions be fulfilled. Then a bounded solution to problem* (70) *exists and stabilizes to a constant at the exponential rate. Moreover, the following estimates hold:*

$$\|\nabla u\|_{L^2(G)} < \infty, \quad \|u\|_{L^\infty(G_1^\infty)} < \infty. \tag{72}$$

Proof. To prove the existence of a bounded solution we use the sequence of auxiliary problems in growing finite cylinders:

$$\begin{cases} -\mathrm{div}(\hat{a}(x)\nabla u^k) - (\hat{b}(x), \nabla u^k) = 0, & x \in G_0^k, \\ \dfrac{\partial u^k}{\partial n_{\hat{a}}} = 0, & x \in \Sigma_0^k, \\ u^k(0, x') = \varphi(x'), \quad u^k(k, x') = 0, & x' \in Q. \end{cases} \tag{73}$$

Let us recall that according to Remark 1 for measurable bounded coefficients $\hat{a}_{ij}$ and $\hat{b}_j$, not necessary periodic, estimates (10) and (11) hold true. Using the standard elliptic estimates for u^k, we conclude that

$$\|u^k\|_{H^1(G_N^{N+1})} \le C, \quad \forall N > 0,$$

and thus, along a subsequence, u^k converges weakly in $L^2_{\mathrm{loc}}(G)$ to some function $u \in H^1_{\mathrm{loc}}(G)$, as $k \to \infty$, and ∇u^k converges weakly to ∇u in $L^2_{\mathrm{loc}}(G)$. This allows us to pass to the limit in the integral identity and establish the existence of a bounded solution to problem (70) such that

$$\|u\|_{H^1(G_N^{N+1})} \le C, \quad \forall N > 0. \tag{74}$$

However, these estimates do not imply the finiteness of $L^2(G)$ norm of ∇u.

We will proceed as follows. First, making use of Theorem 8.6, we will show that a bounded solution to problem (70) stabilizes to a constant, and then, with the help of this result, we will obtain an estimate for ∇u.

Obviously, problem (70) can be rewritten in the form

$$\begin{cases} -\mathrm{div}(a(x)\nabla u) - (b(x), \nabla u) = \mathrm{div}(a^\circ(x)\nabla u) + (b^\circ(x), \nabla u), & x \in G, \\ \dfrac{\partial u}{\partial n_a} = -\dfrac{\partial u}{\partial n_{a^\circ}}, & x \in \Sigma, \\ u(0, x') = \varphi(x'), & x' \in Q. \end{cases} \tag{75}$$

Consider the following problem in G

$$\begin{cases} -\mathrm{div}(a(x)\nabla w) - (b(x), \nabla w) = \mathrm{div}(a^\circ(x)\nabla u) + (b^\circ(x), \nabla u), & x \in G, \\ \dfrac{\partial w}{\partial n_a} = -\dfrac{\partial u}{\partial n_{a^\circ}}, & x \in \Sigma, \\ w(0, x') = \varphi(x'), & x' \in Q; \end{cases} \tag{76}$$

here w is an unknown function and u is the solution of (75). Taking into account (74), it is easy to see that under our assumptions on a° and b° all the conditions of Theorem 8.6 are fulfilled, and, therefore, any bounded solution $w(x)$ to problem (76) stabilizes to a constant at the exponential rate. Since u is a solution of (76), it stabilizes to a constant exponentially. Moreover, the inequality holds

$$\int\limits_G |\nabla u|^2\, dx = \sum_{n=0}^{\infty} \int\limits_{G_n^{n+1}} |\nabla u|^2\, dx \le C_0 \sum_{n=0}^{\infty} e^{-\gamma n} \le C.$$

Lemma 9.1 is proved. □

One of the principal results of this section is given by the following lemma, which states that the uniqueness property is invariant under exponentially decaying perturbations of the coefficients.

Lemma 9.2.
- *$\bar b_1 < 0$ iff for any $\varphi(x') \in H^{1/2}(Q)$ and any $l \in \mathbb{R}^1$ there exists a bounded solution to problem (70) stabilizes to l, as $x_1 \to \infty$;*
- *$\bar b_1 \ge 0$ iff for any $\varphi(x') \in H^{1/2}(Q)$ there exists a unique bounded solution to problem (70) and it stabilizes to a constant $m = m(\varphi)$, as $x_1 \to \infty$.*

Proof. First, assume that for any $\varphi(x')$ there exists a unique solution to problem (70) which stabilizes to a constant $m = m(\varphi)$, as $x_1 \to \infty$. In particular, for $\varphi = 1$ a solution u to problem (70) is unique and $u \equiv 1$. This solution can be obtained as the limit of solutions u^k of (73). Since $u = 1$, then u^k converges to 1 uniformly on each compact subset of G, as $k \to \infty$. Let us show that in this case $\bar b_1 \ge 0$. Multiplying the equation in (73) by $u^k p$ and integrating by parts over G_ξ^k we obtain

$$\int\limits_{G_\xi^k} (\hat a \nabla u^k, \nabla u^k)\, p\, dx + \int\limits_{G_\xi^k} (a^\circ \nabla u^k, \nabla p)\, u^k\, dx - \int\limits_{G_\xi^k} (b^\circ, \nabla u^k)\, p\, u^k\, dx$$
$$-\frac{1}{2}\int\limits_{S_\xi} \left(\frac{\partial p}{\partial n_a} - (b,n)p\right)(u^k)^2\, dx' + \int\limits_{S_\xi} \frac{\partial u^k}{\partial n_{\hat a}} u^k\, p\, dx' = 0. \tag{77}$$

The integral containing $a^\circ(x)$, admits the following upper bound

$$\begin{aligned}\left|\int\limits_{G_\xi^k} (a^\circ \nabla u^k, \nabla p)\, u^k\, dx\right| &\le \sum_{n=\xi}^{k-1} \left| \int\limits_{G_n^{n+1}} (a^\circ \nabla u^k, \nabla p)\, u^k\, dx\right| \\ &\le C \sum_{n=\xi}^{k-1} e^{-\gamma_1 n} \|\nabla p\|_{L^2(G_n^{n+1})}.\end{aligned}$$

For any $\delta > 0$ we can choose sufficiently large ξ_0 so that for all $\xi > \xi_0$

$$C \sum_{n=\xi}^{k-1} e^{-\gamma_1 n} \|\nabla p\|_{L^2(G_n^{n+1})} < \delta.$$

Similarly, for large enough ξ,

$$\left|\int\limits_{G_\xi^k} (b^\circ, \nabla u^k)\, p\, u^k\, dx\right| < \delta.$$

Taking into account the convergence of u^k to 1, coerciveness of the matrix $\hat a$ and the definition of $\bar b_1$, we obtain the following inequality:

$$\bar b_1 \ge -C\delta, \quad \forall \delta > 0,$$

which implies that $\bar b_1 \ge 0$.

Now let us suppose that for any constant there exists a solution of (70) converging to this constant. Then for any $\xi \ge 0$ there is a solution $v(x)$ to problem

$$\begin{cases} -\mathrm{div}\,(\hat a(x)\, \nabla\, v(x)) - (\hat b(x), \nabla\, v(x)) = 0, & x \in G_\xi^\infty, \\ \dfrac{\partial v}{\partial n_{\hat a}} = 0, & x \in \Sigma_\xi^\infty, \\ v(\xi, x') = 0, & x' \in Q, \end{cases} \tag{78}$$

such that

$$|v-1| \le C(\xi)\, e^{-\gamma(x_1-\xi)}, \quad x_1 > \xi.$$

It is clear that uniformly in ξ for any $n > 0$

$$\|v\|_{H^1(G_n^{n+1})} \le C.$$

We rewrite problem (78) in the form

$$\begin{cases} -\mathrm{div}(a(x)\nabla w) - (b(x), \nabla w) = \mathrm{div}(a^\circ(x)\nabla v) + (b^\circ(x), \nabla v), & x \in G_\xi^\infty, \\ \dfrac{\partial w}{\partial n_a} = -\dfrac{\partial v}{\partial n_{a^\circ}}, & x \in \Sigma_\xi^\infty, \\ w(\xi, x') = 0, & x' \in Q; \end{cases} \tag{79}$$

If we assume that $\bar{b}_1 \ge 0$, then (79) has a unique bounded solution which coincides with $v(x)$ and converges to a constant. The uniqueness of solution allows us to estimate $\|v\|_{L^2(G_n^{n+1})}$ in terms of the norm of the right-hand side:

$$\|v\|_{L^2(G_n^{n+1})} \le C\|(1+\sqrt{x_1})a^\circ \nabla v\|_{L^2(G_\xi^\infty)}$$

$$\le C \sum_{N=\xi}^{\infty} \|(1+\sqrt{x_1})a^\circ \nabla v\|_{L^2(G_n^{n+1})} \le C \sum_{N=\xi}^{\infty} (1+\sqrt{N})\, e^{-\gamma_1 N}.$$

For any positive δ, choosing sufficiently large ξ, we obtain

$$\|v\|_{L^2(G_n^{n+1})} < \delta, \quad n > \xi.$$

This contradicts our assumption that v converges to 1. Therefore, $\bar{b}_1 < 0$. Lemma 9.2 is proved. □

Remark 6. It turns out that in the case when $a_{ij}^\circ(x)$ and $b_j^\circ(x)$ do not decay exponentially, the statements of Lemma 9.1 and 9.2 may fail to hold. To illustrate this, let us consider the following problem

$$\begin{cases} -\Delta u - b_1^\circ(x_1)\partial_1 u = 0, & x \in G, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \Sigma, \\ u(0, x') = 1, & x' \in Q, \end{cases} \tag{80}$$

with $b_1^\circ = 2/(1+x_1)$. Observe that, in contrast with the non-perturbed problem, which has a unique solution, problem (80) possesses two bounded solutions: $u_1 = 1$ and $u_2 = 1/(1+x_1)$. The last one stabilizes to zero, as $x_1 \to \infty$, but not at the exponential rate.

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