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Nonlinear “double porosity” type model

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Abstract We consider a variational problem $\inf_{u\in H^1(\Omega)}\int_\Omega\{a^\varepsilon|\nabla u^\varepsilon|^m + g|u^\varepsilon|^m - mf^\varepsilon u^\varepsilon\}\,\mathrm{d}x$ in a bounded domain $\Omega = \mathcal{F}^{(\varepsilon)} \cup \overline{\mathcal{M}}^{(\varepsilon)}$ with a microstructure $\mathcal{F}^{(\varepsilon)}$ which is not in general periodic; $a^\varepsilon = a^\varepsilon(x)$ is of order 1 in $\mathcal{F}^{(\varepsilon)}$ and $\sup_{x\in\mathcal{M}^{(\varepsilon)}} a^\varepsilon(x) \to 0$ as $\varepsilon \to 0$. A homogenized model is constructed. ***To cite this article: L. Pankratov, A. Piatnitski, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 435–440.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Un modèle non linéaire de type double porosité

Résumé On étudie un problème variationnel $\inf_{u\in H^1(\Omega)}\int_\Omega\{a^\varepsilon|\nabla u^\varepsilon|^m + g|u^\varepsilon|^m - mf^\varepsilon u^\varepsilon\}\,\mathrm{d}x$ dans un ouvert borné $\Omega = \mathcal{F}^{(\varepsilon)} \cup \overline{\mathcal{M}}^{(\varepsilon)}$ avec une microstructure $\mathcal{F}^{(\varepsilon)}$ non périodique ; $a^\varepsilon = a^\varepsilon(x)$ vaut 1 dans $\mathcal{F}^{(\varepsilon)}$ et $\sup_{x\in\mathcal{M}^{(\varepsilon)}} a^\varepsilon(x) \to 0$ lorsque $\varepsilon \to 0$. Un modèle homogénéisé est construit. ***Pour citer cet article : L. Pankratov, A. Piatnitski, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 435–440.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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On considère le problème variationnel (1), où $a^\varepsilon(x)$ est une fonction mesurable telle que $a^\varepsilon(x)$ vaut 1 dans $\mathcal{F}^{(\varepsilon)}$ et $\sup_{x\in\mathcal{M}^{(\varepsilon)}} a^\varepsilon(x) \to 0$ lorsque $\varepsilon \to 0$. Dans le cas $m = 2$ ce problème peut modéliser la densité d'un fluide faiblement compressible s'écoulant dans un milieu poreux Ω composé de blocs (ou matrices) $\mathcal{M}^{(\varepsilon)}$ entourés de fissures $\mathcal{F}^{(\varepsilon)}$, $\Omega = \mathcal{F}^{(\varepsilon)} \cup \overline{\mathcal{M}}^{(\varepsilon)}$. Le domaine Ω est un milieu dispersé vérifiant les conditions (C.1), (C.2) avec des caractéristiques locales définies par (3), (4). Le résultat essentiel est le Théorème 1. *Soit* $u^\varepsilon \in \mathrm{W}^{1,m}(\Omega)$ *la solution unique de* (1) *avec* $f^\varepsilon \in \mathrm{L}^{m'}(\Omega)$, $m' = m/(m-1)$. *Si* f^ε *est nulle dans* $\mathcal{M}^{(\varepsilon)}$ *et converge vers* f *dans* $\mathrm{L}^{m'}(\mathcal{F}^{(\varepsilon)})$, *i.e.*, $\lim_{\varepsilon\to 0}\|f^\varepsilon - f\|_{m',\mathcal{F}^{(\varepsilon)}} = 0$, *alors* $\mathbf{P}_\varepsilon(u^\varepsilon\chi_{\mathcal{F}^{(\varepsilon)}})$ *converge vers* u *solution unique du problème variationnel* (5) *dans l'espace* $\mathrm{L}^m(\Omega)$. *De plus, la fonction* $u^\varepsilon|u^\varepsilon|^{m-2}\chi_{\mathcal{M}^{(\varepsilon)}}$ *converge faiblement vers* $\omega_0 = (b/g)u|u|^{m-2}$ *dans* $\mathrm{L}^{m'}(\Omega)$. La démonstration du Théorème 1 (*voir* § 4) se decompose en trois étapes ; elle utilise des résultats auxiliaires (*voir* § 3) :

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(1) On définit la fonction $w_h^\varepsilon(x)$ par (6) et on obtient (9). (2) Pour $\{u^\varepsilon\}$ suite de solutions du problème (1) qui converge vers une fonction u dans $\mathrm{L}^m(\mathcal{F}^{(\varepsilon)})$ on obtient (10). En raison de (9), (10), la fonction u est la solution de (5). (3) Utilisant la forme explicite de $w_h^\varepsilon(x)$, (6), on construit la fonction $u_h^\varepsilon(x)$ qui vérifie (15). D'après les Lemmes 1, 2 et les formules (15), (16) on déduit alors la convergence faible de $u^\varepsilon|u^\varepsilon|^{m-2}\chi_{\mathcal{M}^{(\varepsilon)}}$ vers ω_0 dans $\mathrm{L}^{m'}(\Omega)$.

1. Introduction

This paper deals with a model homogenization problem for high contrast media whose characteristics can be described in terms of elliptic equations with a gradient nonlinearity. More precisely, we consider the variational formulation of the Neumann boundary value problem for a nonlinear elliptic equation generalizing the stationary version of the well-known "double porosity" problem. The classical double porosity model involves linear second order parabolic equation that can be reduced, by means of the Laplace transform, to a variational problem for a proper quadratic functional (*see* [3]). Here we study a similar variational problem for a wider class of functionals given by (1) below. The problems of this kind describe, for example, the combustion in a gradiently nonlinear medium (*see*, e.g., [13]) and also the non-Newtonian flows in porous media (*see*, e.g., [7], Chapter 4). The main feature of such models, in particular of the model considered in this Note, is a high contrast ratio for the permeability tensor in the matrices set and in the fissures system. The homogenized model in our case has an additional term related to the so-called source term in the classical double porosity model (*see* (5) and (8) in [3]). Most of the studies on the subject were carried out in the framework of periodic homogenization (*see*, e.g., [2]), except in [4], where stationarity and ergodicity are assumed for the random field describing the matrices set, and in [3], where due to the variational homogenization technique used, there is no need to make any periodicity assumptions.

In this Note we extend to the nonlinear case the ideas proposed earlier in [8] (*see* also [9]) for the case of linear problems in perforated domains.

Previously, a nonlinear periodic double porosity type model for a two-phase flow was investigated in [7], Chapter 5. Various variational problems in porous media have been widely studied in the existing literature, here we refer to [5,14] and the bibliography there.

Notation. – We denote by C a generic independent of ε constant; $\|\cdot\|_{s,D}$ and $\|\cdot\|_{s,D}^{(1)}$ stand for the norms in the spaces $\mathrm{L}^s(D)$ and $\mathrm{W}^{1,s}(D)$, respectively. Also, $m' = m/(m-1)$, and $\chi_{\mathcal{F}^{(\varepsilon)}}(x)$ and $\chi_{\mathcal{M}^{(\varepsilon)}}(x)$ are the indicators of $\mathcal{F}^{(\varepsilon)}$ and $\mathcal{M}^{(\varepsilon)}$.

2. Setting the problem and formulation of the main result

Let $\Omega = \mathcal{F}^{(\varepsilon)} \cup \mathcal{M}^{(\varepsilon)} \cup \partial\mathcal{M}^{(\varepsilon)}$ be a bounded domain in $\mathbf{R}^n$ with piecewise smooth boundary $\partial\Omega$, and suppose $\mathcal{F}^{(\varepsilon)} \cap \mathcal{M}^{(\varepsilon)} = \emptyset$. Consider the variational problem:

$$J^{(\varepsilon)}\big[u^\varepsilon\big] = \int_\Omega \big\{a^\varepsilon(x)\big|\nabla u^\varepsilon\big|^m + g(x)\big|u^\varepsilon\big|^m - mf^\varepsilon(x)u^\varepsilon\big\}\,\mathrm{d}x \to \inf, \quad u^\varepsilon \in \mathrm{W}^{1,m}(\Omega), \tag{1}$$

whose data satisfy the conditions: $g(x)$ is a smooth strictly positive function in Ω; $f^\varepsilon \in \mathrm{L}^{m'}(\Omega)$ is assumed to be zero on $\mathcal{M}^{(\varepsilon)}$ and to converge in $\mathrm{L}^{m'}(\mathcal{F}^{(\varepsilon)})$ towards f, i.e.,

$$\lim_{\varepsilon\to 0}\big\|f^\varepsilon - f\big\|_{m',\mathcal{F}^{(\varepsilon)}} = 0; \tag{2}$$

$a^\varepsilon(x)$ is a measurable function that admits, for any $\varepsilon > 0$, $a^\varepsilon(x) > c(\varepsilon) > 0$; moreover, $0 < a_0 \leqslant a^\varepsilon(x) \leqslant a_0^{-1}$ in $\mathcal{F}^{(\varepsilon)}$ and $\sup_{x\in\mathcal{M}^{(\varepsilon)}} a^\varepsilon(x) = \alpha^\varepsilon > 0$, $\alpha^\varepsilon \to 0$ as $\varepsilon \to 0$.

It is known (*see* [11], Chapter 5) that, for any $\varepsilon > 0$, there exists a unique solution $u^\varepsilon \in \mathrm{W}^{1,m}(\Omega)$ of problem (1), and that u^ε solves the Neumann boundary value problem for the corresponding Euler equation $\operatorname{div}(a^\varepsilon|\nabla u^\varepsilon|^{m-2}\nabla u^\varepsilon) + g|u^\varepsilon|^{m-2}u^\varepsilon = f^\varepsilon$.

Suppose that $\mathcal{F}^{(\varepsilon)} \subset \Omega$ is a disperse medium, i.e., the following assumptions hold:

(C.1) the local concentrations of the matrices set $\mathcal{M}^{(\varepsilon)}$ and of the fissures part $\mathcal{F}^{(\varepsilon)}$ have positive continuous limits, that is the indicators of $\mathcal{M}^{(\varepsilon)}$ and $\mathcal{F}^{(\varepsilon)}$ converge weakly in $\mathrm{L}^2(\Omega)$ to continuous positive limits. This implies that there exists a continuous function $\rho(x) > 0$ such that $\lim_{h\to 0}\lim_{\varepsilon\to 0} h^{-n}\operatorname{meas}[K_h^x \cap \mathcal{F}^{(\varepsilon)}] = \rho(x)$, where $K_h^x = x + (-\frac{h}{2}, \frac{h}{2})^n$, $x \in \Omega$.

Notice that under this condition $\operatorname{meas}\mathcal{F}^{(\varepsilon)} \nrightarrow 0$ as $\varepsilon \to 0$.

(C.2) there exists a family of extension operators $\mathbf{P}_\varepsilon : \mathrm{W}^{1,m}(\mathcal{F}^{(\varepsilon)}) \to \mathrm{W}^{1,m}(\Omega)$ such that $\|\mathbf{P}_\varepsilon v^\varepsilon\|^{(1)}_{m,\Omega} \leqslant C\|v^\varepsilon\|^{(1)}_{m,\mathcal{F}^{(\varepsilon)}}$ uniformly in $\varepsilon > 0$, and $\mathbf{P}_\varepsilon v^\varepsilon = v^\varepsilon$ on $\mathcal{F}^{(\varepsilon)}$.

Remark 1. – Condition (C.2) ("strong connectedness condition") has been introduced in [8] for $m = 2$ related to the case of linear operators. Later various extension conditions have been widely discussed in the existing literature (*see*, e.g., [1,6,12]).

Instead of the periodicity assumption on the microstructure of the disperse media, we impose certain conditions on local energy characteristics of the sets $\mathcal{F}^{(\varepsilon)}$ and $\mathcal{M}^{(\varepsilon)}$.

For $y \in \Omega$ we define:

– the functional associated to the energy of $\mathcal{F}^{(\varepsilon)}$

$$E_h^{y(\varepsilon)}(p) = \inf_{v^\varepsilon} h^{-n}\int_{\mathcal{F}^{(\varepsilon)}\cap K_h^y}\left\{a^\varepsilon\left|\nabla v^\varepsilon\right|^m + h^{-m-\gamma}\left|v^\varepsilon - (x-y,p)\right|^m\right\}\mathrm{d}x, \tag{3}$$

where $0 < \gamma < m$ and the infimum is taken over $v^\varepsilon \in \mathrm{W}^{1,m}(\mathcal{F}^{(\varepsilon)} \cap K_h^y)$;

– the functional associated to the energy in $\mathcal{M}^{(\varepsilon)}$

$$b_h^{y(\varepsilon)} = \inf_{w^\varepsilon} h^{-n}\int_{K_h^y}\left\{\Phi\left(w^\varepsilon\right) + h^{-m-\gamma}\left|w^\varepsilon - 1\right|^m\chi_{\mathcal{F}^{(\varepsilon)}}\right\}\mathrm{d}x, \tag{4}$$

where $\Phi(w) = a^\varepsilon|\nabla w|^m + g|w|^m\chi_{\mathcal{M}^{(\varepsilon)}}$ and the infimum is taken over $w^\varepsilon \in \mathrm{W}^{1,m}(K_h^y)$.

The functionals (3) and (4) characterize the local averaged permeability of the fissures and local accumulating properties of the matrices, respectively.

Our further assumptions are as follows: *for any* $x \in \Omega$

(C.3) there exists $\lim_{h\to 0}\underline{\lim}_{\varepsilon\to 0}E_h^{x(\varepsilon)}(p) = \lim_{h\to 0}\overline{\lim}_{\varepsilon\to 0}E_h^{x(\varepsilon)}(p) = A(x,p)$, with $A(x,p)$, $A(x,\cdot) \in \mathrm{C}^{2+\beta}(\mathbf{R}^n)$, $\beta > 0$, and $A(\cdot,p) \in \mathrm{C}(\Omega)$; moreover, $\mu_1|p|^{m-2}|\xi|^2 \geqslant A_{p_ip_j}(p)\xi_i\xi_j \geqslant \mu_2|p|^{m-2}|\xi|^2$, $\mu_1, \mu_2 > 0$;

(C.4) there exists $\lim_{h\to 0}\underline{\lim}_{\varepsilon\to 0}b_h^{x(\varepsilon)} = \lim_{h\to 0}\overline{\lim}_{\varepsilon\to 0}b_h^{x(\varepsilon)} = b(x)$ with $b \in \mathrm{C}(\Omega)$.

THEOREM 1. – *Let u^ε be a solution of problem* (1), *and assume that conditions* (C.1)–(C.4) *are fulfilled. Then $\mathbf{P}_\varepsilon(u^\varepsilon\chi_{\mathcal{F}^{(\varepsilon)}})$ converges in $\mathrm{L}^m(\Omega)$ to a unique solution $u(x)$ of*

$$J_{\mathrm{hom}}[u] = \int_\Omega\left\{A(x,\nabla u) + B(x)|u|^m - mf(x)\rho(x)u\right\}\mathrm{d}x \to \inf, \quad u \in \mathrm{W}^{1,m}(\Omega), \tag{5}$$

where $B(x) = (g\rho + b)(x)$. Moreover, $u^\varepsilon|u^\varepsilon|^{m-2}\chi_{\mathcal{M}^{(\varepsilon)}}$ converges weakly in $\mathrm{L}^{m'}(\Omega)$ to $\omega_0 = (b/g)u|u|^{m-2}$.

Remark 2. – In the periodic and locally periodic framework the existence of the limits in (C.3), (C.4) for $m = 2$ was obtained in [3]. Similar result holds for $m > 2$. For example, let $\mathcal{M}^{(\varepsilon)}$ be a periodic set of balls of radius $r\varepsilon$, $r < 1/2$, centered at points of $\varepsilon\mathbf{Z}^n$, $a^\varepsilon(x) = a\varepsilon^m \cdot \chi_{\mathcal{M}^{(\varepsilon)}} + 1 \cdot \chi_{\mathcal{F}^{(\varepsilon)}}$. Then $b(x) = b = \int_{\mathcal{M}} gw|w|^{m-2}\,\mathrm{d}x$ with $w = w(x)$ the solution of the cell problem: $a\operatorname{div}(|\nabla w|^{m-2}\nabla w) - gw|w|^{m-2} = 0$, $x \in \mathcal{M} \equiv \{x \in \mathbf{R}^n : |x| < r\}$; $w = 1$, $x \in \partial\mathcal{M}$.

3. Auxiliary results

In this section we introduce functions $Y_h^\varepsilon(x)$ and $V_h^\varepsilon(x)$, which will be used to construct convenient approximations for the solution of (1) in $\mathcal{F}^{(\varepsilon)}, \mathcal{M}^{(\varepsilon)} \subset \Omega$.

The following assertions can be proved in the same way as Lemmas 4.1–4.4 in [10].

LEMMA 1. – *Let* (C.4) *be satisfied. Then there are a set* $\mathcal{B}_h^{(\varepsilon)} \subset \mathcal{F}^{(\varepsilon)}$ *and a function* Y_h^ε *such that*
(a) $\overline{\lim}_{\varepsilon\to 0} \operatorname{meas}(\mathcal{B}_h^{(\varepsilon)}) = \mathrm{O}(h^{\gamma/(m+1)})$;
(b) $0 \leqslant Y_h^\varepsilon(x) \leqslant 1$; $Y_h^\varepsilon(x) = 1$ *for* $x \in \mathcal{F}^{(\varepsilon)} \setminus \mathcal{B}_h^{(\varepsilon)}$;
(c) $\overline{\lim}_{\varepsilon\to 0} \int_\Omega \Phi(Y_h^\varepsilon)|w|^m \,\mathrm{d}x \leqslant \int_\Omega b|w|^m \,\mathrm{d}x + \mathrm{o}(1)$, $h \to 0$, *for any* $w \in \mathrm{C}^2(\Omega)$.

LEMMA 2. – *Let* (C.3) *be satisfied and let* $\mathcal{B}_h^{(\varepsilon)}$ *be the set defined in Lemma* 1; $w \in \mathrm{C}^2(\Omega)$. *Then there are a set* $\mathcal{D}_h^{(\varepsilon)} \subset \Omega$ *and a function* $V_h^\varepsilon \equiv V_h^\varepsilon(x, w(x))$, $V_h^\varepsilon \in \mathrm{W}^{1,m}(\Omega)$ *such that*
(a) $\mathcal{B}_h^{(\varepsilon)} \subset \mathcal{D}_h^{(\varepsilon)}$, $\overline{\lim}_{\varepsilon\to 0} \operatorname{meas}(\mathcal{D}_h^{(\varepsilon)}) = \mathrm{o}(1)$, $h \to 0$;
(b) $\max_{x\in\Omega} |V_h^\varepsilon(x) - w(x)| \leqslant Ch$;
(c) $\lim_{\varepsilon\to 0} \int_{\mathcal{D}_h^{(\varepsilon)} \cup \mathcal{M}^{(\varepsilon)}} a^\varepsilon |\nabla V_h^\varepsilon|^m \,\mathrm{d}x = 0$, $\overline{\lim}_{\varepsilon\to 0} \int_{\mathcal{F}^{(\varepsilon)}} a^\varepsilon |\nabla V_h^\varepsilon|^m \,\mathrm{d}x \leqslant \int_\Omega A(x, \nabla w)\,\mathrm{d}x + \mathrm{o}(1)$, $h \to 0$.

LEMMA 3. – *Let* (C.4) *be satisfied. Assume that a sequence* $\{u^\varepsilon\} \subset \mathrm{W}^{1,m}(\Omega)$ *converges in* $\mathrm{L}^m(\mathcal{F}^{(\varepsilon)})$ *to* $u \in \mathrm{C}^2(\Omega)$, *and* $\int_\Omega \{a^\varepsilon(x)|\nabla u^\varepsilon|^m + |u^\varepsilon|^m\}\,\mathrm{d}x \leqslant C$. *Then there are a set* $\mathcal{B}^{(\varepsilon)} \subset \Omega$ *and* $\mathcal{M}^{(\varepsilon)} \subset \mathcal{B}^{(\varepsilon)}$, *a function* $\hat{u}^\varepsilon(x)$ *and a subsequence* $\varepsilon_k \to 0$ *still denoted by* ε *such that*
(a) $\lim_{\varepsilon\to 0} \operatorname{meas} \mathcal{B}_1^{(\varepsilon)} = 0$, *where* $\mathcal{B}_1^{(\varepsilon)} = \mathcal{B}^{(\varepsilon)} \cap \mathcal{F}^{(\varepsilon)}$;
(b) $\hat{u}^\varepsilon(x) = u^\varepsilon(x)$ *for* $x \in \mathcal{F}^{(\varepsilon)} \setminus \mathcal{B}_1^{(\varepsilon)}$ *and* $\lim_{\varepsilon\to 0} \|\hat{u}^\varepsilon\|^{(1)}_{m,\mathcal{B}_1^{(\varepsilon)}} = 0$;
(c) $\lim_{\varepsilon\to 0} \int_{\mathcal{B}^{(\varepsilon)}} \Phi(u^\varepsilon)\,\mathrm{d}x \geqslant \int_\Omega b(x)|u(x)|^m \,\mathrm{d}x$.

4. Proof of Theorem 1

Consider the variational problem (1). Since $J^{(\varepsilon)}[0] = 0$, we have $J^{(\varepsilon)}[u^\varepsilon] \leqslant 0$. Now (2) and (C.2) imply that the sequence $\{\mathbf{P}_\varepsilon(u^\varepsilon \chi_{\mathcal{F}^{(\varepsilon)}})\}$ is uniformly bounded in $\mathrm{W}^{1,m}(\Omega)$. Therefore, one can extract a subsequence weakly converging in $\mathrm{W}^{1,m}(\Omega)$ to $u \in \mathrm{W}^{1,m}(\Omega)$. Let us show that u is a solution of (5).

Let $w \in \mathrm{C}^2(\Omega)$ and let $Y_h^\varepsilon(x)$, $V_h^\varepsilon(x) \equiv V_h^\varepsilon(x, w(x))$, $\mathcal{B}_h^{(\varepsilon)}$, $\mathcal{D}_h^{(\varepsilon)}$ ($\mathcal{B}_h^{(\varepsilon)} \subset \mathcal{D}_h^{(\varepsilon)}$) be the functions and the sets defined in Lemmas 1, 2. Consider the function

$$w_h^\varepsilon(x) = V_h^\varepsilon\big(x, w(x)\big) Y_h^\varepsilon(x). \tag{6}$$

It is clear that $w_h^\varepsilon \in \mathrm{W}^{1,m}(\Omega)$. Since $u^\varepsilon(x)$ is the solution of (1), we have

$$J^{(\varepsilon)}\big[u^\varepsilon\big] \leqslant J^{(\varepsilon)}\big[w_h^\varepsilon\big]. \tag{7}$$

It follows now from Lemmas 1 and 2 that

$$\overline{\lim_{\varepsilon\to 0}} \int_\Omega \big\{a^\varepsilon(x)\big|\nabla w_h^\varepsilon\big|^m + g(x)\big|w_h^\varepsilon\big|^m\big\}\,\mathrm{d}x \leqslant \int_\Omega \big\{A(x, \nabla w) + B(x)|w|^m\big\}\,\mathrm{d}x + j_1(\varepsilon, h), \tag{8}$$

where $\overline{\lim}_{\varepsilon\to 0} j_1(\varepsilon, h) = \mathrm{o}(1)$ as $h \to 0$. Finally, (2) implies the inequality

$$\overline{\lim_{\varepsilon\to 0}}\, J^{(\varepsilon)}\big[u^\varepsilon\big] \leqslant J_{\mathrm{hom}}[w] \tag{9}$$

for any $w \in \mathrm{C}^2(\Omega)$. Then it is clear that (9) holds for any $w \in \mathrm{W}^{1,m}(\Omega)$.

Let us show now that if a sequence $\{u^\varepsilon\}$ converges in $\mathrm{L}^m(\mathcal{F}^{(\varepsilon)})$ to $u \in \mathrm{W}^{1,m}(\Omega)$ then

$$\underline{\lim}_{\varepsilon\to 0}\, J^{(\varepsilon)}\big[u^\varepsilon\big] \geqslant J_{\mathrm{hom}}[u]. \tag{10}$$

To this end we approximate u by $u_\delta \in \mathrm{C}^2(\Omega)$ verifying $\|u_\delta - u\|^{(1)}_{m,\Omega} < \delta$. The sequence $\{u^\varepsilon\}$ converges in $\mathrm{L}^m(\mathcal{F}^{(\varepsilon)})$ to u, then it is clear that there exists a sequence $\{u_\delta^\varepsilon\} \subset \mathrm{W}^{1,m}(\Omega)$ that converges to u_δ in the same sense and

$$\int_\Omega \left\{a^\varepsilon \left|\nabla\left(u_\delta^\varepsilon - u^\varepsilon\right)\right|^m + \left|u_\delta^\varepsilon - u^\varepsilon\right|^m\right\} \mathrm{d}x \leqslant C\delta^m; \qquad \int_\Omega \left\{a^\varepsilon \left|\nabla u_\delta^\varepsilon\right|^m + \left|u_\delta^\varepsilon\right|^m\right\} \mathrm{d}x \leqslant C. \tag{11}$$

Now from (11) and the inequality $|A(x,u^\varepsilon) - A(x,u_\delta^\varepsilon)| \leqslant C(1+|\nabla u^\varepsilon| + |\nabla u_\delta^\varepsilon|)^{m-1}|\nabla(u^\varepsilon - u_\delta^\varepsilon)|$ (*see* [12]), we see that (10) immediately follows from

$$\varliminf_{\varepsilon\to 0} J^{(\varepsilon)}\left[u_\delta^\varepsilon\right] \geqslant J_{\mathrm{hom}}[u_\delta]. \tag{12}$$

Apply Lemma 3 to $u_\delta^\varepsilon(x)$ and $u_\delta(x)$. Then there are a set $\mathcal{B}^{(\varepsilon)}$ and a function $\hat u_\delta^\varepsilon \in \mathrm{W}^{1,m}(\Omega)$ such that $\mathcal{M}^{(\varepsilon)} \subset \mathcal{B}^{(\varepsilon)}$, $\hat u_\delta^\varepsilon(x) = u_\delta^\varepsilon(x)$ for $x \in \mathcal{F}^{(\varepsilon)} \setminus \mathcal{B}^{(\varepsilon)}$ and relations (a)–(c) of Lemma 3 hold. It is clear that $\hat u_\delta^\varepsilon$ converges in $\mathrm{L}^m(\mathcal{F}^{(\varepsilon)})$ to u_δ.

Cover Ω by cubes $K_h^\alpha = x^\alpha + (-h/2, +h/2)^n$ centered at points $h\mathbf{Z}^n$ and consider the function $v_\alpha^\varepsilon(x) = \hat u_\delta^\varepsilon(x) - u_\delta(x^\alpha)$, $x \in K_h^\alpha \cap \mathcal{F}^{(\varepsilon)} \subset \Omega$. Since $\hat u_\delta^\varepsilon$ converges in $\mathrm{L}^m(\mathcal{F}^{(\varepsilon)})$ to u_δ, by the definition of the functional $E_h^{x^\alpha(\varepsilon)}$ and Lemma 3 we get

$$\sum_\alpha \int_{(\mathcal{F}^{(\varepsilon)}\setminus\mathcal{B}^{(\varepsilon)})\cap K_h^\alpha} a^\varepsilon(x)\left|\nabla \hat u_\delta^\varepsilon\right|^m \mathrm{d}x \geqslant \sum_\alpha E_h^{x^\alpha(\varepsilon)}\left(\nabla u_\delta^\alpha\right) + \mathrm{o}(1), \quad \text{as } h\to 0 \text{ and } h \gg \varepsilon, \tag{13}$$

where $\nabla u_\delta^\alpha = \nabla u_\delta(x^\alpha)$. By the definition $\hat u_\delta^\varepsilon(x) = u_\delta^\varepsilon(x)$ for $x \in \mathcal{F}^{(\varepsilon)} \setminus \mathcal{B}^{(\varepsilon)}$, therefore,

$$J^{(\varepsilon)}\left[u_\delta^\varepsilon\right] \geqslant \sum_\alpha E_h^{x^\alpha(\varepsilon)}\left(\nabla u_\delta^\alpha\right) + \sum_\alpha \int_{\mathcal{F}^{(\varepsilon)}\cap K_h^\alpha} G\left(u_\delta^\varepsilon\right) \mathrm{d}x + \int_{\mathcal{B}^{(\varepsilon)}} \Phi\left(u_\delta^\varepsilon\right) \mathrm{d}x + j_2(\varepsilon,h), \tag{14}$$

where $G(u_\delta^\varepsilon) = |u_\delta^\varepsilon|^m - mf^\varepsilon u_\delta^\varepsilon$, $j_2(\varepsilon,h) = \mathrm{o}(1)$ as $h\to 0$ and $h \gg \varepsilon$. We fix δ and pass in (14) to the limit as $\varepsilon, h \to 0$. We obtain (12), and, therefore, (10). It follows from (9), (10) that if u^ε the solution of (1) converges in $\mathrm{L}^m(\mathcal{F}^{(\varepsilon)})$ to u, then u is the solution of (5).

It remains to prove that $u^\varepsilon|u^\varepsilon|^{m-2}\chi_{\mathcal{M}^{(\varepsilon)}}$ converges weakly in $\mathrm{L}^{m'}(\Omega)$ to $\omega_0 = (b/g)u|u|^{m-2}$. Let u_h^ε be the function defined by (6) with $w = u$, where u is the solution of (5). Then it follows from (9), (10) that $\lim_{\varepsilon\to 0} J^{(\varepsilon)}[u_h^\varepsilon] = \lim_{\varepsilon\to 0} J^{(\varepsilon)}[u^\varepsilon] = J[u]$, where u^ε is the solution of (1). Now using the inequality $F(u+v, p+q) \geqslant F(u,p) + \theta(m)F(v,q) + F_{p_i}(u,p)q_i + F_u(u,p)v$, with $F(u,p) = |p|^m + |u|^m$ and $0 < \theta(m) \leqslant 1$ we get

$$\lim_{\varepsilon\to 0} \left\|u_h^\varepsilon - u^\varepsilon\right\|_{m,\Omega} = 0. \tag{15}$$

Let $w_h^{y(\varepsilon)}(x)$ be the minimizer of (4) and suppose that $\varepsilon = \varepsilon(h) \downarrow 0$ as $h \to 0$. Denote by h_ε the function inverse to $\varepsilon(h)$ and set $w^{y(\varepsilon)}(x) \equiv w_h^{y(\varepsilon)}(x)$ for $h = h_\varepsilon$. Consider the Euler equation corresponding to the functional (4), by condition (C.4) we get

$$\lim_{\varepsilon\to 0} h_\varepsilon^{-n} \int_{K_{h_\varepsilon}^y} g\chi_{\mathcal{M}^{(\varepsilon)}} w^{y(\varepsilon)}\left|w^{y(\varepsilon)}\right|^{m-2} \mathrm{d}x = b(y). \tag{16}$$

Now the weak convergence of $u^\varepsilon|u^\varepsilon|^{m-2}\chi_{\mathcal{M}^{(\varepsilon)}}$ in $L^{m'}(\Omega)$ to $\omega_0 = (b/g)u|u|^{m-2}$ follows from (15), (16), and Lemmas 1, 2. Theorem 1 is proved.

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References

[1] E. Acerbi, V. Chiadò Piat, G. Dal Maso, D. Percivale, An extension theorem from connected sets, and homogenization in general periodic domains, Nonlinear Anal. 18 (1992) 481–496.

[2] T. Arbogast, J. Douglas, U. Hornung, Derivation of the double porosity model of single phase flow via homogenization theory, SIAM J. Appl. Math. 21 (1990) 823–826.

[3] A. Bourgeat, M. Goncharenko, M. Panfilov, L. Pankratov, A general double porosity model, C. R. Acad. Sci. Paris, Série IIb 327 (1999) 1245–1250.
[4] A. Bourgeat, A. Mikelic, A. Piatnitski, Modèle de double porosité aléatoire, C. R. Acad. Sci. Paris, Série I 327 (1998) 99–104.
[5] A. Braides, A. Defranceschi, Homogenization of Multiple Integrals, Oxford Lecture Ser. Math. Appl., Vol. 12, Clarendon Press, Oxford, 1998.
[6] D. Cioranescu, J. Saint Jean Paulin, Homogenization of Reticulated Structures, Appl. Math. Sci., Vol. 136, Springer-Verlag, New York, 1999.
[7] Homogenization and Porous Media, in: U. Hornung (Ed.), Interdisciplinary Appl. Math., Vol. 6, Springer-Verlag, New York, 1997.
[8] E. Khruslov, Asymptotic behavior of the solutions of the second boundary value problem in the case of the refinement of the boundary of the domain, Mat. Sb. 106 (1978) 604–621.
[9] E. Khruslov, Averaged models of diffusion in fractured–porous media, Dokl. Akad. Nauk SSSR 309 (1989) 332–335. English translation in Soviet Phys. Dokl. 34 (1989) 980–981.
[10] E. Khruslov, L. Pankratov, Homogenization of boundary value problems for the Ginzburg–Landau equation in weakly connected domains, in: V. Marchenko (Ed.), Spectral Theory and Related Topics, Vol. 19, American Mathematical Society, Providence, RI, 1994, pp. 233–268.
[11] O. Ladyzhenskaya, N. Ural'tseva, Linear and Quasilinear Elliptic Equations, Academic Press, New York, 1968.
[12] L. Pankratov, Homogenization of nonlinear Neumann elliptic and parabolic problems, in: Homogenization and Applications to Material Sciences, Math. Sci. Appl., Vol. 9, 1997, pp. 341–353.
[13] A. Samarskii, V. Galaktionov, S. Kurdyumov, A. Mikhailov, Blow-up in Quasilinear Parabolic Equations, De Gruyter, Berlin, 1995.
[14] V. Zhikov, S. Kozlov, O. Oleinik, Homogenization of Differential Operators and Integral Functionals, Springer-Verlag, New York, 1994.