DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **18**, Number **5**, July **2013**

doi:10.3934/dcdsb.2013.18.1217

pp. **1217–1251**

THE EXISTENCE OF WEAK SOLUTIONS TO IMMISCIBLE COMPRESSIBLE TWO-PHASE FLOW IN POROUS MEDIA: THE CASE OF FIELDS WITH DIFFERENT ROCK-TYPES

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(Communicated by Gregoire Allaire)

Abstract. We study a model describing immiscible, compressible two-phase flow, such as water-gas, through heterogeneous porous media taking into account capillary and gravity effects. We will consider a domain made up of several zones with different characteristics: porosity, absolute permeability, relative permeabilities and capillary pressure curves. This process can be formulated as a coupled system of partial differential equations which includes a nonlinear parabolic pressure equation and a nonlinear degenerate diffusion-convection saturation equation. Moreover the transmission conditions are nonlinear and the saturation is discontinuous at interfaces separating different media. There are two kinds of degeneracy in the studied system: the first one is the degeneracy of the capillary diffusion term in the saturation equation, and the second one appears in the evolution term of the pressure equation. Under some realistic assumptions on the data, we show the existence of weak solutions with the help of an appropriate regularization and a time discretization. We use suitable test functions to obtain a priori estimates. We prove a new compactness result in order to pass to the limit in nonlinear terms. This passage to the limit is nontrivial due to the degeneracy of the system.

2010 *Mathematics Subject Classification.* Primary: 76S05, 76T10; Secondary: 35D30, 35K65, 35Q35.

Key words and phrases. Immiscible compressible, nonlinear degenerate system, two-phase flow, porous media, nuclear waste, water-gas.

The research was supported by FP7 EURATOM Fund 230357, the FORGE project & MoMaS.

1. **Introduction.** Two-phase flow in porous media is important to many practical problems, including those in petroleum reservoir engineering, unsaturated zone hydrology, and soil sciences. More recently, modeling multiphase flow received an increasing attention in connection with the disposal of radioactive waste and sequestration of CO_2. The modeling and numerical simulation of two-phase flow in porous media represents an important tool in the design of cost-efficient and safe methods of studying the mentioned practical problems. It can reduce the number of laboratory and field experiments, help to identify the significant mechanisms, optimize existing strategies and give indications of possible risks.

This paper focuses on the modeling of immiscible compressible two-phase flow in heterogeneous porous media, in the framework of the geological disposal of radioactive waste. As a matter of fact, one of the solutions envisaged for managing waste produced by nuclear industry is to dispose it in deep geological formations chosen for their ability to prevent and attenuate possible releases of radionuclides in the geosphere. In the frame of designing nuclear waste geological repositories appears a problem of possible two-phase flow of water and gas mainly hydrogen, for more details see for instance [10, 33].

The mathematical analysis of two-phase flow in porous media has been a problem of interest for many years and many methods have been developed. There is an extensive literature on this subject. We will not attempt a literature review here, but merely mention a few references. Here we restrict ourselves to two-phase flow in porous media. We refer for instance to [1, 11, 12, 14, 15, 16, 19, 20, 21, 31, 37, 38] for more information on the analysis, especially on the existence of solutions, of immiscible incompressible two-phase flow in porous media, and to [7, 8, 9, 18, 22] for miscible compressible flow in porous media.

However, as reported in [6], the situation is quite different for immiscible compressible two-phase flow in porous media, where, only recently few results have been obtained. In the case of immiscible two-phase flows with one (or more) compressible fluids without any exchange between the phases, some approximate models were studied in [23, 24, 25]. Namely, in [23] certain terms related to the compressibility are neglected, and in [24, 25] the mass densities are assumed to depend not on the physical pressure, but on Chavent's global pressure. As shown in [5] the models based on the mass density approximation can be suitable in oil reservoir simulations but are inadequate in many underground gas and water flows where the difference between the phase pressures and the global pressure can be significant. In the articles [26, 29], a more general immiscible compressible two-phase flow model in porous media is considered for fields with a single rock type, which is too restrictive for some realistic problems, such as gas migration through engineered and geological barriers for a deep repository for radioactive waste. The immiscible compressible two-phase flows models studied in [23, 24, 25, 26, 29, 30] are based on phase formulations, i.e. the main unknowns are the phase pressures and the saturation of one phase, and the feature of the global pressure as introduced in [11, 15] for incompressible immiscible flows is used to establish a priori estimates.

Let us also mention that, recently, a new global pressure concept was introduced in [4, 5] for modeling immiscible, compressible two-phase flow in porous media without any simplifying assumptions. The resulting equations are written in a fractional flow formulation and lead to a coupled system which consists of a nonlinear parabolic equation (the global pressure equation) and a nonlinear diffusion-convection one (the saturation equation). This new formulation is fully equivalent to the original phase equations formulation, i.e. where the phase pressures and the phase saturations are primary unknowns. The fractional flow approach treats the two-phase flow problem as a total fluid flow of a single mixed fluid, and then describes the individual phases as fractions of the total flow. For this model, an existence result, under realistic assumptions on the data, is obtained in [6].

In the case of immiscible two-phase flows with one (or more) compressible fluids with exchange between the phases, i.e. a multicomponent model, existence of weak solutions to these equations under some assumptions on the compressibility of the fluids has been recently established in [32, 35, 36].

The paper is concerned with a nonlinear degenerate system of diffusion-convection equations modeling the flow and transport of immiscible compressible fluids through heterogeneous porous media, capillary and gravity effects being taken into account. We will consider a domain made up of several zones with different characteristics: porosity, absolute permeability, relative permeabilities and capillary pressure curves. In the literature, this may be rephrased by saying that we consider a field containing several types of rocks. We restrict our attention to water (incompressible) and gas such as hydrogen (compressible) in the context of gas migration through engineered and geological barriers for a deep repository for radioactive waste. For more details on the formulation of such problems see, e.g., the Couplex-Gas benchmark [10] which was proposed by ANDRA and MoMaS to improve the modeling of the migration of hydrogen produced by the corrosion of nuclear waste packages in an underground storage. This is a system of two-phase (water–hydrogen) flow in a porous medium.

For notational convenience we only consider a field which contains two different rock types. But it is easy to see that all the results are valid in a domain with several rock types. We will restrict our attention to water (incompressible) and gas such a hydrogen (compressible), however the methodology and the analysis can be extended to problems where both fluids are assumed to be compressible. The model to be presented herein is formulated in terms of the wetting (water) saturation phase and the non-wetting (gas) pressure phase. The governing equations are derived from the mass conservation laws of both fluids, along with constitutive relations relating the velocities to the pressures gradients and gravitational effects. Traditionally, the standard Muskat-Darcy law provides this relationship. This formulation leads to a coupled system consisting of a nonlinear parabolic equation for the gas pressure and a nonlinear degenerate parabolic diffusion-convection equation for the water saturation, subject to appropriate transmission, boundary and initial conditions.

There are two kinds of degeneracy in the studied system. The first one is the classical degeneracy of the diffusion operator. This degeneracy is due to the capillary effects, it can be observed even in the case of incompressible immiscible two-phase flow. The second one represents the evolution term degeneracy. It occurs in the region where the gas saturation vanishes: the gas density cannot be determined by its evolution and has no physical meaning since the gas phase is missing. In both cases the presence of degeneracy weakens the energy estimates and makes a proof of compactness results more involved. Our aim is to establish existence of weak solutions for this system of equations under realistic assumptions. Let us mention that the main difficulties related to the mathematical analysis of such equations are the coupling, the degeneracy of the diffusion term in the saturation equation and the degeneracy of the temporal term in the pressure equation. Moreover the transmission conditions are nonlinear and the saturation is discontinuous at the interface separating the two media. In contrast to the case of a single rock-type model, the transmission conditions lead to additional difficulties in the proof of the existence result for the system under consideration, see Remark 3 and Section 7 below. Although we follow the strategy used in [29], that is we first regularize our model and then using the discretization in time, apply the fixed point arguments, still the presence of discontinuity at the interface brings additional difficulties in obtaining a priori estimates and passage to the limit, and makes the proof essentially more involved. Our approach also relies on the compactness result proved in our previous work [3]. Thus we extend the results of [29] in the case of porous media with different rock types. This study was intended as a first step to the homogenization of immiscible compressible two-phase flow through heterogeneous reservoirs with several rock types.

The rest of the paper is organized as follows. In the next Section, we give a short description of the mathematical and physical model used in this study. Following [15], we introduce a global pressure and give some useful relations. Then we provide the detailed assumptions on data and formulate the main result of the paper on the existence of a weak solution of the studied problem. Note that the existence result is proved with the help of regularization, time discretization, a priori estimates and compactness arguments. Section 3 deals with some properties of the solutions and with two compactness results which play a crucial rule in studying our degenerate system. It should be mentioned that the compactness result used for studying single rock type models (see [29] and the references therein) fails to apply to several rock types models. Therefore, we prove a new compactness result adapted to the problem under consideration. In Section 4 we present a short description of the scheme of proving the main result of the paper. In section 5 we introduce the regularized problem with a regularization parameter $\eta > 0$, and its time discretization with a small parameter $h > 0$, and, using the Leray-Shauder fixed point theorem, we establish, as in [29], the existence of a weak solution of this problem. Section 6 is devoted to the study of the non degenerate system. We use suitable test functions introduced in [26] to get uniform estimates with respect to h. These estimates allow us to pass to the limit, as h tends to zero, and to justify the existence of a weak solution of the regularized problem with continuous time. In Section 7 we complete the proof of the main result. To this end, we perform the limit as η tends to zero and obtain a solution of the non-regularized system. This part of the proof relies on the compactness results established in Section 3. Lastly, some concluding remarks are forwarded.

2. Mathematical model and the main result.

For notational convenience we only consider a field which contains two different types of rock. We consider a reservoir $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ which is a bounded Lipschitz domain. We suppose that Ω is made of two parts Ω_1 and Ω_2 with $\Omega_2 \subset \Omega$ and such that

$$\Omega = \Omega_1 \cup \Gamma_{1,2} \cup \Omega_2, \tag{1}$$

where $\Gamma_{1,2}$ is the interface between the subdomains Ω_1, Ω_2 which is assumed to be sufficiently smooth, say Lipschitz continuous. We also introduce the following notation:

$$\Omega_T \stackrel{\text{def}}{=} \Omega\times]0,T[, \quad \Omega_{\ell,T} \stackrel{\text{def}}{=} \Omega_\ell\times]0,T[, \quad \Sigma_T \stackrel{\text{def}}{=} \Gamma_{1,2}\times]0,T[, \tag{2}$$

where $T > 0$ is fixed; here and in what follows $\ell = 1, 2$, the subscript ℓ refers to the ℓ^{th} type of rock.

We consider an immiscible compressible two-phase flow process in porous media. We focus on the phases water and gas, but the consideration below is also valid for a general wetting phase and a non-wetting phase, each phase consisting of a component.

The water–gas flow in porous reservoirs can be described in terms of the following characteristics:

- $\Phi = \Phi(x)$ is the porosity of the medium Ω;
- $K = K(x)$ is the absolute permeability tensor of Ω;
- ϱ_w, ϱ_g are the densities of water and gas, respectively.
- $S_{\ell,w} = S_{\ell,w}(x,t)$, $S_{\ell,g} = S_{\ell,g}(x,t)$ are the saturations of water and gas in Ω_ℓ, respectively;
- $k^{(\ell)}_{r,w} = k^{(\ell)}_{r,w}(S_{\ell,w})$, $k^{(\ell)}_{r,g} = k^{(\ell)}_{r,g}(S_{\ell,g})$ are the relative permeabilities of water and gas in the medium Ω_ℓ, respectively;
- $p_{\ell,w} = p_{\ell,w}(x,t)$, $p_{\ell,g} = p_{\ell,g}(x,t)$ are the pressures of water and gas in Ω_ℓ, respectively.

The conservation of mass in each phase can be written as (see, e.g., [13, 15, 17, 28]):

$$\begin{cases} \Phi(x)\dfrac{\partial}{\partial t}(S_{\ell,w}\,\varrho_w(p_{\ell,w})) + \operatorname{div}\left(\varrho_w(p_{\ell,w})\,\vec{q}_{\ell,w}\right) = Q_{\ell,w}(x,t) \quad \text{in } \Omega_{\ell,T}; \\ \Phi(x)\dfrac{\partial}{\partial t}(S_{\ell,g}\,\varrho_g(p_{\ell,g})) + \operatorname{div}\left(\varrho_g(p_{\ell,g})\,\vec{q}_{\ell,g}\right) = Q_{\ell,g}(x,t) \quad \text{in } \Omega_{\ell,T}, \end{cases} \tag{3}$$

where the velocities of the water and gas $\vec{q}_{\ell,w}$, $\vec{q}_{\ell,g}$ are defined by the Darcy-Muskat's law:

$$\vec{q}_{\ell,w} = -K(x)\lambda_{\ell,w}(S_{\ell,w})\left(\nabla p_{\ell,w} - \varrho_w(p_{\ell,w})\vec{g}\right), \quad \text{with } \lambda_{\ell,w}(S_{\ell,w}) = \frac{k^{(\ell)}_{r,w}}{\mu_w}(S_{\ell,w}); \tag{4}$$

$$\vec{q}_{\ell,g} = -K(x)\lambda_{\ell,g}(S_{\ell,g})\left(\nabla p_{\ell,g} - \varrho_g(p_{\ell,g})\vec{g}\right), \quad \text{with } \lambda_{\ell,g}(S_{\ell,g}) = \frac{k^{(\ell)}_{r,g}}{\mu_g}(S_{\ell,g}). \tag{5}$$

Here $\vec{g}$, μ_w, μ_g are the gravity vector and the viscosities of the water and gas, respectively. The source terms $Q_{\ell,w}, Q_{\ell,g}$ are given by:

$$Q_{\ell,w} \stackrel{\text{def}}{=} \varrho_w(p_{\ell,w})S^I_{\ell,w}f_I(x,t) - \varrho_w(p_{\ell,w})S_{\ell,w}f_P(x,t); \tag{6}$$

$$Q_{\ell,g} \stackrel{\text{def}}{=} \varrho_g(p_{\ell,g})S^I_{\ell,g}f_I(x,t) - \varrho_g(p_{\ell,g})S_{\ell,g}f_P(x,t), \tag{7}$$

where the functions $f_I \geq 0$ and $f_P \geq 0$ are injection and productions terms, respectively, and $S^I_{\ell,w}, S^I_{\ell,g}$ are known injection saturations. In the proof of the main result of the paper no additional complications arise from the source terms with respect to other nonlinear terms in system (3). Therefore, for the sake of brevity, and without loss of generality, we assume that

$$f_I = f_P \equiv 0, \tag{8}$$

i.e., we assume no source/sink terms.

From now on we assume that the density of the water is constant, which for the sake of simplicity will be taken equal to one, i.e. $\varrho_w(p_{\ell,w}) = \text{Const} = 1$, and the gas density ϱ_g is a smooth monotone function such that

$$\begin{gathered}\varrho_g(p) = \varrho_{\min} \quad \text{for } p \le p_{\min}; \qquad \varrho_g(p) = \varrho_{\max} \quad \text{for } p \ge p_{\max}; \\ \varrho_{\min} < \varrho_g(p) < \varrho_{\max} \quad \text{for } p_{\min} < p < p_{\max}.\end{gathered} \tag{9}$$

Here the pairs of constants $\varrho_{\min}, \varrho_{\max}$ and $p_{\min}, p_{\max}$ satisfy the bounds:

$$0 < \varrho_{\min} < \varrho_{\max} < +\infty \quad \text{and} \quad 0 < p_{\min} < p_{\max} < +\infty. \tag{10}$$

In what follows we make use of the following notation:

$$\varrho_{\ell,g} = \varrho_g(p_{\ell,g}). \tag{11}$$

The model is completed as follows. By the definition of saturations, one has

$$S_{\ell,w} + S_{\ell,g} = 1 \quad \text{with } S_{\ell,w}, S_{\ell,g} \ge 0. \tag{12}$$

We set:

$$S_\ell \stackrel{\text{def}}{=} S_{\ell,w}. \tag{13}$$

Then the curvature of the contact surface between the two fluids links the jump of pressure of two phases to the saturation by the capillary pressure law:

$$P_{\ell,c}(S_\ell) = p_{\ell,g} - p_{\ell,w} \quad \text{with } P'_{\ell,c}(s) < 0 \text{ for all } s \in [0,1] \text{ and } P_{\ell,c}(1) = 0, \tag{14}$$

where $P'_{\ell,c}(s)$ denotes the derivative of the function $P_{\ell,c}(s)$.

Now due to (8), (11), (13) and the assumption on the density of the water phase, we rewrite the system (3) as follows:

$$\begin{cases} \Phi \dfrac{\partial \mathsf{S}}{\partial t} - \operatorname{div}\, (K(x)\lambda_w(x,\mathsf{S})\,(\nabla \mathsf{p}_w - \vec{g})) = 0 \quad \text{in } \Omega_T; \\[2mm] \Phi \dfrac{\partial \Theta}{\partial t} - \operatorname{div}\, (K(x)\lambda_g(x,\mathsf{S})\varrho_g(\mathsf{p}_g)\,(\nabla \mathsf{p}_g - \varrho_g(\mathsf{p}_g)\vec{g})) = 0 \quad \text{in } \Omega_T; \\[2mm] P_c\,(x,\mathsf{S}) = \mathsf{p}_g - \mathsf{p}_w \quad \text{in } \Omega_T, \end{cases} \tag{15}$$

where

$$\mathsf{S} \stackrel{\text{def}}{=} S_1\,\mathbf{I}_1 + S_2\,\mathbf{I}_2; \quad \mathsf{p}_g \stackrel{\text{def}}{=} p_{1,g}\,\mathbf{I}_1 + p_{2,g}\,\mathbf{I}_2, \quad \mathsf{p}_w \stackrel{\text{def}}{=} p_{1,w}\,\mathbf{I}_1 + p_{2,w}\,\mathbf{I}_2; \tag{16}$$

$$\Theta \stackrel{\text{def}}{=} \varrho_g(\mathsf{p}_g)(1-\mathsf{S}) = \Theta_1\,\mathbf{I}_1 + \Theta_2\,\mathbf{I}_2; \text{ with } \Theta_\ell \stackrel{\text{def}}{=} \varrho_{\ell,g}[1-S_\ell]; \tag{17}$$

$$P_c\,(x,\mathsf{S}) \stackrel{\text{def}}{=} P_{1,c}\,(S_\ell)\;\mathbf{I}_1 + P_{2,c}\,(S_\ell)\;\mathbf{I}_2; \tag{18}$$

$$\begin{aligned}\lambda_w(x,\mathsf{S}) &\stackrel{\text{def}}{=} \lambda_{1,w}(S_1)\,\mathbf{I}_1 + \lambda_{2,w}(S_2)\,\mathbf{I}_2, \\ \lambda_g(x,\mathsf{S}) &\stackrel{\text{def}}{=} \lambda_{1,g}(S_1)\,\mathbf{I}_1 + \lambda_{2,g}(S_2)\,\mathbf{I}_2,\end{aligned} \tag{19}$$

and $\mathbf{I}_\ell = \mathbf{I}_\ell(x)$ is the characteristic function of the subdomain Ω_ℓ.

The continuity of the physical quantities at the interface $\Gamma_{1,2}$, i.e. the phase fluxes and the pressures of the water and gas, gives the following transmission conditions:

$$\begin{cases} \vec{q}_{1,w}\cdot\vec{\nu} = \vec{q}_{2,w}\cdot\vec{\nu} \text{ and } \vec{q}_{1,g}\cdot\vec{\nu} = \vec{q}_{2,g}\cdot\vec{\nu} \quad \text{on } \Sigma_T; \\ p_{1,w} = p_{2,w} \text{ and } p_{1,g} = p_{2,g} \quad \text{on } \Sigma_T, \end{cases} \tag{20}$$

where Σ_T is defined in (2), $\vec{\nu}$ is the unit outer normal on $\Gamma_{1,2}$, and the velocities $\vec{q}_{\ell,w}, \vec{q}_{\ell,g}$, in the notation (11), (13) are given by:

$$\vec{q}_{\ell,w} = -K(x)\lambda_{\ell,w}(S_\ell)\,(\nabla p_{\ell,w} - \vec{g}) \quad \text{and} \quad \vec{q}_{\ell,g} = -K(x)\lambda_{\ell,g}(S_\ell)\,(\nabla p_{\ell,g} - \varrho_{\ell,g}\vec{g})\,.$$

Remark 1. It is important to notice that in contrast to the functions $\mathsf{p}_g, \mathsf{p}_w$, the saturation S may have a jump at the interface $\Gamma_{1,2}$. Namely, it is easy to see from the transmission conditions (20) for the phase pressures that $P_{1,c}(S_1) = P_{2,c}(S_2)$ on Σ_T which gives the discontinuity of the saturation at the interface.

Now we specify the boundary and initial conditions. We suppose that the boundary $\partial\Omega$ consists of two parts Γ_{inj} and Γ_{imp} such that $\Gamma_{\text{inj}} \cap \Gamma_{\text{imp}} = \emptyset$, $\partial\Omega = \overline{\Gamma}_{\text{inj}} \cup \overline{\Gamma}_{\text{imp}}$. The boundary conditions are given by:

$$\begin{cases} p_{1,g}(x,t) = p_{1,w}(x,t) = 0 & \text{on } \Gamma_{\text{inj}} \times]0,T[; \\ \vec{q}_{1,w} \cdot \vec{\nu} = \vec{q}_{1,g} \cdot \vec{\nu} = 0 & \text{on } \Gamma_{\text{imp}} \times]0,T[. \end{cases} \tag{21}$$

Remark 2. Without loss of generality, we restrict the presentation to the case where the subdomain $\Omega_2 \subset \Omega$, but it is easy to see that all results also hold in the case $\partial\Omega_2 \cap \partial\Omega \neq \emptyset$. This study was intended as a first step to the homogenization of immiscible compressible two-phase flow through heterogeneous reservoirs with several rock types where this assumption is essential to apply the extension technique.

Finally, the initial conditions read:

$$\mathsf{p}_w(x,0) = \mathsf{p}_w^{\mathbf{0}}(x) \quad \text{and} \quad \mathsf{p}_g(x,0) = \mathsf{p}_g^{\mathbf{0}}(x) \quad \text{in } \Omega. \tag{22}$$

2.1. **Global pressure and useful relations.** In what follows we will make use of the so called *global pressure* introduced in [11, 15], see also [17]. It plays a crucial mathematical role, in particular, for compactness results. For compressible fluids the global pressure was used in [24, 26, 29, 30]. However, in contrast with the models studied in these papers, due to the presence of two types of rock in the model studied here, the saturation and global pressure may be discontinuous. The idea of the introduction of the global pressure is as follows, see [11, 15]. We want to replace the water–gas flow by a flow of a fictive fluid obeying the Darcy law with a non–degenerating coefficient. Namely, we are looking for a pressure P_ℓ and the coefficient $\gamma_\ell(S_\ell)$ such that $\gamma_\ell(S_\ell) > 0$ for all $S_\ell \in [0,1]$ and

$$\lambda_{\ell,w}(S_\ell)\nabla p_{\ell,w} + \lambda_{\ell,g}(S_\ell)\nabla p_{\ell,g} = \gamma_\ell(S_\ell)\nabla \mathsf{P}_\ell. \tag{23}$$

Then, for each subdomain Ω_ℓ, the global pressure, P_ℓ, is defined by:

$$p_{\ell,w} \stackrel{\text{def}}{=} \mathsf{P}_\ell + \mathsf{G}_{\ell,w}(S_\ell) \quad \text{and} \quad p_{\ell,g} \stackrel{\text{def}}{=} \mathsf{P}_\ell + \mathsf{G}_{\ell,g}(S_\ell); \tag{24}$$

the functions $\mathsf{G}_{\ell,w}(s)$ and $\mathsf{G}_{\ell,g}(s)$ will be introduced later on, see (27), (28).
Now it is easy to see that

$$\lambda_{\ell,w}(S_\ell)\nabla p_{\ell,w} + \lambda_{\ell,g}(S_\ell)\nabla p_{\ell,g} = \lambda_\ell(S_\ell)\nabla \mathsf{P}_\ell + \\ + \{\lambda_{\ell,g}(S_\ell)\nabla \mathsf{G}_{\ell,g}(S_\ell) + \lambda_{\ell,w}(S_\ell)\nabla \mathsf{G}_{\ell,w}(S_\ell)\},$$

where

$$\lambda_\ell(s) \stackrel{\text{def}}{=} \lambda_{\ell,w}(s) + \lambda_{\ell,g}(s) \tag{25}$$

We set:

$$\lambda_{\ell,g}(S_\ell)\nabla \mathsf{G}_{\ell,g}(S_\ell) + \lambda_{\ell,w}(S_\ell)\nabla \mathsf{G}_{\ell,w}(S_\ell) = 0. \tag{26}$$

Then $\gamma_\ell(S_\ell) = \lambda_\ell(S_\ell)$. The standard assumption on the function $\lambda_\ell(S_\ell)$ is that $\lambda_\ell(S_\ell) > 0$ for all $S_\ell \in [0,1]$ (see the condition **(A.5)** below). Thus the relation

(23) is established. Now we specify the functions $\mathsf{G}_{\ell,w}$, $\mathsf{G}_{\ell,g}$. We define $\mathsf{G}_{\ell,g}$ as follows:

$$\mathsf{G}_{\ell,g}(S_\ell) \stackrel{\text{def}}{=} \mathsf{G}_{\ell,g}(0) + \int_0^{S_\ell} \frac{\lambda_{\ell,w}(s)}{\lambda_\ell(s)}\, P'_{\ell,c}(s)\, ds. \tag{27}$$

The functions $\mathsf{G}_{\ell,w}$ are then defined by

$$\mathsf{G}_{\ell,w}(S_\ell) \stackrel{\text{def}}{=} \mathsf{G}_{\ell,g}(S_\ell) - P_{\ell,c}(S_\ell) \quad \text{with } \nabla \mathsf{G}_{\ell,w}(S_\ell) = -\frac{\lambda_{\ell,g}(S_\ell)}{\lambda_\ell(S_\ell)} P'_{\ell,c}(S_\ell)\, \nabla S_\ell. \tag{28}$$

Notice that from (27), (28) we get:

$$\lambda_{\ell,w}(s)\nabla \mathsf{G}_{\ell,w}(s) = \alpha_\ell(s)\nabla s \quad \text{and} \quad \lambda_{\ell,g}(s)\nabla \mathsf{G}_{\ell,g}(s) = -\alpha_\ell(s)\nabla s, \tag{29}$$

where

$$\alpha_\ell(s) \stackrel{\text{def}}{=} \frac{\lambda_{\ell,g}(s)\, \lambda_{\ell,w}(s)}{\lambda_\ell(s)} \left| P'_{\ell,c}(s) \right|. \tag{30}$$

Now we link the capillary pressure and the mobilities in the following way. We define two scalar functions $\mathsf{A}_{\ell,g}, \mathsf{A}_{\ell,w}$ as follows:

$$\sqrt{\lambda_{\ell,g}(s)}\; \mathsf{G}'_{\ell,g}(s) = \mathsf{A}'_{\ell,g}(s) \quad \text{and} \quad \sqrt{\lambda_{\ell,w}(s)}\; \mathsf{G}'_{\ell,w}(s) = \mathsf{A}'_{\ell,w}(s). \tag{31}$$

Then, following the lines of [26, 29], due to (24), (26), (25), and (31) we have the following identity:

$$\lambda_{\ell,g}(S_\ell)|\nabla p_{\ell,g}|^2 + \lambda_{\ell,w}(S_\ell)|\nabla p_{\ell,w}|^2 = \lambda_\ell(S_\ell)|\nabla \mathsf{P}_\ell|^2 + \left|\nabla \mathfrak{b}_\ell(S_\ell)\right|^2, \tag{32}$$

where

$$\mathfrak{b}_\ell(S_\ell) \stackrel{\text{def}}{=} \int_0^s \mathfrak{a}_\ell(\xi)\, d\xi \quad \text{with } \mathfrak{a}_\ell(s) \stackrel{\text{def}}{=} \sqrt{\frac{\lambda_{\ell,g}(s)\, \lambda_{\ell,w}(s)}{\lambda_\ell(s)}}\, \left| P'_{\ell,c}(s) \right|. \tag{33}$$

To make the assumptions on the data of our problem, we introduce the function β_ℓ,

$$\beta_\ell(s) \stackrel{\text{def}}{=} \int_0^s \alpha_\ell(\xi)\, d\xi, \tag{34}$$

where the function α_ℓ is defined in (30). Notice that by the definition of the global pressure, (33), (34), and by the boundedness of $\lambda_{\ell,w}, \lambda_{\ell,g}$ (see the condition **(A.5)** below) the following relations holds:

$$\left|\nabla \beta_\ell(S_\ell)\right|^2 \leq C \left|\nabla \mathfrak{b}_\ell(S_\ell)\right|^2, \tag{35}$$

$$\begin{aligned} \lambda_{\ell,w}(s)\nabla p_{\ell,w} &= \lambda_{\ell,w}(s)\nabla \mathsf{P}_\ell + \nabla \beta_\ell(s), \\ \lambda_{\ell,g}(s)\nabla p_{\ell,g} &= \lambda_{\ell,g}(s)\nabla \mathsf{P}_\ell - \nabla \beta_\ell(s). \end{aligned} \tag{36}$$

2.2. **Main assumptions.** The main assumptions on the data are as follows:

(A.1) The porosity $\Phi \in L^\infty(\Omega)$, and there are positive constants ϕ_-, ϕ^+ such that $0 < \phi_- < \phi^+$ and

$$0 < \phi_- \leq \Phi(x) \leq \phi^+ < 1 \quad \text{a.e. in } \Omega. \tag{37}$$

(A.2) The tensor $K \in (L^\infty(\Omega))^{d\times d}$ and there exist constants K_-, K^+ such that $0 < K_- < K^+$ and

$$K_-|\xi|^2 \leq (K(x)\xi, \xi) \leq K^+|\xi|^2 \text{ for all } \xi \in \mathbb{R}^{\mathrm{d}}, \text{ a.e. in } \Omega. \tag{38}$$

(A.3) The function $\varrho_g = \varrho_g(p)$ given by (9) is a monotone C^1–function in $\mathbb{R}$.

(A.4) The capillary pressure function $P_{\ell,c}(s) \in C^1([0,1];\mathbb{R}^+)$. Moreover, $P'_{\ell,c}(s) < 0$ in $[0,1]$ and the following two relations hold: $P_{\ell,c}(S_\ell = 1) = 0$ and $P_{1,c}(S_\ell = 0) = P_{2,c}(S_\ell = 0)$.

(A.5) The functions $\lambda_{\ell,w}, \lambda_{\ell,g}$ belong to the space $C([0,1];\mathbb{R}^+)$ and satisfy the following properties:
(i) $0 \le \lambda_{\ell,w}, \lambda_{\ell,g} \le 1$ in $[0,1]$; **(ii)** $\lambda_{\ell,w}(0) = 0$ and $\lambda_{\ell,g}(1) = 0$; **(iii)** there is a positive constant L_0 such that $\lambda_\ell(s) = \lambda_{\ell,w}(s) + \lambda_{\ell,g}(s) \ge L_0 > 0$ in $[0,1]$.

(A.6) The function $\alpha_\ell \in C^1([0,1];\mathbb{R}^+)$. Moreover, $\alpha_\ell(0) = \alpha_\ell(1) = 0$ and $\alpha_\ell > 0$ in $]0,1[$.

(A.7) The function β_ℓ^{-1}, inverse of β_ℓ defined in (34) is a Hölder function of order θ with $\theta \in (0,1)$ on the interval $[0,\beta_\ell(1)]$. Namely, there exists a positive constant C_β such that for all $s_1, s_2 \in [0,\beta(1)]$ the following inequality holds:

$$\left|\beta_\ell^{-1}(s_1) - \beta_\ell^{-1}(s_2)\right| \le C_\beta \left|s_1 - s_2\right|^\theta.$$

(A.8) The initial data for the pressures are such that $\mathsf{p}_g^{\mathbf{0}}, \mathsf{p}_w^{\mathbf{0}} \in L^2(\Omega)$.

(A.9) The initial data for the saturation is such that $\mathsf{S}^{\mathbf{0}} \in L^\infty(\Omega)$ and $0 \le \mathsf{S}^{\mathbf{0}} \le 1$ a.e.in Ω.

The assumptions **(A.1)**–**(A.9)** are classical for two-phase flow in porous media.

2.3. **Formulation of the main result.** In order to formulate the main result of the paper, we introduce the following Sobolev space:

$$H^1_{\Gamma_{\text{inj}}}(\Omega) \stackrel{\text{def}}{=} \left\{u \in H^1(\Omega) \,:\, u = 0 \text{ on } \Gamma_{\text{inj}}\right\}.$$

The space $H^1_{\Gamma_{\text{inj}}}(\Omega)$ is a Hilbert space. The norm in this space is given by

$$\|u\|_{H^1_{\Gamma_{\text{inj}}}(\Omega)} = \|\nabla u\|_{(L^2(\Omega))^d}.$$

The main result of the paper is as follows.

Theorem 2.1. *Let assumptions* **(A.1)**–**(A.9)** *be fulfilled. Then there exist the triple of functions* $\langle \mathsf{p}_g, \mathsf{p}_w, \mathsf{S}\rangle$ *such that:*

(I): *The functions* $\mathsf{p}_g, \mathsf{p}_w, \mathsf{S}$ *have the following regularity properties:*

$$\begin{aligned} \mathsf{p}_w, \mathsf{p}_g &\in L^2(0,T;L^2(\Omega)), \\ \sqrt{\lambda_w(x,\mathsf{S})}\,\nabla \mathsf{p}_w, \sqrt{\lambda_g(x,\mathsf{S})}\,\nabla \mathsf{p}_g &\in L^2(0,T;L^2(\Omega)); \end{aligned} \tag{39}$$

$$\beta_\ell(S_\ell) \in L^2(0,T;H^1(\Omega_\ell)) \quad \text{and} \quad \mathsf{P}_\ell \in L^2(0,T;H^1(\Omega_\ell)); \tag{40}$$

$$\Phi\frac{\partial S_\ell}{\partial t} \in L^2(0,T;H^{-1}(\Omega_\ell)) \quad \text{and} \quad \Phi\frac{\partial \Theta_\ell}{\partial t} \in L^2(0,T;H^{-1}(\Omega_\ell)); \tag{41}$$

where the function Θ_ℓ *is given in (17).*

(II): *the maximum principle holds:*

$$0 \le \mathsf{S} \le 1 \text{ a.e. in } \Omega_T. \tag{42}$$

(III): *For any* $\varphi_w, \varphi_g \in C^1([0,T];H^1(\Omega))$ *satisfying*

$$\varphi_w = \varphi_g = 0 \quad \text{on } \Gamma_{\text{inj}}\times]0,T[, \quad \text{and} \quad \varphi_w(x,T) = \varphi_g(x,T) = 0,$$

we have:

$$
\begin{aligned}
&-\int_{\Omega_T} \Phi(x)\mathsf{S}\frac{\partial \varphi_w}{\partial t}\, dx\, dt - \int_{\Omega} \Phi(x)\mathsf{S}^{\mathbf{0}}(x)\varphi_w(x,0)\, dx + \\
&\int_{\Omega_T} K(x)\lambda_w(x,\mathsf{S})\nabla \mathsf{p}_w \cdot \nabla\varphi_w\, dx\, dt - \int_{\Omega_T} K(x)\lambda_w(x,\mathsf{S})\vec{g}\cdot\nabla\varphi_w\, dx\, dt = 0,
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
&-\int_{\Omega_T} \Phi(x)\Theta\frac{\partial \varphi_g}{\partial t}\, dx\, dt - \int_{\Omega} \Phi(x)\Theta(x,0)\varphi_g(x,0)\, dx + \\
&+\int_{\Omega_T} K(x)\lambda_g(x,\mathsf{S})\varrho_g(\mathsf{p}_g)\nabla \mathsf{p}_g \cdot \nabla\varphi_w\, dx\, dt \\
&-\int_{\Omega_T} K(x)\lambda_g(x,\mathsf{S})\left[\varrho_g(\mathsf{p}_g)\right]^2 \vec{g}\cdot\nabla\varphi_w\, dx\, dt = 0
\end{aligned}
\tag{44}
$$

with Θ *defined in (17), and*

$$P_{\ell,c}(S_\ell) = p_{\ell,g} - p_{\ell,w}.$$

(IV): *The initial conditions are satisfied in a weak sense as follows:*

$$\forall\psi \in H^1_{\Gamma_{\rm inj}}(\Omega), \quad \int_{\Omega} \Phi(x)\mathsf{S}(x,t)\psi(x)\, dx, \int_{\Omega} \Phi(x)\Theta(x,t)\psi(x)\, dx \in C\left([0,T]\right). \tag{45}$$

Furthermore, we have

$$\left(\int_{\Omega} \Phi(x)\mathsf{S}\,\psi\, dx\right)(0) = \int_{\Omega} \Phi(x)\mathsf{S}^{\mathbf{0}}\,\psi\, dx \tag{46}$$

and

$$\left(\int_{\Omega} \Phi(x)\Theta\,\psi\, dx\right)(0) = \int_{\Omega} \Phi(x)\Theta^{\mathbf{0}}\,\psi\, dx \quad \text{with } \Theta^{\mathbf{0}} \stackrel{\text{def}}{=} \Theta(x,0). \tag{47}$$

The proof of Theorem 2.1 will be done in several steps. The scheme of the proof is given below in Section 4. The next section is devoted to the proof of the regularity and compactness properties of the solutions to problem (15) under the assumption that there exists at least one weak solution of (15). The maximum principle (42) will be discussed below in Lemma 5.5.

3. Properties of the solutions to system (15)–(22).

The outline of the section is as follows. First, we establish *a priori* estimates for solution of (15)–(22). These estimates explain clearly the origins of the requirements (39)–(41). The derivation of the *a priori* estimates is essentially based on the *energy equality*. Notice that this equality was introduced for the first time in the case of a homogeneous porous medium in [26]. In Section 3.1 we generalize the *energy equality* to the case of a porous media made of two types of rock. In Section 3.2 we will show that the weak formulation of the problem (43)–(44) contains an interpretation of the initial conditions. Finally, in Section 3.3 we establish some compactness results which will be used in the proof of the main result of the paper.

3.1. **Regularity properties (39)–(41).** First, we obtain the *energy equality*. To this end, following [26], we introduce the functions:

$$\mathsf{R}_{\ell,w}(p_{\ell,w}) \stackrel{\text{def}}{=} \int_0^{p_{\ell,w}} d\xi = p_{\ell,w} \quad \text{and} \quad \mathsf{R}_{\ell,g}(p_{\ell,g}) \stackrel{\text{def}}{=} \int_0^{p_{\ell,g}} \frac{d\xi}{\varrho_g(\xi)}. \tag{48}$$

It is clear that

$$\nabla \mathsf{R}_{\ell,w}(p_{\ell,w}) = \nabla p_{\ell,w} \quad \text{and} \quad \nabla \mathsf{R}_{\ell,g}(p_{\ell,g}) = \frac{1}{\varrho_{\ell,g}} \nabla p_{\ell,g}, \text{ where } \varrho_{\ell,g} = \varrho_g(p_{\ell,g}).$$

Then following the lines of [26, 3], one can prove the following lemma.

Lemma 3.1 (*Energy equality*)**.** *Let* $\langle \mathsf{p}_g, \mathsf{p}_w \rangle$ *be a solution to (15)–(22). Then*

$$\begin{aligned} \frac{d}{dt} \int_\Omega \Phi(x) \mathcal{E}(x,t)\, dx + \int_\Omega K(x) \lambda_w(x, \mathsf{S}) \nabla \mathsf{p}_w \cdot (\nabla \mathsf{p}_w - \vec{g})\, dx \\ + \int_\Omega K(x) \lambda_g(x, \mathsf{S}) \varrho_g(\mathsf{p}_g) \nabla \mathsf{p}_g \cdot (\nabla \mathsf{p}_g - \varrho_g(\mathsf{p}_g)\vec{g})\, dx = 0 \end{aligned} \tag{49}$$

in the sense of distributions. Here $\mathcal{E} \stackrel{\text{def}}{=} \mathcal{E}_1 \mathbf{I}_1 + \mathcal{E}_2 \mathbf{I}_2$ *with*

$$\mathcal{E}_\ell \stackrel{\text{def}}{=} (1 - S_\ell)\mathcal{R}_\ell(p_{\ell,g}) - \digamma(S_\ell), \quad \text{where } \digamma_\ell(s) \stackrel{\text{def}}{=} \int_0^s P_{\ell,c}(\xi)\, d\xi \tag{50}$$

and

$$\mathcal{R}_\ell(p) \stackrel{\text{def}}{=} \varrho_g(p) \mathsf{R}_{\ell,g}(p) - p. \tag{51}$$

Moreover, for all $p \in \mathbb{R}$, $\mathcal{R}_\ell(p) \geq 0$.

Let us prove that the function $\mathcal{E}_\ell$ is bounded from below. The boundedness of $\mathcal{E}_\ell$ is a consequence of Lemma 3.1 and condition **(A.4)**. Namely, we have

$$\mathcal{E}_\ell = (1 - S_\ell)\mathcal{R}_\ell(p_{\ell,g}) - \digamma(S_\ell) \geq -\digamma(1) \geq - \max_{S_\ell \in [0,1]} P_{\ell,c}(S_\ell). \tag{52}$$

Now *a priori* estimates for the solutions to (15)–(22) are a simple corollary of Lemma 3.1 and (52). Namely, we have:

Corollary 1. *Let* $\langle \mathsf{p}_g, \mathsf{p}_w \rangle$ *be a solution to (15)–(22) and and let* P_ℓ *and* β_ℓ *be the functions defined in (24) and (34), respectively. Then*

$$\int_{\Omega_T} \left\{ \lambda_w(x, \mathsf{S}) |\nabla \mathsf{p}_w|^2 + \lambda_g(x, \mathsf{S}) |\nabla \mathsf{p}_g|^2 \right\} dx < +\infty; \tag{53}$$

$$\int_{\Omega_{\ell,T}} \left\{ |\nabla \mathsf{P}_\ell|^2 + |\nabla \beta_\ell(S_\ell)|^2 \right\} dx < +\infty; \tag{54}$$

$$\|\partial_t(\Phi \Theta_\ell)\|_{L^2(0,T;H^{-1}(\Omega_\ell))} + \|\partial_t(\Phi S_\ell)\|_{L^2(0,T;H^{-1}(\Omega_\ell))} < +\infty. \tag{55}$$

3.2. **Interpretation of the initial conditions.** In this section we show that the weak formulation (43)–(44) contains the interpretation of the initial conditions (45)–(47).

Let $\varphi_w = \chi(t)\,\omega(x)$, where $\chi \in \mathcal{D}]0,T[$ and $\omega \in H^1_{\Gamma_{\rm inj}}(\Omega)$. Then, from (43) we get:

$$\frac{d}{dt}\int\limits_{\Omega} \Phi(x)\mathsf{S}(x,t)\omega(x)\ dx + \int\limits_{\Omega} K(x)\lambda_w(x,\mathsf{S})(\nabla \mathsf{p}_w - \vec{g})\cdot\nabla\omega\ dx = 0 \tag{56}$$

in the sense of distributions. From the regularity properties of the solutions, we deduce that

$$\int\limits_{\Omega} \Phi(x)\mathsf{S}(x,t)\omega(x)\,dx \in L^1(]0,T[), \qquad \frac{d}{dt}\int\limits_{\Omega} \Phi(x)\mathsf{S}(x,t)\omega(x)\,dx \in L^1(]0,T[). \tag{57}$$

This means that the function $\int_\Omega \Phi(x)\mathsf{S}(x,t)\omega(x)\ dx \in W^{1,1}(]0,T[)$ and consequently this function is continuous.

Now, we multiply (56) by $\chi \in C^\infty([0,T])$ such that $\chi(0) = 1$ and $\chi(T) = 0$. Then, integrating by parts we get:

$$\begin{aligned} -\left(\int\limits_{\Omega} \Phi(x)\mathsf{S}\,\omega(x)\ dx\right)(0) = &-\int\limits_{\Omega_T} K(x)\lambda_w(x,\mathsf{S})(\nabla \mathsf{p}_w - \vec{g})\cdot\nabla\varphi_w\ dx\,dt \\ &+\int\limits_{\Omega_T} \Phi(x)\mathsf{S}\,\omega(x)\frac{d\chi}{dt}\ dx\,dt. \end{aligned}$$

Comparing now this equation and (43), where $\varphi_w(x,t) = \chi(t)\,\omega(x)$, we observe that

$$\left(\int\limits_{\Omega} \Phi(x)\mathsf{S}\,\omega(x)\ dx\right)(0) = \int\limits_{\Omega} \Phi(x)\mathsf{S}^{\mathbf{0}}(x)\omega(x)\ dx \tag{58}$$

which makes the initial condition at $t = 0$ well defined.

In a similar way we show the relation (47).

3.3. **Compactness results.** In this section we obtain two compactness results that will be used in the proof of the main existence theorem. Notice that the previous results obtained in [24] and [29, 30] are not sufficient for our purposes. In these papers the method proposed earlier in [15] for the constant porosity function is generalized to the case of the porosity function belonging to the class $W^{1,\infty}$. The proof is essentially based on Simon's embedding theorem for the spaces of functions depending on the space and time variables (see [34]). However, the assumption that the porosity function is from the space $W^{1,\infty}$ is not admissible for the homogenization of the water–gas flow in porous media made of different types of rock. Below we propose our own approach to this problem. This approach was developed for the first time in [3] in the context of the homogenization of a single rock type model. Namely, we have the following compactness lemma.

Lemma 3.2 (Compactness lemma)**.** *Let the function $\Phi = \Phi(x)$, $\Phi \in L^\infty(\Omega)$, and there are positive constants ϕ_1, ϕ_2 such that $0 < \phi_1 \le \Phi(x) \le \phi_2 < 1$ a.e. in Ω and let $\{v^\varepsilon\}_{\varepsilon>0} \subset L^2(\Omega_T)$ be a family of functions satisfying the properties:*

1. *the function v^ε is uniformly bounded in the space $L^\infty(\Omega_T)$, i.e.,*

$$0 \le v^\varepsilon \le C; \tag{59}$$

2. *there exists a function ϖ such that $\varpi(\xi) \to 0$ as $\xi \to 0$ and the following inequality holds true:*

$$\int_{\Omega_T} |v^\varepsilon(x+\Delta x,\tau) - v^\varepsilon(x,\tau)|^2 \, dx\, d\tau \le C\,\varpi(|\Delta x|); \tag{60}$$

3. *the function v^ε is such that*

$$\|\partial_t(\Phi v^\varepsilon)\|_{L^2(0,T;H^{-1}(\Omega))} \le C. \tag{61}$$

Then the family $\{v^\varepsilon\}_{\varepsilon>0}$ is a compact set in $L^2(\Omega_T)$.

This compactness result is particular case of the Lemma 4.2 which was proved in [3]. We apply Lemma 3.2 in order to obtain the compactness results for the sequences $\{\Theta_\ell^\varepsilon\}_{\varepsilon>0}$, $\{S_\ell^\varepsilon\}_{\varepsilon>0}$. The first compactness result reads.

Proposition 1 (*First compactness result*)**.** *Let $\{\Theta_\ell^\varepsilon\}_{\varepsilon>0} \subset L^2(\Omega_{\ell,T})$ be defined by*

$$\Theta_\ell^\varepsilon \stackrel{\text{def}}{=} \varrho_g(p_{\ell,g}^\varepsilon)(1-S_\ell^\varepsilon),$$

where ε is a small positive parameter which goes to zero. Suppose that

$$\begin{aligned} &\left\|\sqrt{\lambda_{\ell,w}(S_\ell^\varepsilon)}\,\nabla p_{\ell,w}^\varepsilon\right\|_{L^2(\Omega_{\ell,T})} + \left\|\sqrt{\lambda_{\ell,g}(S_\ell^\varepsilon)}\,\nabla p_{\ell,g}^\varepsilon\right\|_{L^2(\Omega_{\ell,T})} \\ &\qquad + \|\partial_t(\Phi\Theta_\ell^\varepsilon)\|_{L^2(0,T;H^{-1}(\Omega_\ell))} \le C, \end{aligned} \tag{62}$$

where C is a constant that does not depend on ε. Then $\{\Theta_\ell^\varepsilon\}_{\varepsilon>0}$ is a compact set in the space $L^2(\Omega_{\ell,T})$.

Proof. The idea of the proof is to apply the Compactness lemma 3.2. Namely, we check the conditions of the lemma. First, it follows from the definition of the function ϱ_g and condition **(A.1)** that

$$0 \le \Theta_\ell^\varepsilon = \varrho_g(\mathsf{P}_\ell^\varepsilon + \mathsf{G}_g(S_\ell^\varepsilon))\,(1-S_\ell^\varepsilon) \le \varrho_{\max} < \infty. \tag{63}$$

Now we exploit (62) and (32). As in Corollary 1 we get:

$$\int_{\Omega_{\ell,T}} \left\{|\nabla \mathsf{P}_\ell^\varepsilon|^2 + |\nabla\beta_\ell(S_\ell^\varepsilon)|^2\right\} dx \le C, \tag{64}$$

where C is a constant that does not depend on ε. Then from (64) and the condition **(A.7)** we have:

$$\int_{\Omega_{\ell,T}} |\Theta_\ell^\varepsilon(x+y,\tau) - \Theta_\ell^\varepsilon(x,\tau)|^2 \, dx\, d\tau \le \omega_1(y) \text{ with } \omega_1(\xi)\to 0 \text{ as } \xi \to 0. \tag{65}$$

Finally, we observe that condition 3 of the Compactness lemma is fulfilled due to (62) which concludes the proof of Proposition 1. □

As a consequence of the L^2–compactness and the uniform L^∞–bound for Θ_ℓ^ε we have the following result:

Corollary 2. *The family $\{\Theta_\ell^\varepsilon\}_{\varepsilon>0}$ is a compact set in the space $L^q(\Omega_{\ell,T})$ for all $q \in [1,+\infty[$.*

By similar arguments we prove the second compactness result.

Proposition 2 (*Second compactness result*)**.** *Let $\{S_\ell^\varepsilon\}_{\varepsilon>0} \subset L^2(\Omega_{\ell,T})$, where ε is small positive parameter which goes to zero. Assume that*

$$\begin{aligned} \Big\|\sqrt{\lambda_{\ell,w}(S_\ell^\varepsilon)}\,\nabla p_{\ell,w}\Big\|_{L^2(\Omega_{\ell,T})} + \Big\|\sqrt{\lambda_{\ell,g}(S_\ell^\varepsilon)}\,\nabla p_{\ell,g}\Big\|_{L^2(\Omega_{\ell,T})} \\ + \|\partial_t(\Phi S_\ell^\varepsilon)\|_{L^2(0,T;H^{-1}(\Omega_\ell))} \le C, \end{aligned} \tag{66}$$

where C is a constant that does not depend on ε. Then, for all $q \in [1,+\infty[$, $\{S_\ell^\varepsilon\}_{\varepsilon>0}$ is a compact set in the space $L^q(\Omega_{\ell,T})$.

4. **Scheme of the proof of Theorem 2.1.** The goal of this section is to give a short scheme of the proof of Theorem 2.1. It will be done in two main steps. First, we consider the following non–degenerate system:

$$\begin{cases} \Phi \dfrac{\partial \mathsf{S}^\eta}{\partial t} - \operatorname{div}\Big(K\lambda_w(x,\mathsf{S}^\eta)\,(\nabla \mathsf{p}_w^\eta - \vec g) + \eta\,\nabla(\mathsf{p}_w^\eta - \mathsf{p}_g^\eta)\Big) = 0 \quad \text{in } \Omega_T; \\ \Phi \dfrac{\partial \Theta^\eta}{\partial t} - \operatorname{div}\Big(K\lambda_g(x,\mathsf{S}^\eta)\varrho_g(\mathsf{p}_g^\eta)\,(\nabla \mathsf{p}_g^\eta - \varrho_g(\mathsf{p}_g^\eta)\vec g) \\ \qquad\qquad + \eta\,\varrho_g(\mathsf{p}_g^\eta)\nabla(\mathsf{p}_g^\eta - \mathsf{p}_w^\eta)\Big) = 0 \text{ in } \Omega_T; \\ P_c(x,\mathsf{S}^\eta) = \mathsf{p}_g^\eta - \mathsf{p}_w^\eta \quad \text{in } \Omega_T, \end{cases} \tag{67}$$

where η is a small positive parameter; for the sake of brevity we do not write down again the initial and boundary conditions.

Notational convention. In what follows the upper index corresponds to the parameter for which we study the limit behavior of the corresponding functions.

The existence result for system (67) will be formulated and proved in Section 6. The proof of this result is based on the existence result for a system with a time discretization. Namely, we will consider the following non–degenerate elliptic problem:

$$\begin{cases} \Phi\Delta_h \mathsf{S}_\eta^h - \operatorname{div}\Big(K\lambda_w(x,\mathsf{S}_\eta^h)\,(\nabla \mathsf{p}_{w,\eta}^h - \vec g) + \eta\nabla\,(\mathsf{p}_{w,\eta}^h - \mathsf{p}_{g,\eta}^h)\Big) = 0; \\ \Phi\Delta_h \Theta_\eta^h - \operatorname{div}\Big(K\lambda_g(x,\mathsf{S}_\eta^h)\varrho_{g,\eta}^h\,(\nabla \mathsf{p}_{g,\eta}^h - \varrho_{g,\eta}^h\vec g) + \eta\varrho_{g,\eta}^h\nabla\,(\mathsf{p}_{g,\eta}^h - \mathsf{p}_{w,\eta}^h)\Big) = 0; \\ P_c(x,\mathsf{S}_\eta^h) = \mathsf{p}_{g,\eta}^h - \mathsf{p}_{w,\eta}^h, \end{cases} \tag{68}$$

where

$$\varrho_{g,\eta}^h = \varrho_g(\mathsf{p}_{g,\eta}^h), \quad \Delta_h \mathsf{S}_\eta^h \overset{\text{def}}{=} \frac{\mathsf{S}_\eta^h - \mathsf{S}_\eta^\star}{h}, \quad \Delta_h \Theta_\eta^h \overset{\text{def}}{=} \frac{\Theta_\eta^h - \Theta_\eta^\star}{h} \tag{69}$$

and where $\mathsf{S}_\eta^\star, \Theta_\eta^\star$ are given functions.

Remark 3 (On the interface and boundary conditions)**.** In the case of fields with different rock-types, when deriving the limit problem, we have to take care of the continuity of the phase pressures at the interface $\Gamma_{1,2}$ and of the Dirichlet boundary condition on Γ_{inj}. If the gradients of the phase pressures are uniformly bounded with respect to the corresponding small parameter, the continuity of the pressures and the boundary condition on Γ_{inj} are evident (see, e.g., the proof of Theorem 5.1). However, this uniform boundedness is violated when we pass to the limit as $\eta \to 0$. In this case we obtain the desired continuity of the phase pressures and the boundary condition on $\Gamma_{\rm inj}$ using the notion of the global pressure and the equality (32). For more details see Section 7.2.

The rest of the paper is organized as follows. In Section 5 we are dealing with the time discrete model. The existence result is proved in two main steps. In the first step we consider system (68) with non–degenerate mobilities $\lambda_w^\epsilon \stackrel{\rm def}{=} \lambda_w + \epsilon$, $\lambda_g^\epsilon \stackrel{\rm def}{=} \lambda_g + \epsilon$ with $\epsilon > 0$ and then apply the Leray–Schauder fixed point theorem. In the second step we pass to the limit as $\epsilon \to 0$. In Section 6 we pass to the limit as $h \to 0$. This proves the existence result for non–degenerate system (67). Finally, in Section 7 we pass to the limit as $\eta \to 0$ to prove the main result of the paper.

5. **Existence result for system (68).** In this section we deal with the time discrete non–degenerate model (68), where the dependence on the parameter η is not indicated explicitly for the sake of brevity:

$$\begin{cases} \Phi\Delta_h \mathsf{S}^h - {\rm div}\left(K\lambda_w(x,\mathsf{S}^h)\left(\nabla \mathsf{p}_w^h - \vec{g}\right) + \eta\nabla\left(\mathsf{p}_w^h - \mathsf{p}_g^h\right)\right) = 0; \\ \Phi\Delta_h \Theta^h - {\rm div}\left(K\lambda_g(x,\mathsf{S}^h)\varrho_g^h\left(\nabla \mathsf{p}_g^h - \varrho_g^h\vec{g}\right) + \eta\varrho_g^h\nabla\left(\mathsf{p}_g^h - \mathsf{p}_w^h\right)\right) = 0; \\ P_c(x,\mathsf{S}^h) = \mathsf{p}_g^h - \mathsf{p}_w^h. \end{cases} \tag{70}$$

As before, we impose the following interface conditions on $\Gamma_{1,2}$:

$$\begin{cases} \vec{q}_{1,w}^{\,h}\cdot\vec{\nu} = \vec{q}_{2,w}^{\,h}\cdot\vec{\nu} \quad \text{and} \quad \vec{q}_{1,g}^{\,h}\cdot\vec{\nu} = \vec{q}_{2,g}^{\,h}\cdot\vec{\nu} \quad \text{on } \Sigma_T; \\ p_{1,w}^h = p_{2,w}^h \text{ and } p_{1,g}^h = p_{2,g}^h \quad \text{on } \Sigma_T, \end{cases} \tag{71}$$

where

$$\vec{q}_{\ell,w}^{\,h} \stackrel{\rm def}{=} -K(x)\lambda_{\ell,w}(S_\ell^h)\left(\nabla p_{\ell,w}^h - \vec{g}\right) - \eta\nabla(p_{\ell,w}^h - p_{\ell,g}^h);$$

$$\vec{q}_{\ell,g,\eta}^{\,h} \stackrel{\rm def}{=} -K(x)\lambda_{\ell,g}(S_\ell^h)\varrho_{\ell,g}^h\left(\nabla p_{\ell,g,\eta} - \varrho_{\ell,g}^h\vec{g}\right) - \eta\varrho_{\ell,g}^h\nabla(p_{\ell,g}^h - p_{\ell,w}^h).$$

Also, we equip this system with the following boundary conditions:

$$\begin{cases} p_{1,g}^h = p_{1,w}^h = 0 \quad \text{on } \Gamma_{\rm inj}; \\ \vec{q}_{1,w}^{\,h}\cdot\vec{\nu} = \vec{q}_{1,g}^{\,h}\cdot\vec{\nu} = 0 \quad \text{on } \Gamma_{\rm imp}. \end{cases} \tag{72}$$

The main result of the section is given by the following theorem.

Theorem 5.1. *Let assumptions* **(A.1)**–**(A.9)** *be fulfilled and let η be a fixed positive parameter. Then for all $h > 0$, there exists a pair of functions $\langle \mathsf{p}_g^h, \mathsf{p}_w^h\rangle$ such that*

(I): *The functions $\mathsf{p}_g^h, \mathsf{p}_w^h$, and S_ℓ^h have the following regularity properties:*

$$\mathsf{p}_w^h, \mathsf{p}_g^h \in H^1_{\Gamma_{\rm inj}}(\Omega) \quad \text{and} \quad S_\ell^h \in H^1(\Omega_\ell). \tag{73}$$

(II): *The maximum principle holds:*

$$0 \le \mathsf{S}^h \le 1 \ \text{ a.e. in } \Omega. \tag{74}$$

(III): *For any $\varphi_w, \varphi_g \in H^1_{\Gamma_{\rm inj}}(\Omega)$,*

$$\int_\Omega \left\{\Phi(x)\Delta_h \mathsf{S}^h\,\varphi_w + K(x)\lambda_w(x,\mathsf{S}^h)\left(\nabla \mathsf{p}_w^h - \vec{g}\right)\cdot\nabla\varphi_w\right\}dx + \int_\Omega \eta\nabla\left(\mathsf{p}_w^h - \mathsf{p}_g^h\right)\cdot\nabla\varphi_w\,dx = 0;$$

$$\int_\Omega \left\{\Phi(x)\Delta_h \Theta^h\, \varphi_g + K(x)\lambda_g(x,\mathsf{S}^h)\left(\nabla \mathsf{p}_g^h - \varrho_g^h \vec{g}\right)\varrho_g^h \cdot \nabla\varphi_g\right\} dx - \int_\Omega \eta \varrho_g^h \nabla\left(\mathsf{p}_w^h - \mathsf{p}_g^h\right)\cdot \nabla\varphi_g \, dx = 0.$$

5.1. **Proof of Theorem 5.1.** First, we shortly describe the scheme of the proof of Theorem 5.1. We follow the steps developed in [29] for the single rock type model. Taking into account this fact, for the sake of brevity, we will omit the proofs of several propositions and lemmata given below. Before establishing Theorem 5.1 which is the main goal of this section, we consider a regularized problem. Namely, we consider the system (70)–(72) with non–degenerate mobilities $\lambda^\epsilon_{\ell,w}, \lambda^\epsilon_{\ell,g}$ given by:

$$\lambda^\epsilon_{\ell,w} \stackrel{\text{def}}{=} \lambda_{\ell,w} + \epsilon \quad \text{and} \quad \lambda^\epsilon_{\ell,g} \stackrel{\text{def}}{=} \lambda_{\ell,g} + \epsilon, \tag{75}$$

with $\epsilon > 0$. In addition, we replace the regularization terms in (70) with their projections on finite-dimensional subspaces defined in terms of the eigenbasis of the Laplace operator in Ω with Dirichlet boundary conditions. This further regularization allows us to truncate high frequencies in the additional terms containing the parameter η, and makes it possible to apply Leray–Schauder fixed point theorem.

The passage to the non–degenerate mobilities leads to the loss of the maximum principle for the saturation S^h_ℓ. In this connection, the functions $\lambda^\epsilon_{\ell,w}, \lambda^\epsilon_{\ell,g}$ are extended on $(-\infty, 0]$ and $[1, +\infty)$ by constants in such a way that the extended functions are continuous. It is clear that they are bounded in $\mathbb{R}$. For the same reason we introduce the extension of the functions S^h_ℓ. Namely,

$$\mathsf{Z}_\ell(S^h_\ell) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{for } S^h_\ell \le 0; \\ S^h_\ell & \text{for } S^h_\ell \in [0,1]; \\ 1 & \text{for } S^h_\ell \ge 1 \end{cases} \quad \text{and} \quad \mathsf{Z}(\mathsf{S}^h) \stackrel{\text{def}}{=} \mathsf{Z}_1(S^h_1)\,\mathbf{I}_1 + \mathsf{Z}_2(S^h_2)\,\mathbf{I}_2. \tag{76}$$

Similarly, in order to write the saturations S^h_ℓ as functions of the principle unknowns $p^h_{\ell,w}$ and $p^h_{\ell,g}$, we extend the capillary pressure function $P_{\ell,c}$ on the complement to the interval $[0,1]$ as follows:

$$\overline{P}_{\ell,c}(s) \stackrel{\text{def}}{=} \begin{cases} P_{\ell,c}(0) - s & \text{for } s \le 0; \\ P_{\ell,c}(s) & \text{for } s \in [0,1]; \\ -(s-1) & \text{for } s \ge 1. \end{cases} \tag{77}$$

This is possible due to the condition **(A.4)**. Finally, we note that

$$S^h_\ell = \overline{P}^{\,-1}_{\ell,c}\left(p^h_{\ell,g} - p^h_{\ell,w}\right).$$

The existence of solution to (70)–(72) is proved in three steps. At the **first step** assuming that the parameters $\epsilon, N, h, \eta > 0$ are fixed we study the following regularized elliptic system (the dependence on the parameters h, $\eta > 0$ is omitted for brevity):

$$
\begin{aligned}
&\int_\Omega \Phi(x)\Delta_h \mathsf{S}^{\epsilon,N}\,\varphi_w\,dx + \int_\Omega K(x)\lambda_w^\epsilon(x,\mathsf{S}^{\epsilon,N})\nabla \mathsf{p}_w^{\epsilon,N}\cdot\nabla\varphi_w\,dx \\
&\qquad - \int_\Omega K(x)\lambda_w(x,\mathsf{S}^{\epsilon,N})\vec{g}\cdot\nabla\varphi_w\,dx \\
&\qquad + \eta\int_\Omega \nabla\Big(\mathcal{P}_N\left[\mathsf{p}_w^{\epsilon,N}\right] - \mathcal{P}_N\left[\mathsf{p}_g^{\epsilon,N}\right]\Big)\cdot\nabla\varphi_w\,dx = 0;
\end{aligned} \tag{78}
$$

$$
\begin{aligned}
&\int_\Omega \Phi(x)\Delta_h \Theta^{\epsilon,N}\,\varphi_g\,dx + \int_\Omega K(x)\lambda_g^\epsilon(x,\mathsf{S}^{\epsilon,N})\varrho_g(\mathsf{p}_g^{\epsilon,N})\nabla \mathsf{p}_g^{\epsilon,N}\cdot\nabla\varphi_g\,dx \\
&\qquad - \int_\Omega K(x)\lambda_g(x,\mathsf{S}^{\epsilon,N})\left[\varrho_g(\mathsf{p}_g^{\epsilon,N})\right]^2\vec{g}\cdot\nabla\varphi_g\,dx \\
&\qquad + \eta\int_\Omega \varrho_g(\mathsf{p}_g^{\epsilon,N})\nabla\Big(\mathcal{P}_N\left[\mathsf{p}_g^{\epsilon,N}\right] - \mathcal{P}_N\left[\mathsf{p}_w^{\epsilon,N}\right]\Big)\cdot\nabla\varphi_g\,dx = 0
\end{aligned} \tag{79}
$$

for all $\varphi_w, \varphi_g \in H^1_{\Gamma_{\rm inj}}(\Omega)$. Here $\mathcal{P}_N$ is the orthogonal projector of $L^2(\Omega)$ on the first N eigenvectors of the operator $u \to -\Delta u$ with homogeneous Dirichlet boundary conditions;

$$
\Delta_h \mathsf{S}^{\epsilon,N} \stackrel{\rm def}{=} \frac{\mathsf{Z}(\mathsf{S}^{\epsilon,N}) - \mathsf{S}^\star}{h} \quad \text{and} \quad \Delta_h \Theta^{\epsilon,N} \stackrel{\rm def}{=} \frac{\Theta^{\epsilon,N} - \Theta^\star}{h}
$$

with

$$
\begin{aligned}
&\Theta^{\epsilon,N} = \varrho_g(\mathsf{p}_g^{\epsilon,N})(1-\mathsf{Z}(\mathsf{S}^{\epsilon,N})); \quad \lambda_w^\epsilon(x,\mathsf{S}^{\epsilon,N}) \stackrel{\rm def}{=} \lambda_{1,w}^\epsilon(\mathsf{S}^{\epsilon,N})\mathbf{I}_1 + \lambda_{2,w}^\epsilon(\mathsf{S}^{\epsilon,N})\mathbf{I}_2; \\
&\lambda_g^\epsilon(x,\mathsf{S}^{\epsilon,N}) \stackrel{\rm def}{=} \lambda_{1,g}^\epsilon(\mathsf{S}^{\epsilon,N})\mathbf{I}_1 + \lambda_{2,g}^\epsilon(\mathsf{S}^{\epsilon,N})\mathbf{I}_2; \quad \mathsf{S}^{\epsilon,N} = \sum_{\ell=1}^{2} \overline{P}_{\ell,c}^{-1}\left(\mathsf{p}_g^{\epsilon,N} - \mathsf{p}_w^{\epsilon,N}\right)\mathbf{I}_\ell.
\end{aligned} \tag{80}
$$

The **second step** is concerned with the passage to the limit in (78)–(80) as $N \to \infty$, while the **third step** with the passage to the limit as ϵ goes to zero.

5.1.1. *Step 1: Application of a fixed point theorem.* In this section, for fixed $N > 0$ and $\epsilon > 0$, we prove the existence of solutions to system (78)–(79). For the sake of brevity we omit here the dependence of the solutions on the parameters N, ϵ.

We apply the following version of the Leray–Schauder fixed point theorem (see, e.g. [27]).

Theorem 5.2 (*Leray–Schauder's fixed point theorem.*)**.** *Let $\mathcal{M}$ be a continuous and compact map of a Banach space $\mathcal{B}$ into itself. Suppose that the set of $x \in \mathcal{B}$ such that $x = \sigma\mathcal{M}x$ for some $\sigma \in [0,1]$, is bounded. Then the map $\mathcal{M}$ has a fixed point.*

The main result of Section 5.1.1 is the following proposition.

Proposition 3. *Assume that $\mathsf{S}^\star, \Theta^\star \in L^2(\Omega)$ and $\mathsf{S}^\star, \Theta^\star \geq 0$ in Ω. Then there exists a pair of functions $\langle \mathsf{p}_w, \mathsf{p}_g\rangle \in H^1_{\Gamma_{\rm inj}}(\Omega) \times H^1_{\Gamma_{\rm inj}}(\Omega)$, solution to (78)–(80).*

5.1.2. *Step 2: Passage to the limit as $N \to +\infty$.* In this Section we pass to the limit as $N \to +\infty$. For the sake of simplicity we omit the dependence on the parameter ϵ in the functions depending on N. It follows from the previous Section that the pair of functions $\langle \mathsf{p}_w^N, \mathsf{p}_g^N \rangle \in H^1_{\Gamma_{\rm inj}}(\Omega) \times H^1_{\Gamma_{\rm inj}}(\Omega)$ is the solution of the following system of equations:

$$\int_\Omega \Phi(x)\Delta_h \mathsf{S}^N \varphi_w\, dx + \int_\Omega K(x)\lambda_w^\epsilon(x, \mathsf{S}^N)\nabla \mathsf{p}_w^N \cdot \nabla\varphi_w\, dx$$
$$- \int_\Omega K(x)\lambda_w(x, \mathsf{S}^N)\vec{g} \cdot \nabla\varphi_w\, dx + \eta \int_\Omega \nabla\Big(\mathcal{P}_N\left[\mathsf{p}_w^N\right] - \mathcal{P}_N\left[\mathsf{p}_g^N\right]\Big) \cdot \nabla\varphi_w\, dx = 0; \tag{81}$$

$$\begin{aligned} \int_\Omega \Phi(x)\Delta_h \Theta^N \varphi_g\, dx + \int_\Omega K(x)\lambda_g^\epsilon(x, \mathsf{S}^N)\varrho_g(\mathsf{p}_g^N)\nabla \mathsf{p}_g^N \cdot \nabla\varphi_g\, dx \\ - \int_\Omega K(x)\lambda_g(x, \mathsf{S}^N)\left[\varrho_g(\mathsf{p}_g^N)\right]^2 \vec{g} \cdot \nabla\varphi_g\, dx \\ +\eta \int_\Omega \nabla\Big(\mathcal{P}_N\left[\mathsf{p}_g\right] - \mathcal{P}_N\left[\mathsf{p}_w\right]\Big) \cdot \nabla\varphi_g\, dx = 0 \end{aligned} \tag{82}$$

for all $\varphi_w, \varphi_g \in H^1_{\Gamma_{\rm inj}}(\Omega)$.

Choosing

$$\varphi_w = \mathsf{R}_w(\mathsf{p}_w) = \mathsf{p}_w \quad \text{and} \quad \varphi_g = \mathsf{R}_g(\mathsf{p}_g) = \int_0^{\mathsf{p}_g} \frac{d\xi}{\varrho_g(\xi)},$$

as test functions, then we get the following estimate:

$$\epsilon \int_\Omega \left\{|\nabla \mathsf{p}_w^N|^2 + |\nabla \mathsf{p}_g^N|^2\right\} dx + \eta \int_\Omega \left|\nabla\left(\mathcal{P}_N\left[\mathsf{p}_w\right] - \mathcal{P}_N\left[\mathsf{p}_g\right]\right)\right|^2 dx \le C, \tag{83}$$

where C is a constant which does not depend on N. Then (up to subsequences) we obtain the following convergence results:

$$\mathsf{p}_w^N \longrightarrow \mathsf{p}_w^\epsilon \text{ weakly in } H^1_{\Gamma_{\rm inj}}(\Omega), \text{ strongly in } L^2(\Omega), \text{ and a.e. in } \Omega; \tag{84}$$

$$\mathsf{p}_g^N \longrightarrow \mathsf{p}_g^\epsilon \text{ weakly in } H^1_{\Gamma_{\rm inj}}(\Omega), \text{ strongly in } L^2(\Omega), \text{ and a.e. in } \Omega. \tag{85}$$

Taking into account that $S_\ell = \overline{P}_{\ell,c}^{\,-1}(p_{\ell,g} - p_{\ell,w})$, we also have:

$$\mathsf{S}^N \longrightarrow \mathsf{S}^\epsilon \text{ strongly in } L^2(\Omega) \text{ and a.e. in } \Omega. \tag{86}$$

Now we pass to the limit in (81)–(82) as $N \to +\infty$ using the convergence results (84)– (86). The corresponding system of equations is as follows:

$$\int_\Omega \Phi(x)\Delta_h\, \mathsf{S}^\epsilon \varphi_w\, dx + \int_\Omega K(x)\lambda_w^\epsilon(x, \mathsf{S}^\epsilon)\nabla \mathsf{p}_w^\epsilon \cdot \nabla\varphi_w\, dx -$$
$$- \int_\Omega K(x)\lambda_w(x, \mathsf{S}^\epsilon)\vec{g} \cdot \nabla\varphi_w\, dx + \eta \int_\Omega \nabla\left(\mathsf{p}_w^\epsilon - \mathsf{p}_g^\epsilon\right) \cdot \nabla\varphi_w\, dx = 0; \tag{87}$$

$$\int_\Omega \Phi(x)\Delta_h \Theta^\epsilon \varphi_g\, dx + \int_\Omega K(x)\lambda_g^\epsilon(x, \mathsf{S}_\epsilon)\varrho_g(\mathsf{p}_g^\epsilon)\nabla \mathsf{p}_g^\epsilon \cdot \nabla\varphi_g\, dx -$$

$$-\int\limits_{\Omega} K(x)\lambda_g(x,\mathsf{S}_\epsilon)\left[\varrho_g(\mathsf{p}_g^\epsilon)\right]^2 \vec{g}\cdot\nabla\varphi_g\,dx-\eta\int\limits_{\Omega}\varrho_g(\mathsf{p}_g^\epsilon)\nabla\left(\mathsf{p}_w^\epsilon-\mathsf{p}_g^\epsilon\right)\cdot\nabla\varphi_g\,dx=0 \quad (88)$$

for all $\varphi_w,\varphi_g\in H^1_{\Gamma_{\rm inj}}(\Omega)$.

Notice that the continuity of the phase pressures on $\Gamma_{1,2}$ and the Dirichlet boundary condition on $\Gamma_{\rm inj}$ are a consequence of the following relations:

$$\|\mathsf{p}_w^\epsilon\|_{H^1(\Omega)}\le \varliminf_{N\to+\infty}\|\mathsf{p}_w^N\|_{H^1(\Omega)}<+\infty \text{ and } \|\mathsf{p}_g^\epsilon\|_{H^1(\Omega)}\le \varliminf_{N\to+\infty}\|\mathsf{p}_g^N\|_{H^1(\Omega)}<+\infty$$

along with the continuity of the pressures $\mathsf{p}_w^N,\mathsf{p}_g^N$ on $\Gamma_{1,2}$.

5.1.3. *Step 3: Passage to the limit as $\epsilon\to 0$.* First, we notice that as in the previous sections we omit the dependence of the corresponding functions on η,h and keep the dependence on the small parameter ϵ, only.

It follows from the results of Section 5.1.2 that for any $\epsilon>0$, there is $\left\langle \mathsf{p}_w^\epsilon,\mathsf{p}_g^\epsilon\right\rangle\in H^1_{\Gamma_{\rm inj}}(\Omega)\times H^1_{\Gamma_{\rm inj}}(\Omega)$ which is the solution of (87)–(88). First, we obtain uniform estimates (with respect to ϵ) for the solutions in order to pass to the limit in (87)–(88) as $\epsilon\to 0$. These estimates are given by the following Lemma whose proof use the same arguments as in [3].

Lemma 5.3. *Let $\left\langle \mathsf{p}_w^\epsilon,\mathsf{p}_g^\epsilon\right\rangle$ be a solution to (87)–(88) and let P_ℓ^ϵ be the global pressure defined in (24). Then we have:*

$$\{\mathsf{P}_\ell^\epsilon\}_{\epsilon>0} \text{ is uniformly bounded in } H^1(\Omega_\ell); \quad (89)$$
$$\{\beta_\ell(S_\ell^\epsilon)\}_{\epsilon>0} \text{ is uniformly bounded in } H^1(\Omega_\ell); \quad (90)$$
$$\{\nabla\overline{P}_{\ell,c}(S_\ell^\epsilon)\}_{\epsilon>0} \text{ is uniformly bounded in } L^2(\Omega_\ell); \quad (91)$$
$$\{\mathsf{p}_w^\epsilon\}_{\epsilon>0} \text{ is uniformly bounded in } H^1_{\Gamma_{\rm inj}}(\Omega); \quad (92)$$
$$\{\mathsf{p}_g^\epsilon\}_{\epsilon>0} \text{ is uniformly bounded in } H^1_{\Gamma_{\rm inj}}(\Omega). \quad (93)$$

The uniform estimates established in the previous lemma imply the following convergence results.

Lemma 5.4. *Let $\{S_\ell^\epsilon\}_{\epsilon>0}$ and $\{\mathsf{p}_w^\epsilon\}_{\epsilon>0}$, $\{\mathsf{p}_g^\epsilon\}_{\epsilon>0}$ be the sequences of saturation and the phase pressures, respectively. Then we have:*

$$\beta_\ell(S_\ell^\epsilon)\longrightarrow\beta_\ell(S_\ell) \quad \text{weakly in } H^1(\Omega_\ell) \text{ and a.e. in } \Omega_\ell\,; \quad (94)$$
$$S_\ell^\epsilon\longrightarrow S_\ell \quad \text{strongly in } L^2(\Omega_\ell) \text{ and a.e. in } \Omega_\ell\,; \quad (95)$$
$$\mathsf{p}_w^\epsilon\longrightarrow\mathsf{p}_w \quad \text{weakly in } H^1(\Omega_\ell) \text{ and a.e. in } \Omega_\ell\,; \quad (96)$$
$$\mathsf{p}_g^\epsilon\longrightarrow\mathsf{p}_g \quad \text{weakly in } H^1(\Omega_\ell) \text{ and a.e. in } \Omega_\ell\,. \quad (97)$$

5.1.4. *End of the proof of Theorem 5.1.* Now we are in position to complete the proof of Theorem 5.1. To this end we have to passe to the limit in (87)–(88) as $\epsilon\to 0$ and then prove the maximum principle for the saturations.

The passage to the limit as $\epsilon\to 0$ yields the existence of a pair of functions $\langle \mathsf{p}_w^h,\mathsf{p}_g^h\rangle\in H^1_{\Gamma_{\rm inj}}(\Omega)\times H^1_{\Gamma_{\rm inj}}(\Omega)$ such that, for any $\varphi_w,\varphi_g\in H^1_{\Gamma_{\rm inj}}(\Omega)$, we have:

$$\int\limits_{\Omega}\left\{\Phi(x)\Delta_h\mathsf{S}^h\,\varphi_w\,dx+\left[K(x)\lambda_w(x,\mathsf{S}^h)\left(\nabla\mathsf{p}_w^h-\vec{g}\right)+\eta\nabla\left(\mathsf{p}_w^h-\mathsf{p}_g^h\right)\right]\cdot\nabla\varphi_w\right\}\,dx=0; \quad (98)$$

$$\int\limits_{\Omega}\left\{\Phi(x)\Delta_h\Theta^h\,\varphi_g+\left[K(x)\lambda_g(x,\mathsf{S}^h)\left(\nabla\mathsf{p}_g^h-\varrho_g^h\vec{g}\right)\varrho_g^h-\eta\varrho_g^h\nabla\left(\mathsf{p}_w^h-\mathsf{p}_g^h\right)\right]\cdot\nabla\varphi_g\right\}\,dx=0. \quad (99)$$

Now we prove the maximum principle for the system (98)–(99). The following result holds.

Lemma 5.5 (*Maximum principle*)**.** *Let* $\mathsf{S}^\star, \Theta^\star \geq 0$. *Then under the conditions of Theorem 5.1 we have:*

$$0 \leq \mathsf{S}^h \leq 1 \quad \text{a.e. in } \Omega. \tag{100}$$

Proof of Lemma 5.5. First, let us show that $\mathsf{S}^h \geq 0$ a.e. in Ω. In contrast to the case considered in [29] we cannot use S^h as a test function in (98) because S^h might have a jump at the interface $\Gamma_{1,2}$. Instead we let

$$S_\ell^{h,-} \stackrel{\text{def}}{=} \min\{S_\ell^h, 0\} \tag{101}$$

and define the test function $\varphi_w(x)$ as follows:

$$\varphi_w(x) \stackrel{\text{def}}{=} \begin{cases} \overline{P}_{1,c}(S_1^{h,-}(x)) - P_{1,c}(0) & \text{in } \Omega_1; \\ \overline{P}_{2,c}(S_2^{h,-}(x)) - P_{2,c}(0) & \text{in } \Omega_2. \end{cases} \tag{102}$$

Due to the definition of the capillary pressure, we have $\varphi_w \in H^1(\Omega)$, and $\varphi_w \geq 0$ in Ω. Moreover, by the definitions of Z_ℓ and the mobility extension, we have:

$$\mathsf{Z}_\ell(S_\ell^h)\varphi_w \equiv 0 \qquad \text{and} \qquad \lambda_{\ell,w}(S_\ell^h)\varphi_w \equiv 0. \tag{103}$$

Now we plug the function φ_w in (98). Then it follows from the definitions of the functions φ_w, $S_\ell^{h,-}$, the definition of the extension of the capillary pressure function (77), and (103) that

$$\frac{1}{h}\int_\Omega \Phi(x)\mathsf{S}^*\varphi_w(x)\,dx + \eta\sum_{\ell=1}^{2}\int_{\Omega_\ell} \nabla\overline{P}_{\ell,c}(S_\ell^h)\nabla\overline{P}_{\ell,c}(S_\ell^{h,-})\,dx = 0.$$

Since the first integral here is positive this yields

$$\sum_{\ell=1}^{2}\int_{\Omega_\ell} \nabla\overline{P}_{\ell,c}(S_\ell^h)\nabla\overline{P}_{\ell,c}(S_\ell^{h,-})\,dx \leq 0.$$

Notice that for $x \in \{x \in \Omega \,:\, S_\ell^h \geq 0\}$ we have $S_\ell^{h,-} = 0$, and thus $\nabla\overline{P}_{\ell,c}(S_\ell^{h,-}) = 0$. Therefore,

$$\sum_{\ell=1}^{2}\int_{\Omega_\ell} \nabla\overline{P}_{\ell,c}(S_\ell^{h,-})\nabla\overline{P}_{\ell,c}(S_\ell^{h,-})\,dx, \leq 0.$$

and consequently

$$\overline{P}_{\ell,c}(S_\ell^{h,-}) = const \qquad \text{in } \Omega.$$

Combining this relation with the boundary condition on Γ_{inj} we obtain $\overline{P}_{\ell,c}(S_\ell^{h,-}) = P_{\ell,c}(0)$ and $S_\ell^{h,-} = 0$. This inequality implies that $S_\ell^h \geq 0$ a.e. in Ω_ℓ, and consequently $\mathsf{S}^h \geq 0$ a.e. in Ω.

In a similar way we can prove that $\mathsf{S}^h \leq 1$ a.e. in Ω and Lemma 5.5 is proved.

As in Section 5.1.2 the continuity of the phase pressures on $\Gamma_{1,2}$ and the Dirichlet boundary condition on Γ_{inj} are the consequence of the following relations:

$$\|\mathsf{p}_w^h\|_{H^1(\Omega)} \leq \varliminf_{N\to+\infty} \|\mathsf{p}_w^\epsilon\|_{H^1(\Omega)} < +\infty \quad \text{and} \quad \|\mathsf{p}_g^h\|_{H^1(\Omega)} \leq \varliminf_{N\to+\infty} \|\mathsf{p}_g^\epsilon\|_{H^1(\Omega)} < +\infty$$

along with the continuity of the pressures $\mathsf{p}_w^\epsilon, \mathsf{p}_g^\epsilon$ on $\Gamma_{1,2}$. This completes the proof of Theorem 5.1

6. **Proof of the existence result for the non–degenerate system (67).** In this section we prove the existence result for the non–degenerate system:

$$\begin{cases} \Phi \dfrac{\partial \mathsf{S}^\eta}{\partial t} - \operatorname{div}\left(K(x)\lambda_w(x,\mathsf{S}^\eta)\left(\nabla \mathsf{p}_w^\eta - \vec{g}\right) + \eta\, \nabla(\mathsf{p}_w^\eta - \mathsf{p}_g^\eta)\right) = 0 \quad \text{in } \Omega_T; \\ \Phi \dfrac{\partial \Theta^\eta}{\partial t} - \operatorname{div}\left(K(x)\lambda_g(x,\mathsf{S}^\eta)\left(\nabla \mathsf{p}_g^\eta - \varrho_g^\eta \vec{g}\right)\, \varrho_g^\eta + \eta\, \varrho_g^\eta \nabla(\mathsf{p}_g^\eta - \mathsf{p}_w^\eta)\right) = 0 \quad \text{in } \Omega_T; \\ P_c\,(x,\mathsf{S}^\eta) = \mathsf{p}_g^\eta - \mathsf{p}_w^\eta \quad \text{in } \Omega_T, \end{cases} \tag{104}$$

where $\varrho_g^\eta \stackrel{\text{def}}{=} \varrho_g(\mathsf{p}_g^\eta)$. The interface, initial, and boundary conditions for the system (104) read:

$$\begin{cases} \vec{q}_{1,w}^{\,\eta} \cdot \vec{\nu} = \vec{q}_{2,w}^{\,\eta} \cdot \vec{\nu} \quad \text{and} \quad \vec{q}_{1,g}^{\,\eta} \cdot \vec{\nu} = \vec{q}_{2,g}^{\,\eta} \cdot \vec{\nu} \quad \text{on } \Sigma_T; \\ p_{1,w}^\eta = p_{2,w}^\eta \text{ and } p_{1,g}^\eta = p_{2,g}^\eta \quad \text{on } \Sigma_T, \end{cases} \tag{105}$$

where

$$\vec{q}_{\ell,w}^{\,\eta} \stackrel{\text{def}}{=} -K(x)\lambda_{\ell,w}(S_\ell^\eta)\left(\nabla p_{\ell,w}^\eta - \vec{g}\right) - \eta \nabla(p_{\ell,w}^\eta - p_{\ell,g}^\eta);$$

$$\vec{q}_{\ell,g}^{\,\eta} \stackrel{\text{def}}{=} -K(x)\lambda_{\ell,g}(S_\ell^\eta)\varrho_{\ell,g}^\eta\left(\nabla p_{\ell,g}^\eta - \varrho_{\ell,g}^\eta \vec{g}\right) - \eta \varrho_{\ell,g}^\eta \nabla(p_{\ell,g}^\eta - p_{\ell,w}^\eta);$$

$$\mathsf{p}_w^\eta(x,0) = \mathsf{p}_{w,\eta}^{\mathbf{0}}(x) \quad \text{and} \quad \mathsf{p}_g^\eta(x,0) = \mathsf{p}_{g,\eta}^{\mathbf{0}}(x) \quad \text{in } \Omega; \tag{106}$$

$$\begin{cases} p_{1,g}^\eta = p_{1,w}^\eta = 0 \quad \text{on } \Gamma_{\text{inj}} \times (0,T); \\ \vec{q}_{1,w}^{\,\eta} \cdot \vec{\nu} = \vec{q}_{1,g}^{\,\eta} \cdot \vec{\nu} = 0 \quad \text{on } \Gamma_{\text{imp}} \times (0,T). \end{cases} \tag{107}$$

The main result of this section is the following theorem.

Theorem 6.1. *Let assumptions* **(A.1)**–**(A.9)** *be fulfilled. Then there exists* $\langle \mathsf{p}_g^\eta, \mathsf{p}_w^\eta \rangle$ *such that:*

(I): *The functions* $\mathsf{p}_g^\eta, \mathsf{p}_w^\eta, S_\ell^\eta$ *have the following regularity properties:*

$$\mathsf{p}_g^\eta \in L^2(0,T;H^1_{\Gamma_{\text{inj}}}(\Omega)) \quad \text{and} \quad \mathsf{p}_w^\eta \in L^2(0,T;H^1_{\Gamma_{\text{inj}}}(\Omega)); \tag{108}$$

$$S_\ell^\eta \in L^2(0,T;H^1(\Omega_\ell)); \tag{109}$$

$$\Phi \frac{\partial S_\ell^\eta}{\partial t} \in L^2(0,T;H^{-1}(\Omega_\ell)) \quad \text{and} \quad \Phi \frac{\partial \Theta_\ell^\eta}{\partial t} \in L^2(0,T;H^{-1}(\Omega_\ell)). \tag{110}$$

(II): *The maximum principle holds:*

$$0 \le S_\ell^\eta \le 1 \ \text{ a.e. in } \Omega_{\ell,T}. \tag{111}$$

(III): *For any* $\varphi_w, \varphi_g \in C^1([0,T];H^1(\Omega))$ *satisfying* $\varphi_w = \varphi_g = 0$ *on* $\Gamma_{\text{inj}} \times (0,T)$ *and* $\varphi_w(x,T) = \varphi_g(x,T) = 0$*, we have:*

$$\begin{aligned} &-\int\limits_{\Omega_T} \Phi(x)\mathsf{S}^\eta \frac{\partial \varphi_w}{\partial t}\, dx\, dt - \int\limits_{\Omega} \Phi(x)\mathsf{S}^{\mathbf{0}}(x)\varphi_w(x,0)\, dx \\ &+ \int\limits_{\Omega_T} K(x)\lambda_w(x,\mathsf{S}^\eta)\nabla \mathsf{p}_w^\eta \cdot \nabla\varphi_w\, dx\, dt \\ &- \int\limits_{\Omega_T} K(x)\lambda_w(x,\mathsf{S}^\eta)\vec{g} \cdot \nabla\varphi_w\, dx\, dt + \eta \int\limits_{\Omega_T} \nabla(\mathsf{p}_w^\eta - \mathsf{p}_g^\eta) \cdot \nabla\varphi_w\, dx\, dt = 0; \end{aligned} \tag{112}$$

$$
\begin{aligned}
&-\int_{\Omega_T} \Phi(x)\Theta^\eta \frac{\partial \varphi_g}{\partial t}\, dx\, dt - \int_{\Omega} \Phi(x)\Theta^{\mathbf{0}}(x)\varphi_g(x,0)\, dx \\
&+\int_{\Omega_T} K(x)\lambda_g(x,\mathsf{S}^\eta)\varrho_g(\mathsf{p}_g^\eta)\nabla \mathsf{p}_g^\eta \cdot \nabla\varphi_g\, dx\, dt \\
&-\int_{\Omega_T} K(x)\lambda_g(x,\mathsf{S}^\eta)\left[\varrho_g(\mathsf{p}_g^\eta)\right]^2 \vec{g}\cdot\nabla\varphi_g\, dx\, dt \\
&+\eta\int_{\Omega_T} \varrho_g(\mathsf{p}_g^\eta)\nabla(\mathsf{p}_g^\eta - \mathsf{p}_w^\eta)\cdot\nabla\varphi_g\, dx\, dt = 0.
\end{aligned}
\tag{113}
$$

6.1. **Proof of Theorem 6.1.** The outline of the proof is as follows. First, in Section 6.1.1 we establish the uniform estimates for the solutions to the system (98)–(99) and obtain the corresponding compactness results with respect to the parameter h. Then in Section 6.1.2 we complete the proof of Theorem 6.1.

6.1.1. *Uniform estimates and compactness results.* The proof is based on a semi–discretization method in the time variable proposed in [2] and then applied in the study of water–gas flows in [23, 24, 26, 29]. Let $T > 0$, $N \in \mathbb{N}$ and $h = T/N$. For all $n \in [0, N-1]$, we define the sequences:

$$
\mathsf{p}_{w,h}^{\mathbf{0}} = \mathsf{p}_w^{\mathbf{0}} \quad \text{and} \quad \mathsf{p}_{g,h}^{\mathbf{0}} = \mathsf{p}_g^{\mathbf{0}} \quad \text{a.e. in } \Omega. \tag{114}
$$

Consider the pair of functions $\langle \mathsf{p}_{w,h}^n, \mathsf{p}_{g,h}^n\rangle \in L^2(\Omega)\times L^2(\Omega)$ with $S_{\ell,h}^n \geq 0$ and $\varrho_g(p_{\ell,g,h}^n)(1-S_{\ell,h}^n) \geq 0$ and then define $\langle \mathsf{p}_{w,h}^{n+1}, \mathsf{p}_{g,h}^{n+1}\rangle$ as the solution of the following system of equations:

$$
\Phi\Delta_h^n S_{\ell,h}^{n+1} - \mathrm{div}\left(K\lambda_{\ell,w}(S_{\ell,h}^{n+1})\left(\nabla p_{\ell,w,h}^{n+1} - \vec{g}\right) + \eta\nabla\left(p_{\ell,w,h}^{n+1} - p_{\ell,g,h}^{n+1}\right)\right) = 0; \tag{115}
$$

$$
\begin{aligned}
\Phi\Delta_h^n \Theta_{\ell,h}^{n+1} &- \mathrm{div}\left(K\varrho_{\ell,g,h}^{n+1}\lambda_{\ell,g}(S_{\ell,h}^{n+1})\left(\nabla p_{\ell,w,h}^{n+1} - \varrho_{\ell,g,h}^{n+1}\vec{g}\right)\right) \\
&-\mathrm{div}\left(\eta\varrho_{\ell,g,h}^{n+1}\nabla\left(p_{\ell,w,h}^{n+1} - p_{\ell,g,h}^{n+1}\right)\right) = 0,
\end{aligned}
\tag{116}
$$

where

$$
\Delta_h^n S_{\ell,h}^{n+1} \stackrel{\text{def}}{=} \frac{S_{\ell,h}^{n+1} - S_{\ell,h}^n}{h}, \ \Delta_h^n \Theta_{\ell,h}^{n+1} \stackrel{\text{def}}{=} \frac{\varrho_{\ell,g,h}^{n+1}(1-S_{\ell,h}^{n+1}) - \varrho_{\ell,g,h}^n(1-S_{\ell,h}^n)}{h}
$$

with $\varrho_{\ell,g,h}^n \stackrel{\text{def}}{=} \varrho_g(p_{\ell,g,h}^n)$.

The system (115)–(116) is completed with the following interface and boundary conditions:

$$
\begin{cases}
\vec{q}_{1,w,h}^{\,(n+1)}\cdot\vec{\nu} = \vec{q}_{2,w,h}^{\,(n+1)}\cdot\vec{\nu} \quad \text{and} \quad \vec{q}_{1,g,h}^{\,(n+1)}\cdot\vec{\nu} = \vec{q}_{2,g,h}^{\,(n+1)}\cdot\vec{\nu} \quad \text{on } \Gamma_{1,2}; \\
p_{1,g,h}^{n+1} = p_{2,g,h}^{n+1} \quad \text{and} \quad p_{1,w,h}^{n+1} = p_{2,w,h}^{n+1} \quad \text{on } \Gamma_{1,2};
\end{cases}
\tag{117}
$$

$$
\begin{cases}
p_{1,g,h}^{n+1} = p_{1,g,h}^{n+1} = 0 \quad \text{on } \Gamma_{\text{inj}}; \\
\vec{q}_{1,w,h}^{\,(n+1)}\cdot\vec{\nu} = \vec{q}_{1,g,h}^{\,(n+1)}\cdot\vec{\nu} = 0 \quad \text{on } \Gamma_{\text{imp}},
\end{cases}
\tag{118}
$$

where

$$
\vec{q}_{\ell,w,h}^{\,(n+1)} \stackrel{\text{def}}{=} -K(x)\lambda_{\ell,w}(S_{\ell,h}^{n+1})\left(\nabla p_{\ell,w,h}^{n+1} - \vec{g}\right) - \eta\nabla\left(p_{\ell,w,h}^{n+1} - p_{\ell,g,h}^{n+1}\right);
$$

$$\vec{q}_{\ell,g,h}^{\,(n+1)} \stackrel{\text{def}}{=} -K(x)\lambda_{\ell,g}(S_{\ell,h}^{n+1})\varrho_{\ell,g,h}^{n+1}\left(\nabla p_{\ell,g,h}^{n+1} - \varrho_{\ell,g,h}^{n+1}\vec{g}\right) - \eta\varrho_{\ell,g,h}^{n+1}\nabla\left(p_{\ell,g,h}^{n+1} - p_{\ell,w,h}^{n+1}\right).$$

The sequence $\langle \mathsf{p}_{w,h}^{n+1}, \mathsf{p}_{g,h}^{n+1}\rangle$ is well defined for all $n \in [0, N-1]$ due to Theorem 5.1. Thus, for given $S_{\ell,h}^n$, $\varrho_g(p_{\ell,g,h}^n)(1-S_{\ell,h}^n) \geq 0$ and $S_{\ell,h}^n, \varrho_g(p_{\ell,g,h}^n)(1-S_{\ell,h}^n) \in L^2(\Omega_\ell)$, we construct $\langle \mathsf{p}_{w,h}^{n+1}, \mathsf{p}_{g,h}^{n+1}\rangle \in H^1_{\text{inj}}(\Omega) \times H^1_{\text{inj}}(\Omega)$ so that $S_{\ell,h}^{n+1} \in [0,1]$.

In the following Lemma, we obtain uniform with respect to h estimates for $\langle \mathsf{p}_{w,h}^{n+1}, \mathsf{p}_{g,h}^{n+1}\rangle$. For the sake of brevity, in this Lemma we omit the dependence on the parameter h.

Lemma 6.2. *The solutions of (115)–(118) satisfy the bound:*

$$\begin{aligned}
&\frac{1}{h}\int_\Omega \Phi(x)\left\{\mathcal{R}(\mathsf{p}_g^{n+1})(1-\mathsf{S}^{n+1}) - \mathcal{R}(\mathsf{p}_g^n)(1-\mathsf{S}^n)\right\} dx \\
&\quad -\frac{1}{h}\sum_{\ell=1}^{2}\int_{\Omega_\ell} \Phi(x)\left\{\mathcal{F}_\ell(S_{\ell,w}^{n+1}) - \mathcal{F}_\ell(S_{\ell,w}^n)\right\} dx \\
&\quad +\eta\int_\Omega \left|\nabla\left(\mathsf{p}_g^{n+1} - \mathsf{p}_w^{n+1}\right)\right|^2 dx \\
&+\int_\Omega \left\{\lambda_w(x,\mathsf{S}^{n+1})\left|\nabla \mathsf{p}_w^{n+1}\right|^2 + \lambda_g(x,\mathsf{S}^{n+1})\left|\nabla\mathsf{p}_g^{n+1}\right|^2\right\} dx \leq C,
\end{aligned} \tag{119}$$

where C is a constant that does not depend on h, and $\mathcal{R}(\mathsf{p}_g) \stackrel{\text{def}}{=} \mathcal{R}_1(p_{1,g})\,\mathbf{I}_1 + \mathcal{R}_2(p_{2,g})\,\mathbf{I}_2$.

The proof of Lemma 6.2 is similar to the proof of Lemma 3.1 in [29].

Now, for a given sequence $\{u_h^n\}_n$, we define the following functions:

$$u^h(t) \stackrel{\text{def}}{=} \sum_{n=0}^{N-1} u_h^{n+1}\,\mathbf{I}_{]nh,(n+1)h]}(t) \qquad \forall t \in]0,T[\quad \text{with } u^h(0) = u_0^h, \tag{120}$$

where $\mathbf{I}_{(nh,(n+1)h]}(t)$ denotes the characteristic function of the interval $]nh,(n+1)h]$;

$$\widetilde{u}^h(t) \stackrel{\text{def}}{=} \sum_{n=0}^{N-1}\left[\left(1+n-\frac{t}{h}\right)u_h^n + \left(\frac{t}{h}-n\right)u_h^{n+1}\right]\mathbf{I}_{[nh,(n+1)h]}(t) \tag{121}$$

for all $t \in [0,T]$. Then

$$\frac{\partial \widetilde{u}^h}{\partial t} = \frac{1}{h}\sum_{n=0}^{N-1}\left(u_h^{n+1} - u_h^n\right)\mathbf{I}_{(nh,(n+1)h)}(t) \;\; \forall t \in [0,T] \setminus \left\{\cup_{n=0}^N n\,h\right\}. \tag{122}$$

The following uniform estimates hold true.

Lemma 6.3. *Let $\mathsf{p}_w^h, \mathsf{p}_g^h, S_\ell^h, \Theta_\ell^h$ be the functions defined in terms of $\mathsf{p}_{w,h}^n$, $\mathsf{p}_{g,h}^n$, $S_{\ell,h}^n$, $\Theta_{\ell,h}^n$ as in (120), and let $\widetilde{S}_\ell^h$ and $\widetilde{\Theta}_\ell^h$ be the function defined in terms of $S_{\ell,h}^n$ and $\Theta_{\ell,h}^n$ as in (121). Then*

$$\{S_\ell^h\}_{h>0} \text{ is uniformly bounded in } L^2(0,T;H^1(\Omega_\ell)); \tag{123}$$

$$\{\mathsf{p}_w^h\}_{h>0} \text{ is uniformly bounded in } L^2(0,T;H^1_{\text{inj}}(\Omega)); \tag{124}$$

$$\{\mathsf{p}_g^h\}_{h>0} \text{ is uniformly bounded in } L^2(0,T;H^1_{\text{inj}}(\Omega)); \tag{125}$$

$$\{\Theta_\ell^h\}_{h>0} \text{ is uniformly bounded in } L^2(0,T;H^1(\Omega_\ell)); \tag{126}$$
$$\{\Phi\,\partial_t\,\widetilde{\Theta}_\ell^h\}_{h>0} \text{ is uniformly bounded in } L^2(0,T;H^{-1}(\Omega_\ell)); \tag{127}$$
$$\{\Phi\,\partial_t\,\widetilde{S}_\ell^h\}_{h>0} \text{ is uniformly bounded in } L^2(0,T;H^{-1}(\Omega_\ell)). \tag{128}$$

Proof. First, it is easy to calculate that

$$\int\limits_{\Omega_T} \lambda_w(x,\mathsf{S}^h)\,\left|\nabla \mathsf{p}_w^h\right|^2\,dx\,dt = h\sum_{n=0}^{N-1}\int\limits_{\Omega}\lambda_w(x,\mathsf{S}_h^{n+1})\,|\nabla \mathsf{p}_{w,h}^{n+1}|^2\,dx; \tag{129}$$

$$\int\limits_{\Omega_T} \lambda_g(x,\mathsf{S}^h)\,\left|\nabla \mathsf{p}_g^h\right|^2\,dx\,dt = h\sum_{n=0}^{N-1}\int\limits_{\Omega}\lambda_g(x,\mathsf{S}_h^{n+1})\,\left|\nabla \mathsf{p}_{g,h}^{n+1}\right|^2\,dx; \tag{130}$$

$$\int\limits_{\Omega_{\ell,T}} |\nabla P_{\ell,c}(S_\ell^h)|^2\,dx\,dt = h\sum_{n=0}^{N-1}\int\limits_{\Omega_\ell}|\nabla P_{\ell,c}(S_{\ell,h}^{n+1})|^2\,dx. \tag{131}$$

Then from the inequality (119) we get:

$$\begin{aligned}
&\int\limits_{\Omega}\Phi(x)\mathcal{R}(\mathsf{p}_g^h(T))(1-\mathsf{S}^h(T))\,dx \\
&+\int\limits_{\Omega_T}\left\{\lambda_w(x,\mathsf{S}^h)\left|\nabla \mathsf{p}_w^h\right|^2+\lambda_g(x,\mathsf{S}^h)\left|\nabla \mathsf{p}_g^h\right|^2\right\}dx\,dt \\
&+\eta\sum_{\ell=1}^{2}\int\limits_{\Omega_{\ell,T}}|\nabla P_{\ell,c}(S_\ell^h)|^2dxdt \\
&\qquad\le C+\int\limits_{\Omega}\Phi(x)\mathcal{R}(\mathsf{p}_g^h(0))\mathsf{S}^h(0)dx+\sum_{\ell=1}^{2}\int\limits_{\Omega_\ell}\left|\mathcal{F}_\ell(S_\ell^h(0))-\mathcal{F}_\ell(S_\ell^h(T))\right|dx,
\end{aligned} \tag{132}$$

where C is a constant that does not depend on h. It follows from Lemma 3.1 that $\mathcal{R}_\ell \ge 0$. Then the first term in this inequality is positive and we obtain the following uniform estimate:

$$\begin{aligned}
&\int\limits_{\Omega_T}\left\{\lambda_w(x,\mathsf{S}^h)\left|\nabla \mathsf{p}_w^h\right|^2+\lambda_g(x,\mathsf{S}^h)\left|\nabla \mathsf{p}_g^h\right|^2\right\}dx\,dt \\
&\qquad\qquad+\eta\sum_{\ell=1}^{2}\int\limits_{\Omega_{\ell,T}}|\nabla P_{\ell,c}(S_\ell^h)|^2\,dx\,dt\le C,
\end{aligned} \tag{133}$$

where C is a constant that does not depend on h. Taking into account the equality (32), from (133) we get:

$$\sum_{\ell=1}^{2}\int\limits_{\Omega_{\ell,T}}\lambda_\ell(S_\ell^h)|\mathsf{P}_\ell^h|^2\,dx\,dt+\eta\sum_{\ell=1}^{2}\int\limits_{\Omega_{\ell,T}}|\nabla P_{\ell,c}(S_\ell^h)|^2\,dx\,dt\le C, \tag{134}$$

where C is a constant that does not depend on h.

Now we derive the uniform estimates (123) for the saturation S_ℓ^h. Since the gradient of the capillary pressure is uniformly bounded in h, then it follows from the condition **(A.4)** that

$$\int\limits_{\Omega_{\ell,T}}|\nabla S_\ell^h|^2\,dx\,dt\le C, \tag{135}$$

where $C = C(\eta)$ is a constant that does not depend on h. Thus the uniform boundedness of the sequence $\{S_\ell^h\}_{h>0}$ in the space $L^2(0,T;H^1(\Omega_\ell))$ is established.

Remark 4. We also notice that the sequence $\{S_{1,g}^h\}_{h>0}$ is uniformly bounded in $L^2(0,T;H^1_{\rm inj}(\Omega_1))$, where $S_{1,g}^h = 1 - S_1^h$. In fact,

$$p_{1,g}^h(x,t) - p_{1,w}^h(x,t) = P_{1,c}(S_1^h) = 0 \quad \text{on } \Gamma_{\rm inj} \times (0,T).$$

Then from the condition **(A.4)** we obtain that $S_{1,g}^h = 0$ on $\Gamma_{\rm inj} \times (0,T)$. Along with (135) this gives the uniform boundedness of the sequence $\{S_{1,g}^h\}_{h>0}$ in the space $L^2(0,T;H^1_{\rm inj}(\Omega_1))$.

Let us prove now the uniform bounds (124), (125). To this end we recall that

$$\nabla p_{\ell,w}^h = \nabla \mathsf{P}_\ell^h - \frac{\lambda_{\ell,g}(S_\ell^h)}{\lambda_\ell(S_\ell^h)} \nabla P_{\ell,c}(S_\ell^h) \quad \text{and} \quad \nabla p_{\ell,g}^h = \nabla \mathsf{P}_\ell^h + \frac{\lambda_{\ell,w}(S_\ell^h)}{\lambda_\ell(S_\ell^h)} \nabla P_{\ell,c}(S_\ell^h).$$

Then (124), (125) are the consequence of (134) and the condition **(A.5)** on $\lambda_{\ell,w}$, $\lambda_{\ell,g}$, λ_ℓ.

Consider (126). The gradient of the function Θ_ℓ^h reads:

$$\nabla \Theta_\ell^h = \sum_{n=0}^{N-1} \left[\varrho_g'(p_{\ell,g,h}^{n+1})\,(1 - S_{\ell,h}^{n+1})\,\nabla p_{\ell,g,h}^{n+1} - \varrho_g(p_{\ell,g,h}^{n+1})\,\nabla S_{\ell,h}^{n+1} \right] \mathbf{I}_{[nh,(n+1)h]}(t). \tag{136}$$

Then the uniform bound (126) is a consequence of the previous uniform estimates (123), (125) since ϱ_g is a C^1–function in $\mathbb{R}$.

Finally, the uniform estimates (127), (128) follow directly from the weak formulation of the problem and the previous uniform estimates. Lemma 6.3 is proved. $\square$

Our goal is to construct a solution to the evolution problem (104)–(107) by passing to the limit, as $h \to 0$, in the above elliptic problem.

Lemma 6.4 (*Convergence results with respect to h*)**.** *Up to a subsequence, the following convergence results hold as $h \to 0$:*

$$\| S_\ell^h - \widetilde{S}_\ell^h \|^2_{L^2(\Omega_{\ell,T})} \longrightarrow 0; \tag{137}$$

$$\| \Theta_\ell^h - \widetilde{\Theta}_\ell^h \|^2_{L^2(\Omega_{\ell,T})} \longrightarrow 0; \tag{138}$$

$$\mathsf{p}_w^h \longrightarrow \mathsf{p}_w^\eta \text{ weakly in } L^2(0,T;H^1_{\rm inj}(\Omega)); \tag{139}$$

$$\mathsf{p}_g^h \longrightarrow \mathsf{p}_g^\eta \text{ weakly in } L^2(0,T;H^1_{\rm inj}(\Omega)); \tag{140}$$

$$S_\ell^h \longrightarrow S_\ell^\eta \text{ weakly in } L^2(0,T;H^1(\Omega_\ell)) \text{ and a.e. in } \Omega_{\ell,T}; \tag{141}$$

$$\Theta_\ell^h \longrightarrow \Theta_\ell^\eta \text{ strongly in } L^2(\Omega_{\ell,T}) \text{ with } \Theta_\ell^\eta = \varrho_g(p_{\ell,g}^\eta)\,(1 - S_\ell^\eta) \text{ a.e. in } \Omega_{\ell,T}; \tag{142}$$

$$\big(\varrho_g(p_{\ell,g}^h)\big)^k \lambda_{\ell,g}(S_\ell^h) \longrightarrow \Big(\varrho_g(p_{\ell,g}^\eta)\Big)^k \lambda_{\ell,g}(S_\ell^\eta) \quad \text{a.e. in } \Omega_T \qquad (k = 1,2); \tag{143}$$

$$\partial_t(\Phi\,\widetilde{\Theta}_\ell^h) \longrightarrow \partial_t(\Phi\,\Theta_\ell^\eta) \text{ weakly in } L^2(0,T;H^{-1}(\Omega_\ell)); \tag{144}$$

$$\partial_t(\Phi\,\widetilde{S}_\ell^h) \longrightarrow \partial_t(\Phi\,S_\ell^\eta) \text{ weakly in } L^2(0,T;H^{-1}(\Omega_\ell)). \tag{145}$$

Moreover,

$$0 \le \mathsf{S}^\eta \le 1 \quad \text{a.e. in } \Omega_T. \tag{146}$$

Proof. Let us prove the convergence result (137). To this end, consider the equation (115), i.e.,

$$\begin{aligned}&\Phi(x)\Delta_h^n S_{\ell,h}^{n+1} - \mathrm{div}\left(K(x)\lambda_{\ell,w}(S_{\ell,h}^{n+1})\left(\nabla p_{\ell,w,h}^{n+1} - \vec{g}\right)\right)\\ &\qquad +\mathrm{div}\left(\eta\nabla\left(p_{\ell,w,h}^{n+1} - p_{\ell,g,h}^{n+1}\right)\right) = 0 \quad \text{in } \Omega_\ell.\end{aligned} \tag{147}$$

Let $x \in \Omega_\ell$ and let $\mathrm{dist}\,(x, \partial\Omega_\ell)$ denote the distance between x and the boundary of the domain Ω_ℓ which is denoted by $\partial\Omega_\ell$. Then we introduce a sufficiently smooth cut–off function ζ_ℓ defined by:

$$\zeta_\ell(x) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } \mathrm{dist}\,(x, \partial\Omega_\ell) \geq 2\,h^{1/4};\\ 0 & \text{if } \mathrm{dist}\,(x, \partial\Omega_\ell) \leq h^{1/4}.\end{cases} \tag{148}$$

Notice that it follows from the definition of the function ζ_ℓ that $0 \leq \zeta_\ell \leq 1$ and $|\nabla\zeta_\ell| \leq C\,h^{-1/4}$. Now multiplying the equation (147) by $\mathfrak{z}_{\ell,h}^{n+1}$,

$$\begin{aligned}\mathfrak{z}_{\ell,h}^{n+1} &\overset{\text{def}}{=} \left(S_{\ell,h}^{n+1} - S_{\ell,h}^n\right)\zeta_\ell \text{ with } \nabla\mathfrak{z}_{\ell,h}^{n+1}\\ &= \nabla\left(S_{\ell,h}^{n+1} - S_{\ell,h}^n\right)\zeta_\ell + \left(S_{\ell,h}^{n+1} - S_{\ell,h}^n\right)\nabla\zeta_\ell,\end{aligned} \tag{149}$$

we get:

$$\begin{aligned}&\frac{1}{h}\int_{\Omega_\ell}\Phi(x)\,\zeta_\ell(x)\,\left|S_{\ell,h}^{n+1} - S_{\ell,h}^n\right|^2 dx\\ &\qquad = \int_{\Omega_\ell} K(x)\lambda_{\ell,w}(S_{\ell,h}^{n+1})\nabla p_{\ell,w,h}^{n+1}\cdot\nabla\mathfrak{z}_{\ell,h}^{n+1}\,dx\\ &\qquad - \int_{\Omega_\ell} K(x)\lambda_{\ell,w}(S_{\ell,h}^{n+1})\vec{g}\cdot\nabla\mathfrak{z}_{\ell,h}^{n+1}\,dx\\ &\qquad +\eta\int_{\Omega_\ell}\nabla\left(p_{\ell,w,h}^{n+1} - p_{\ell,g,h}^{n+1}\right)\cdot\nabla\mathfrak{z}_{\ell,h}^{n+1}\,dx.\end{aligned} \tag{150}$$

Now taking into account (149) we obtain:

$$\begin{aligned}&\frac{1}{h}\int_{\Omega_\ell}\zeta_\ell(x)\,\left|S_{\ell,h}^{n+1} - S_{\ell,h}^n\right|^2 dx \leq C(\eta)\left(\|\nabla p_{\ell,w,h}^{n+1}\|^2_{L^2(\Omega_\ell)}\right.\\ &\left.+\|\nabla p_{\ell,g,h}^{n+1}\|^2_{L^2(\Omega_\ell)} + \|\nabla S_{\ell,h}^{n+1}\|^2_{L^2(\Omega_\ell)} + \|\nabla S_{\ell,h}^{n}\|^2_{L^2(\Omega_\ell)} + \|\nabla\zeta_\ell\|^2_{L^2(\Omega_\ell)}\right).\end{aligned} \tag{151}$$

The inequality (151) implies the following bound:

$$\begin{aligned}&\sum_{n=0}^{N-1}\left\|\sqrt{\zeta_\ell}\,(S_{\ell,h}^{n+1} - S_{\ell,h}^n)\right\|^2_{L^2(\Omega_\ell)} \leq C_1(\eta)\left(\|\nabla p_{\ell,w}^h\|^2_{L^2(\Omega_{\ell,T})}\right.\\ &\qquad\left.+\|\nabla p_{\ell,g}^h\|^2_{L^2(\Omega_{\ell,T})} + \|\nabla S_\ell^h\|^2_{L^2(\Omega_{\ell,T})}\right) + C_2(\eta)\,h^{-1/2}.\end{aligned} \tag{152}$$

Now it follows from the definitions of the functions $S_\ell^h, \widetilde{S}_\ell^h$ that

$$\begin{aligned}
&\|\sqrt{\zeta_\ell}\,(S_\ell^h - \widetilde{S}_\ell^h)\|^2_{L^2(\Omega_{\ell,T})} \\
&\quad = \sum_{n=0}^{N-1} \int_{nh}^{(n+1)h} \Big\|\sqrt{\zeta_\ell}\Big(1+n-\frac{t}{h}\Big)(S_{\ell,h}^{n+1} - S_{\ell,h}^n)\Big\|^2_{L^2(\Omega_\ell)}\,dt \\
&\quad = \frac{h}{3}\sum_{n=0}^{N-1}\Big\|\sqrt{\zeta_\ell}\,(S_{\ell,h}^{n+1} - S_{\ell,h}^n)\Big\|^2_{L^2(\Omega_\ell)}.
\end{aligned} \tag{153}$$

Taking into account the boundedness of the functions $S_\ell^h, \widetilde{S}_\ell^h$ and the definition of the cut–off function ζ_ℓ, it is easy to see that inequalities (152), (153) imply (137). In a similar way we get the convergence result (138).

The convergence results (139)–(140) immediately follow from (123)–(125).

Consider now the sequences $\{\widetilde{S}_\ell^h\}_{h>0}$, $\{\widetilde{\Theta}_\ell^h\}_{h>0}$. Following the lines of the proof of Propositions 1, 2 we observe that

$$\widetilde{S}_\ell^h \longrightarrow \widehat{S}_\ell \quad \text{strongly in } \Omega_{\ell,T}; \tag{154}$$

$$\widetilde{\Theta}_\ell^h \longrightarrow \widehat{\Theta}_\ell \quad \text{strongly in } \Omega_{\ell,T}; \tag{155}$$

The identification of the limit function $\widehat{S}_\ell$ relies on the previous convergence results. The identification of the limit function $\widehat{\Theta}_\ell$ relies on the standard monotonicity arguments and the convergence results (140)–(141). Thus, we have that $\widehat{S}_\ell = S_\ell^\eta$ and $\widehat{\Theta}_\ell = \Theta_\ell^\eta = \varrho_g(p_{\ell,g}^\eta)\,(1 - S_\ell^\eta)$.

Now we turn to (143). It is clear that when $S_\ell^h \to 1$

$$\big(\varrho_g(p_{\ell,g}^h)\big)^k\,\lambda_{\ell,g}(S_\ell^h) \longrightarrow 0 = \Big(\varrho_g(p_{\ell,g}^\eta)\Big)^k\,\lambda_{\ell,g}(S_\ell^\eta) \quad \text{a.e. in } \Omega_{\ell,T}.$$

If $S_\ell^h \to S_\ell^\eta \neq 1$ then from (155) we have:

$$\big(\varrho_g(p_{\ell,g}^h)\big)^k \longrightarrow \Big(\varrho_g(p_{\ell,g}^\eta)\Big)^k \quad \text{a.e. in } \Omega_{\ell,T}.$$

Now taking into account the regularity properties of the function ϱ_g, we obtain (143).

The convergence results (144), (145) follow from the uniform estimates (127), (128).

Finally, we remark that the maximum principle is conserved through the limit process and (146) is established. This completes the proof of Lemma 6.4. □

6.1.2. *End of the proof of Theorem 6.1.* Now we are in position to complete the proof of Theorem 6.1. First, we observe that in view of definitions the functions S^h, p_w^h, p_g^h, $\widetilde{\mathsf{S}}^h$, and $\widetilde{\Theta}^h$, from system (115)–(116) we obtain the following system of equations:

$$\begin{cases}
\Phi\frac{\partial \widetilde{S}_\ell^h}{\partial t} - \operatorname{div}\Big(K\lambda_{\ell,w}(S_\ell^h)\Big(\nabla p_{\ell,w}^h - \vec{g}\Big) + \eta\,\nabla(p_{\ell,w}^h - p_{\ell,g}^h)\Big) = 0; \\
\Phi\frac{\partial \widetilde{\Theta}_\ell^h}{\partial t} - \operatorname{div}\Big(K\lambda_{\ell,g}(S_\ell^h)\varrho_{\ell,g}^\eta\Big(\nabla p_{\ell,g}^h - \varrho_{\ell,g}^h\vec{g}\Big) + \eta\,\varrho_{\ell,g}^h\nabla(p_{\ell,g}^h - p_{\ell,w}^h)\Big) = 0,
\end{cases} \tag{156}$$

Now considering the weak formulation of the system (156) and taking into account Lemma 6.4, we pass to the limit as $h \to 0$ and obtain (104)–(107). The fulfillment of the interface and boundary conditions on $\Gamma_{\rm inj}$ can be checked as in the previous sections. This ends the proof of Theorem 6.1.

7. Proof of the main result: The degenerate system.

The goal of this section is to prove the main result of this work, i.e., Theorem 2.1. The proof is based on Theorem 6.1 established in the previous section and the compactness results from Propositions 1, 2.

The outline of the proof is as follows. First, in Section 7.1 we establish the uniform estimates for the solutions to system (104)–(107) and obtain the corresponding compactness results with respect to the parameter η. Then in Section 7.2 we complete the proof of Theorem 2.1.

7.1. Uniform estimates and compactness results.

The *a priori* estimates for the solutions of problem (104)–(107) are given by the following lemma.

Lemma 7.1. *The sequences* $\{S_\ell^\eta\}_{\eta>0}$, $\{\mathsf{p}_w^\eta\}_{\eta>0}$, $\{\mathsf{p}_g^\eta\}_{\eta>0}$, $\{\mathsf{P}_\ell^\eta\}_{\eta>0}$ *are such that*

$$0 \le S_\ell^\eta \le 1 \ \text{ a.e. in } \ \Omega_{\ell,T}; \tag{157}$$

$$\{\mathsf{P}_1^\eta\}_{\eta>0} \quad \text{is uniformly bounded in } L^2(0,T;H^1_{\Gamma_{\rm inj}}(\Omega_1)); \tag{158}$$

$$\{\mathsf{P}_2^\eta\}_{\eta>0} \quad \text{is uniformly bounded in } L^2(0,T;H^1(\Omega_2)); \tag{159}$$

$$\{\sqrt{\eta}\ \nabla P_{\ell,c}(S_\ell^\eta)\}_{\eta>0} \quad \text{is uniformly bounded in } L^2(\Omega_{\ell,T}); \tag{160}$$

$$\left\{\sqrt{\lambda_w(x,\mathsf{S}^\eta)}\,\nabla \mathsf{p}_w^\eta\right\}_{\eta>0} \quad \text{is uniformly bounded in } L^2(\Omega_T); \tag{161}$$

$$\left\{\sqrt{\lambda_g(x,\mathsf{S}^\eta)}\,\nabla \mathsf{p}_g^\eta\right\}_{\eta>0} \quad \text{is uniformly bounded in } L^2(\Omega_T); \tag{162}$$

$$\{\beta_\ell(S_\ell^\eta)\}_{\eta>0} \quad \text{is uniformly bounded in } L^2(0,T;H^1(\Omega_\ell)); \tag{163}$$

$$\{\mathfrak{b}_\ell(S_\ell^\eta)\}_{\eta>0} \quad \text{is uniformly bounded in } L^2(0,T;H^1(\Omega_\ell)); \tag{164}$$

$$\{(\Phi\Theta_\ell^\eta)_t\}_{\eta>0} \quad \text{is uniformly bounded in } L^2(0,T;H^{-1}(\Omega_\ell)); \tag{165}$$

$$\{(\Phi S_\ell^\eta)_t\}_{\eta>0} \quad \text{is uniformly bounded in } L^2(0,T;H^{-1}(\Omega_\ell)). \tag{166}$$

Here the function $\mathfrak{b}_\ell$ *is defined in (33).*

The proof of Lemma 7.1 is done by arguments similar to ones used in the proof of Lemma 3.1–Corollary 1. The only result that have to be discussed is the boundedness of the sequence $\{\mathsf{P}_2^\eta\}_{\eta>0}$ in the space $L^2(0,T;H^1(\Omega_2))$. By arguments similar to those used in the proof of Corollary 1 we prove that $\nabla\mathsf{P}_2^\eta$ is uniformly bounded with respect to η in $L^2(\Omega_{2,T})$. In contrast to the function P_1^η we cannot use Friedrichs' inequality for P_2^η. Therefore, we proceed in another way. The global pressure P_2^η is defined in (24), (27) up to an additive constant. Then we choose this constant in such a way that the mean value of P_2^η equals zero in Ω_2. Now applying Poincaré–Wirtinger's inequality we obtain the desired boundedness of P_2^η in $L^2(\Omega_{2,T})$. Lemma 7.1 is proved.

Now from Lemma 7.1 and Propositions 1, 2 we deduce all the convergence results required for the passage to the limit as $\eta \to 0$ in (104)–(107).

Lemma 7.2. *The sequences* $\{S_\ell^\eta\}_{\eta>0}$, $\{\mathsf{p}_w^\eta\}_{\eta>0}$, $\{\mathsf{p}_g^\eta\}_{\eta>0}$, $\{\mathsf{P}_\ell^\eta\}_{\eta>0}$ *are such that up to a subsequence,*

$$S_\ell^\eta \longrightarrow S_\ell \quad \text{strongly in } L^2(\Omega_{\ell,T}) \text{ and a.e. in } \Omega_{\ell,T}; \tag{167}$$

$$0 \le S_\ell \le 1 \quad \text{a.e. in } \Omega_{\ell,T}; \tag{168}$$

$$\mathsf{P}_1^\eta \longrightarrow \mathsf{P}_1 \quad \text{weakly in } L^2(0,T; H^1_{\Gamma_{\rm inj}}(\Omega_1)); \tag{169}$$

$$\mathsf{P}_2^\eta \longrightarrow \mathsf{P}_2 \quad \text{weakly in } L^2(0,T; H^1(\Omega_2)); \tag{170}$$

$$\beta_\ell(S_\ell^\eta) \longrightarrow \beta_\ell(S_\ell) \quad \text{weakly in } L^2(0,T; H^1(\Omega_\ell)). \tag{171}$$

$$\Big(\varrho_g(p_{\ell,g}^\eta)\Big)^k \lambda_{\ell,g}(S_\ell^\eta) \longrightarrow \left(\varrho_g(p_{\ell,g})\right)^k \lambda_{\ell,g}(S_\ell) \quad \text{a.e. in } \Omega_T \qquad (k=1,2); \tag{172}$$

$$\Theta_\ell^\eta \longrightarrow \Theta_\ell \quad \text{strongly in } L^2(\Omega_{\ell,T}) \ \text{ and a.e. in } \Omega_{\ell,T} \atop \text{with } \Theta_\ell = \varrho_g(p_{\ell,g})\,(1-S_\ell); \tag{173}$$

$$\partial_t(\Phi\Theta_\ell^\eta) \longrightarrow \partial_t(\Phi\Theta_\ell) \quad \text{weakly in } L^2(0,T; H^{-1}(\Omega_\ell)); \tag{174}$$

$$\partial_t(\Phi S_\ell^\eta) \longrightarrow \partial_t(\Phi S_\ell) \quad \text{weakly in } L^2(0,T; H^{-1}(\Omega_\ell)). \tag{175}$$

The proof of Lemma 7.2 can be done by arguments similar to ones used in the proof of Lemma 6.4 from the previous section.

Now we are in position to complete the proof of Theorem 2.1.

7.2. **End of the proof of Theorem 2.1.** We have to pass to the limit, as $\eta \to 0$, in the weak formulation of problem (104)–(107):

$$\begin{aligned} -\int_{\Omega_T} \Phi(x)\mathsf{S}^\eta \frac{\partial \varphi_w}{\partial t}\, dx\, dt - \int_\Omega \Phi(x)\mathsf{S}^{\mathbf{0}}(x)\varphi_w(x,0)\, dx \\ + \int_{\Omega_T} K(x)\lambda_w(x,\mathsf{S}^\eta)\nabla \mathsf{p}_w^\eta \cdot \nabla\varphi_w\, dx\, dt \\ - \int_{\Omega_T} K(x)\lambda_w(x,\mathsf{S}^\eta)\vec{g}\cdot\nabla\varphi_w\, dx\, dt + \eta\int_{\Omega_T} \nabla(\mathsf{p}_w^\eta - \mathsf{p}_g^\eta)\cdot\nabla\varphi_w\, dx\, dt = 0; \end{aligned} \tag{176}$$

$$\begin{aligned} -\int_{\Omega_T} \Phi(x)\Theta^\eta \frac{\partial \varphi_g}{\partial t}\, dx\, dt - \int_\Omega \Phi(x)\Theta^{\mathbf{0}}(x)\varphi_g(x,0)\, dx \\ + \int_{\Omega_T} K(x)\lambda_g(x,\mathsf{S}^\eta)\varrho_g(\mathsf{p}_g^\eta)\nabla \mathsf{p}_g^\eta \cdot \nabla\varphi_g\, dx\, dt \\ - \int_{\Omega_T} K(x)\lambda_g(x,\mathsf{S}^\eta)\left[\varrho_g(\mathsf{p}_g^\eta)\right]^2\vec{g}\cdot\nabla\varphi_g\, dx\, dt \\ + \eta\int_{\Omega_T} \varrho_g(\mathsf{p}_g^\eta)\nabla(\mathsf{p}_g^\eta - \mathsf{p}_w^\eta)\cdot\nabla\varphi_g\, dx\, dt = 0. \end{aligned} \tag{177}$$

Consider the equation (176). The first term converges to the desired limit due to (167). To study the convergence of the third term on the left-hand side of (176), we use the relation (36) to rewrite it in terms of the global pressure. We have:

$$\begin{aligned}&\int\limits_{\Omega_T} K(x)\lambda_w(x,\mathsf{S}^\eta)\nabla \mathsf{p}_w^\eta\cdot\nabla\varphi_w\;dx\,dt\\&=\sum_{\ell=1}^{2}\int\limits_{\Omega_{\ell,T}} K(x)\left\{\lambda_{\ell,w}(S_\ell^\eta)\nabla\mathsf{P}_\ell^\eta+\nabla\beta_\ell(S_\ell^\eta)\right\}\cdot\nabla\varphi_w\,dx\,dt.\end{aligned}\tag{178}$$

Taking into account the almost everywhere convergence of the sequence $\{S_\ell^\eta\}_{\eta>0}$ we have that

$$\lambda_{\ell,w}(S_\ell^\eta)\nabla\varphi_w\longrightarrow\lambda_{\ell,w}(S_\ell)\nabla\varphi_w\quad\text{strongly in }\left(L^2(\Omega_{\ell,T})\right)^d.\tag{179}$$

Then we exploit the weak convergence results for the global pressure, the function β_ℓ, and (179). Passing to the limit in the third term in the left-hand side of (176) and taking into account the definition of the global pressure, we get:

$$\begin{aligned}&\lim_{\eta\to0}\int\limits_{\Omega_T} K(x)\lambda_w(x,\mathsf{S}^\eta)\nabla \mathsf{p}_w^\eta\cdot\nabla\varphi_w\;dx\,dt\\&=\sum_{\ell=1}^{2}\int\limits_{\Omega_{\ell,T}} K(x)\left\{\lambda_{\ell,w}(S_\ell)\nabla\mathsf{P}_\ell+\nabla\beta_\ell(S_\ell)\right\}\cdot\nabla\varphi_w\,dx\,dt.\end{aligned}$$

The lower order terms in (176) converge due to (179).

In a similar way taking into account the convergence result (172) we pass to the limit in the third and fourth terms of equation (177).

Finally, consider the last term in the left–hand side of (176). We rewrite it as follows:

$$\eta\int\limits_{\Omega_T}\nabla(\mathsf{p}_w^\eta-\mathsf{p}_g^\eta)\cdot\nabla\varphi_w\;dx\,dt=\sqrt{\eta}\sum_{\ell=1}^{2}\int\limits_{\Omega_{\ell,T}}\left(\sqrt{\eta}\;\nabla P_{\ell,c}(S_\ell^\eta)\right)\nabla\varphi_w\;dx\,dt.$$

Then from Cauchy's inequality and the uniform estimate (160) we obtain that this term goes to zero as $\eta\to0$. The convergence of the last term on the right–hand side of (177) is studied in a similar way.

This yields the existence of $\langle\mathsf{P}_\ell,S_\ell\rangle$ such that for any $\varphi_w,\varphi_g\in C^1([0,T];H^1(\Omega))$ satisfying $\varphi_w=\varphi_g=0$ on $\Gamma_{\text{inj}}\times(0,T)$ and $\varphi_w(x,T)=\varphi_g(x,T)=0$, we have:

$$\begin{aligned}&-\int\limits_{\Omega_T}\Phi(x)\mathsf{S}\frac{\partial\varphi_w}{\partial t}\;dx\,dt-\int\limits_{\Omega}\Phi(x)\mathsf{S}^{\mathbf{0}}(x)\varphi_w(0,x)\;dx\\&\quad+\sum_{\ell=1}^{2}\int\limits_{\Omega_{\ell,T}}K(x)\lambda_{\ell,w}(S)\nabla\mathsf{P}_\ell\cdot\nabla\varphi_w\,dx\,dt\\&\quad+\sum_{\ell=1}^{2}\int\limits_{\Omega_{\ell,T}}K(x)\nabla\beta_\ell(S_\ell)\cdot\nabla\varphi_w\;dx\,dt\\&\quad-\sum_{\ell=1}^{2}\int\limits_{\Omega_{\ell,T}}K(x)\lambda_{\ell,w}(S)\vec{g}\cdot\nabla\varphi_w\;dx\,dt=0;\end{aligned}\tag{180}$$

$$
\begin{aligned}
-\int\limits_{\Omega_T} \Phi(x)\,\Theta\,\frac{\partial\varphi_g}{\partial t}\;dx\,dt - \int\limits_{\Omega} \Phi(x)\Theta(x,0)\varphi_g(0,x)\;dx \\
+\sum_{\ell=1}^{2}\int\limits_{\Omega_{\ell,T}} K(x)\lambda_{\ell,g}(S)\widetilde{\varrho}_{\ell,g}\nabla \mathsf{P}_\ell\cdot\nabla\varphi_g\;dx\,dt \\
-\sum_{\ell=1}^{2}\int\limits_{\Omega_{\ell,T}} K(x)\,\widetilde{\varrho}_{\ell,g}\,\nabla\beta_\ell(S_\ell)\cdot\nabla\varphi_g\,dx\,dt \\
-\sum_{\ell=1}^{2}\int\limits_{\Omega_{\ell,T}} K(x)\lambda_{\ell,g}(S)\,[\widetilde{\varrho}_{\ell,g}]^2\,\vec{g}\cdot\nabla\varphi_g\,dx\,dt = 0,
\end{aligned}
\tag{181}
$$

where $\widetilde{\varrho}_{\ell,g} \stackrel{\text{def}}{=} \varrho_g(\mathsf{P}_\ell + \mathsf{G}_{\ell,g}(S_\ell))$.

Now taking into account the lower semi–continuity of the norm, by Lemma 7.1, we obtain:

$$
\int\limits_{\Omega_{\ell,T}} |\nabla \mathsf{P}_\ell|^2\;dx\,dt \le \varliminf_{\eta\to 0}\int\limits_{\Omega_{\ell,T}} |\nabla \mathsf{P}_\ell^\eta|^2\;dx\,dt \le C; \tag{182}
$$

$$
\int\limits_{\Omega_{\ell,T}} |\nabla\beta_\ell(S_\ell)|^2\;dx\,dt \le \varliminf_{\eta\to 0}\int\limits_{\Omega_{\ell,T}} |\nabla\beta_\ell(S_\ell^\eta)|^2\;dx\,dt \le C; \tag{183}
$$

$$
\int\limits_{\Omega_{\ell,T}} |\nabla\mathfrak{b}_\ell(S_\ell)|^2\;dx\,dt \le \varliminf_{\eta\to 0}\int\limits_{\Omega_{\ell,T}} |\nabla\mathfrak{b}_\ell(S_\ell^\eta)|^2\;dx\,dt \le C. \tag{184}
$$

Now we set:

$$
p_{\ell,w} \stackrel{\text{def}}{=} \mathsf{P}_\ell + \mathsf{G}_{\ell,w}(S_\ell) \quad\text{and}\quad p_{\ell,g} \stackrel{\text{def}}{=} \mathsf{P}_\ell + \mathsf{G}_{\ell,g}(S_\ell). \tag{185}
$$

We also recall the relation (32):

$$
\lambda_{\ell,g}(S_\ell)|\nabla p_{\ell,g}|^2 + \lambda_{\ell,w}(S_\ell)|\nabla p_{\ell,w}|^2 = \lambda_\ell(S_\ell)|\nabla\mathsf{P}_\ell|^2 + |\nabla\mathfrak{b}_\ell(S_\ell)|^2 .
$$

Then, taking into account (182), (184), and (32) we obtain that the functions $p_{\ell,w}, p_{\ell,g}$ defined in (185) are such that

$$
\int\limits_{\Omega_{\ell,T}} \left\{\lambda_{\ell,w}(S_\ell)\,|\nabla p_{\ell,w}|^2 + \lambda_{\ell,w}(S_\ell)\,|\nabla p_{\ell,w}|^2\right\}\,dx\,dt < +\infty. \tag{186}
$$

Now we rewrite the system (180), (181) in terms of the functions $p_{\ell,w}, p_{\ell,g}$. We have that, for any $\varphi_w, \varphi_g \in C^1([0,T]; H^1(\Omega))$ satisfying $\varphi_w = \varphi_g = 0$ on $\Gamma_{\text{inj}}\times(0,T)$ and $\varphi_w(x,T) = \varphi_g(x,T) = 0$,

$$
\begin{aligned}
-\int\limits_{\Omega_T} \Phi(x)\mathsf{S}\frac{\partial\varphi_w}{\partial t}\;dx\,dt - \int\limits_{\Omega} \Phi(x)\mathsf{S}^{\mathbf{0}}(x)\varphi_w(x,0)\;dx \\
+\sum_{\ell=1}^{2}\int\limits_{\Omega_{\ell,T}} K(x)\lambda_{\ell,w}(S_\ell)\nabla p_{\ell,w}\cdot\nabla\varphi_w\,dx\,dt \\
-\sum_{\ell=1}^{2}\int\limits_{\Omega_{\ell,T}} K(x)\lambda_{\ell,w}(S_\ell)\vec{g}\cdot\nabla\varphi_w\;dx\,dt = 0;
\end{aligned}
\tag{187}
$$

$$
\begin{aligned}
-\int\limits_{\Omega_T} \Phi(x)\Theta\frac{\partial \varphi_g}{\partial t}\, dx\, dt - \int\limits_{\Omega} \Phi(x)\Theta(x,0)\varphi_g(x,0)\, dx \\
+\sum_{\ell=1}^{2} \int\limits_{\Omega_{\ell,T}} \lambda_{\ell,g}(S_\ell)\varrho_{\ell,g}\nabla p_{\ell,g}\cdot\nabla\varphi_w\, dx\, dt \\
-\sum_{\ell=1}^{2} \int\limits_{\Omega_{\ell,T}} \lambda_{\ell,g}(S_\ell)\left[\varrho_{\ell,g}\right]^2 \vec{g}\cdot\nabla\varphi_w\, dx\, dt = 0.
\end{aligned}
\tag{188}
$$

In order to complete the proof of Theorem 2.1 we have to obtain the continuity of the phase pressures at the interface Σ_T as well as the boundary conditions on $\Gamma_{\rm inj}$. We start by obtaining of the phase pressures. The following result holds.

Lemma 7.3. *Let $\langle p_{\ell,w}, p_{\ell,g}\rangle$ be a solution to (187)–(188). Then*

$$
p_{1,w} = p_{2,w} \quad \text{and} \quad p_{1,g} = p_{2,g} \quad \text{on } \Sigma_T. \tag{189}
$$

Proof. The proof of the lemma is based on the definition (185) and the regularity properties of the functions $\mathsf{P}_\ell, \beta_\ell$. Let $\vec{\nu}$ be a unit exterior (with respect to the subdomain Ω_2) vector on Σ_T. Denote:

$$
\Sigma_T^\delta \stackrel{\rm def}{=} \delta\vec{\nu} + \Sigma_T \quad (\delta > 0).
$$

The regularity properties of the functions $\mathsf{P}_\ell, \beta_\ell$ assure the existence of their traces on Σ_T. Moreover, it follows from (182), (183) that

$$
\mathsf{P}_\ell\Big|_{\Sigma_T^\delta} \longrightarrow \mathsf{P}_\ell\Big|_{\Sigma_T} \quad \text{strongly in } L^2(\Sigma_T^\delta); \tag{190}
$$

$$
\beta_\ell(S_\ell)\Big|_{\Sigma_T^\delta} \longrightarrow \beta_\ell(S_\ell)\Big|_{\Sigma_T} \quad \text{strongly in } L^2(\Sigma_T^\delta); \tag{191}
$$

The last convergence result along with the condition **(A.4)** imply that

$$
S_\ell\Big|_{\Sigma_T^\delta} \longrightarrow S_\ell\Big|_{\Sigma_T} \quad \text{strongly in } L^2(\Sigma_T^\delta). \tag{192}
$$

The convergence result (192) and the boundedness of the functions $\mathsf{G}_{\ell,w}(S_\ell), \mathsf{G}_{\ell,g}(S_\ell)$ (see the definitions of these functions in Section 2.1) imply the existence of the traces for $\mathsf{G}_{\ell,w}(S_\ell), \mathsf{G}_{\ell,g}(S_\ell)$. Therefore, the traces of the functions $p_{\ell,w}, p_{\ell,g}$ are well defined.

Now taking into account that

$$
\int\limits_{\Omega_{\ell,T}} \left\{|\nabla \mathsf{P}_\ell^\eta|^2 + |\nabla\beta_\ell(S_\ell^\eta)|^2\right\} dx\, dt \le C,
$$

where C is a constant that does not depend on η, and considering the fact that

$$
p_{1,w}^\eta = p_{2,w}^\eta \quad \text{and} \quad p_{1,g}^\eta = p_{2,g}^\eta \quad \text{on } \Sigma_T,
$$

the definition (185), and (190), (192), we obtain the desired continuity of the phase pressures (189). Lemma 7.3 is proved. □

The Dirichlet condition on the corresponding part of the boundary, i.e.,

$$
p_{1,w} = p_{1,g} = 0 \quad \text{on } \Gamma_{\rm inj}
$$

can be proved by similar arguments.

Thus we can rewrite the system (187)–(188) in the whole domain Ω and obtain first two equations in (15). Passage to the limit in the last equation in (104) does not make any difficulty. This completes the proof of Theorem 2.1.

8. **Concluding remarks.** We have presented a week formulation and an existence result for a degenerate system modeling immiscible compressible two-phase flow through a porous medium made of several types of rocks. We have assumed that the porosity, the absolute permeability, the capillary and relative permeabilities curves are different in each type of porous media. This leads to nonlinear transmission conditions representing the continuity of some physical characteristics such as water and gas pressures, at the interfaces that separate different media. Then the saturation and some other characteristics are getting discontinuous at the interfaces. The study still needs to be improved in several areas such as the cases of unbounded capillary pressure and vanishing $\varrho_{\min}$. These more complicated cases appear in the applications. This study was intended as a first step to the homogenization of immiscible compressible two-phase flow through heterogeneous reservoirs with several rock types. Further work on these important issues is in progress.

Acknowledgments. The research leading to these results has received funding from the European Atomic Energy Community Seventh Framework Program (FP7, 2009-2012) under Grant Agreement n^0 230357, the FORGE project. This work was partially supported by the GnR MoMaS (PACEN/CNRS, ANDRA, BRGM,CEA, EDF, IRSN) France, their supports are gratefully acknowledged. Most of the work on this paper was done when L. Pankratov and A. Piatnitski were visiting the Applied Mathematics Laboratory of the University of Pau & CNRS. They are grateful for the invitations and the hospitality.

REFERENCES

[1] H. W. Alt and E. Di Benedetto, *Nonsteady flow of water and oil through inhomogeneous porous media*, Ann. Scu. Norm. Sup. Pisa Cl. Sci., **12** (1985), 335–392.

[2] H. W. Alt and S. Luckhaus, *Quasilinear elliptic–parabolic differential equations*, Math. Z., **3** (1983), 311–341.

[3] B. Amaziane, S. Antontsev, L. Pankratov and A. Piatnitski, *Homogenization of immiscible compressible two–phase flow in porous media: Application to gas migration in a nuclear waste repository*, SIAM J. Multiscale Model. Simul., **8** (2010), 2023–2047.

[4] B. Amaziane and M. Jurak, *A new formulation of immiscible compressible two-phase flow in porous media*, C. R. Mécanique, **336** (2008), 600–605.

[5] B. Amaziane, M. Jurak and A. Žgaljić–Keko, *Modeling and numerical simulations of immiscible compressible two-phase flow in porous media by the concept of global pressure*, Transport in Porous Media, **84** (2010), 133–152.

[6] B. Amaziane, M. Jurak and A. Žgaljić–Keko, *An existence result for a coupled system modeling a fully equivalent global pressure formulation for immiscible compressible two-phase flow in porous media*, J. Differential Equations, **250** (2011), 1685–1718.

[7] Y. Amirat, K. Hamdache and A. Ziani, *Mathematical analysis for compressible miscible displacement models in porous media*, Math. Models Methods Appl. Sci., **6** (1996), 729–747.

[8] Y. Amirat and M. Moussaoui, *Analysis of a one–dimensional model for compressible miscible displacement in porous media*, SIAM J. Math. Anal., **26** (1995), 659–674.

[9] Y. Amirat and V. Shelukhin, *Global weak solutions to equations of compressible miscible flows in porous media*, SIAM J. Math. Anal., **38** (2007), 1825–1846.

[10] ANDRA, Couplex-Gas Benchmark, 2006. Available online at: `http://www.gdrmomas.org/ex_qualifications.html`.

[11] S. N. Antontsev, A. V. Kazhikhov and V. N. Monakhov, "Boundary Value Problems in Mechanics of Nonhomogeneous Fluids," North-Holland, Amsterdam, 1990.

[12] T. Arbogast, *The existence of weak solutions to single-porosity and simple dual-porosity models of two-phase incompressible flow*, Nonlinear Anal., **19** (1992), 1009–1031.

[13] J. Bear and Y. Bachmat, "Introduction to Modeling of Transport Phenomena in Porous Media," Kluwer Academic Publishers, London, 1991.

[14] A. Bourgeat and A. Hidani, *A result of existence for a model of two-phase flow in a porous medium made of different rock types*, Appl. Anal., **56** (1995), 381–399.

[15] G. Chavent and J. Jaffré, "Mathematical Models and Finite Elements for Reservoir Simulation," North–Holland, Amsterdam, 1986.

[16] Z. Chen, *Degenerate two-phase incompressible flow I: Existence, uniqueness and regularity of a weak solution*, J. Differential Equations, **171** (2001), 203–232.

[17] Z. Chen and G. Huan and Y. Ma, "Computational Methods for Multiphase Flows in Porous Media," SIAM, Philadelphia, 2006.

[18] C. Choquet, *On a fully coupled nonlinear parabolic problem modelling miscible compressible displacement in porous media*, J. Math. Anal. Appl., **339** (2008), 1112–1133.

[19] F. Z. Daïm, R. Eymard and D. Hilhorst, *Existence of a solution for two phase flow in porous media: The case that the porosity depends on the pressure*, J. Math. Anal. Appl., **326** (2007), 332–351.

[20] B. K. Fadimba, *On existence and uniqueness for a coupled system modeling immiscible flow through a porous medium*, J. Math. Anal. Appl., **328** (2007), 1034–1056.

[21] G. Gagneux and M. Madaune-Tort, "Analyse Mathématique de Modèles Non Linéaires de l'Ingénierie Pétrolière," Springer-Verlag, Berlin, 1996.

[22] X. Feng, *Strong solutions to a nonlinear parabolic system modeling compressible miscible displacement in porous media*, Nonlinear Anal., **23** (1994), 1515–1531.

[23] C. Galusinski and M. Saad, *On a degenerate parabolic system for compressible, immiscible, two-phase flows in porous media*, Adv. Differential Equations, **9** (2004), 1235–1278.

[24] C. Galusinski and M. Saad, *Water–gas flow in porous media*, Discrete Contin. Dyn. Syst. Ser. B, **9** (2008), 281–308.

[25] C. Galusinski and M. Saad, *Two compressible immiscible fluids in porous media*, J. Differential Equations, **244** (2008), 1741–1783.

[26] C. Galusinski and M. Saad, *Weak solutions for immiscible compressible multifluid flows in porous media*, C. R. Acad. Sci. Paris, Sér. I, **347** (2009), 249–254.

[27] D. Gilbarg and N. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer–Verlag, 1983.

[28] R. Helmig, "Multiphase Flow and Transport Processes in the Subsurface," Springer, Berlin, 1997.

[29] Z. Khalil and M. Saad, *Solutions to a model for compressible immiscible two phase flow in porous media*, Electronic Journal of Differential Equations, **122** (2010), 1–33.

[30] Z. Khalil and M. Saad, *On a fully nonlinear degenerate parabolic system modeling immiscible gas-water displacement in porous media*, Nonlinear Analysis: Real World Applications, **12** (2011), 1591–1615.

[31] D. Kröner and S. Luckhaus, *Flow of oil and water in a porous medium*, J. Differential Equations, **55** (1984), 276–288.

[32] A. Mikelić, *An existence result for the equations describing a gas–liquid two–phase flow*, C. R. Mécanique, **337** (2009), 226–232.

[33] OECD/NEA, *Safety of geological disposal of high-level and long-lived radioactive waste in France*, An International Peer Review of the "Dossier 2005 Argile" Concerning Disposal in the Callovo-Oxfordian Formation. OECD Publishing (2006). Available online at: `http://www.nea.fr/html/rwm/reports/2006/nea6178-argile.pdf`.

[34] J. Simon, *Compact sets in the space $L^p(0,t;B)$*, Ann. Mat. Pura Appl., IV. Ser., **146** (1987), 65–96.

[35] F. Smaï, *A model of multiphase flow and transport in porous media applied to gas migration in underground nuclear waste repository*, C. R. Math. Acad. Sci. Paris, **347** (2009), 527–532.

[36] F. Smaï, *Existence of solutions for a model of multiphase flow in porous media applied to gas migration in underground nuclear waste repository*, Appl. Anal., **88** (2009), 1609–1616.

[37] L. M. Yeh, *On two-phase flow in fractured media*, Math. Models Methods Appl. Sci., **12** (2002), 1075–1107.

[38] L. M. Yeh, *Hölder continuity for two-phase flows in porous media*, Math. Methods Appl. Sci., **29** (2006), 1261–1289.

Received September 2011; revised December 2012.

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