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A Coupling Approach to Randomly Forced Nonlinear PDE's. II

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Abstract: We consider a class of discrete time random dynamical systems and establish the exponential convergence of its trajectories to a unique stationary measure. The result obtained applies, in particular, to the 2D Navier–Stokes system and multidimensional complex Ginzburg–Landau equation with random kick-force.

1. Main Result

The present paper is an immediate continuation of [KS2] and is devoted to studying the following random dynamical system (RDS) in a Hilbert space H:

$$u^k = S(u^{k-1}) + \eta_k, \quad k \geq 1. \tag{1.1}$$

Here $S : H \to H$ is a locally Lipschitz operator such that $S(0) = 0$ and $\{\eta_k\}$ is a sequence of i.i.d. bounded random variables of the form

$$\eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j, \tag{1.2}$$

where $\{e_j\}$ is an orthonormal basis in H, $b_j \geq 0$ are some constants such that $\sum b_j^2 < \infty$, and ξ_{jk} are scalar random variables. The exact conditions imposed on S can be found in [KS2, Sect. 2] (see Conditions (A) – (C)). Roughly speaking, they mean that S is compact and $S^n(u) \to 0$ as $n \to \infty$ uniformly on bounded subsets of H. Concerning the random variables ξ_{jk}, we assume that they satisfy the following condition:

(D) *For any j, the random variables ξ_{jk}, $k \geq 1$, have the same distribution $\pi_j(dr) = p_j(r)\,dr$, where the densities $p_j(r)$ are functions of bounded variation such that* $\operatorname{supp} p_j \subset [-1, 1]$ *and* $\int_{|r|\leq\varepsilon} p_j(r)\,dr > 0$ *for all $j \geq 1$ and $\varepsilon > 0$. We normalise the functions p_j to be continuous from the right.*

Let us denote by $\mathfrak{P} = \mathfrak{P}(k, v, \cdot)$ the Markov transition function for (1.1) and by $\mathfrak{P}_k$ the associated Markov semigroup acting on the space of bounded continuous functions on H. It was proved in [KS1, KS2] that, under the above conditions, the RDS (1.1) has a unique stationary measure μ, provided that

$$b_j \neq 0 \quad \text{for} \quad 1 \leq j \leq N, \tag{1.3}$$

where $N \geq 1$ is sufficiently large. Moreover, it is shown in [KS2] that any trajectory $\{u^k\}$ of the RDS (1.1) converges to μ (in an appropriate sense) with the rate $e^{-c\sqrt{k}}$. The aim of this paper is to prove that this convergence is exponential:

Theorem 1.1. *There is a constant $c > 0$ and an integer $N \geq 1$ such that if (1.3) holds, then*

$$\left|\mathfrak{P}_k f(u) - (\mu, f)\right| \leq C_R e^{-ck}\big(\sup_H |f| + \mathrm{Lip}(f)\big) \quad \textit{for} \quad k \geq 0, \quad u \in B_H(R), \tag{1.4}$$

where $B_H(R)$ is the ball in H of radius R centred at zero, f is an arbitrary bounded Lipschitz function on H, and $C_R > 0$ is a constant depending on R solely.

As it is shown in [KS1], the conditions (A) – (D) (under which Theorem 1.1 is proved) are satisfied for the 2D Navier–Stokes system and multidimensional complex Ginzburg–Landau equation perturbed by a kick-force of the form

$$\eta(t, x) = \sum_{k=1}^{\infty} \eta_k(x)\delta(t - k),$$

where the kicks η_k are i.i.d. random variables which can be written in the form (1.2) in an appropriate functional space H.

We note that the exponential convergence to the stationary measure was established earlier for the Navier–Stokes system perturbed by a finite-dimensional white noise force. Namely, Bricmont, Kupiainen, Lefevere [BKL] showed that for μ-almost all [1] initial functions u^0 the corresponding trajectory $\{u^k\}$ converges to the stationary measure exponentially fast. Our proof implies the exponential convergence for all initial data and is much shorter. It exploits the coupling approach from [KS2].

For the reader's convenience, we recall some notations used in [KS2].

Notations. We abbreviate a pair of random variables ξ_1, ξ_2 or points u_1, u_2 to $\xi_{1,2}$ and $u_{1,2}$, respectively. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an integer $k \geq 1$ (the case $k = \infty$ is not excluded), we denote by Ω^k the space $\Omega \times \cdots \times \Omega$ (k times) endowed with the σ-algebra $\mathcal{F} \times \cdots \times \mathcal{F}$ and the measure $\mathbb{P} \times \cdots \times \mathbb{P}$. The points of Ω^k will be denoted by $\boldsymbol{\omega}^k = (\omega_1, \ldots, \omega_k)$, where $\omega_j \in \Omega$.

$C_b(H)$ is the space of bounded continuous functions on H with the supremum norm $\|\cdot\|_\infty$.

$L(H)$ is the space of bounded Lipschitz functions on H endowed with the norm $\|f\|_L = \|f\|_\infty + \mathrm{Lip}(f)$, where $\mathrm{Lip}(f)$ is the Lipschitz constant of f.

$\mu_v(k)$ denotes the measure $\mathfrak{P}(k, v, \cdot)$.

$B_H(R)$ is the closed ball of radius $R > 0$ centred at zero.

[1] We denote by μ the unique stationary measure.

2. Proof of the Theorem

Step 1. For any two probability Borel measures μ_1 and μ_2 on H we set

$$\|\mu_1-\mu_2\|_L^* = \sup_{\|f\|_L\le 1}\big|(\mu_1-\mu_2,f)\big|$$

(cf. [D], Sect. 11.3). In view of Lemma 1.2 in [KS2], to prove the theorem it suffices to show that for any $R>0$ there is $C_R>0$ such that

$$\big\|\mu_{u_1}(k)-\mu_{u_2}(k)\big\|_L^* \le C_R e^{-ck} \quad \text{for} \quad u_1,u_2\in B_H(R), \quad k\ge 1,$$

where $c>0$ is a constant not depending on R. As in [KS2], we can restrict our consideration to the compact invariant set $\mathcal{A}$, which contains supports of the measures $\mu_u(k)$, $k\ge 1$, $u\in B_H(R)$ (see formula (2.5) in [KS2]). Moreover, by Lemma 1.3 in [KS2], the required inequality (1.4) will be proved if we show that for any $u_1,u_2\in\mathcal{A}$ and any integer $k\ge 1$ there is a coupling $y_{1,2}(k)=y_{1,2}(k,u_1,u_2)$ for the measures $\mu_{u_{1,2}}(k)$ such that

$$\mathbb{P}\big\{\|y_1(k)-y_2(k)\|\ge C\,e^{-ck}\big\}\le C\,e^{-ck} \quad \text{for} \quad k\ge 1, \tag{2.1}$$

where $\|\cdot\|$ is the norm in H and $C>0$ is a constant not depending on $u_1,u_2\in\mathcal{A}$ and k. Finally, repeating the argument in Step 2 of the proof of Theorem 2.1 in [KS2, Sect. 3.2], we see that it suffices to find an integer $l\ge 1$ and to construct a probability space $(\Omega',\mathcal{F}',\mathbb{P}')$ and a sequence of couplings $y_{1,2}^n(u_1,u_2,\omega)$, $\omega\in\Omega'$, for the measures $\mu_{u_{1,2}}(nl)$, $n\ge 1$, such that the maps $y_{1,2}^n$ are measurable with respect to (u_1,u_2,ω) and satisfy the inequality

$$\mathbb{P}\big\{\|y_1^n-y_2^n\|\ge e^{-c'n}\big\}\le e^{-c'n} \quad \text{for} \quad n\ge 1. \tag{2.2}$$

If (2.2) is established, then (2.1) holds with $c=c'/l$ and some constant $C>1$.

Step 2. To prove (2.2), we shall need the following result, which is a particular case of Lemma 3.3 in [KS2].

Lemma 2.1. *Under the conditions of Theorem 1.1, there is a probability space* $(\Omega,\mathcal{F},\mathbb{P})$*, positive constants* $d_0<1/2$ *and* θ*, and an integer* $l\ge 1$ *such that for any* $u_1,u_2\in\mathcal{A}$ *the measures* $\mu_{u_{1,2}}(l)$ *admit a coupling* $U_{1,2}=U_{1,2}(u_1,u_2;\omega)$ *such that the following assertions hold:*

(i) *The maps* $U_{1,2}(u_1,u_2,\omega)$ *are measurable with respect to* $(u_1,u_2,\omega)\in\mathcal{A}\times\mathcal{A}\times\Omega$.
(ii) *If* $\|u_1-u_2\|>d_0$*, then*

$$\mathbb{P}\big\{\|U_1-U_2\|\le d_0\big\}\ge\theta. \tag{2.3}$$

(iii) *If* $d=\|u_1-u_2\|\le 2^{-r}d_0$ *for some integer* $r\ge 0$*, then*

$$\mathbb{P}\big\{\|U_1-U_2\|\le d/2\big\}\ge 1-2^{-r-3}. \tag{2.4}$$

Remark 2.2. In [KS2], it is proved that the probability on the left-hand side of (2.4) can be estimated from below by $1-2^{-r-1}$. However, it is not difficult to see that the term 2^{-r-1} can be replaced by 2^{-r-3} if the constant d_0 is sufficiently small.

Let us fix arbitrary $u_1,u_2\in\mathcal{A}$ and define a sequence of random variables $y_{1,2}^n = y_{1,2}^n(u_1,u_2,\boldsymbol{\omega}^n)$, $\boldsymbol{\omega}^n=(\boldsymbol{\omega}^{n-1},\omega_n)\in\Omega^n$, by the rule $y_{1,2}^0=u_{1,2}$ and

$$y_{1,2}^n(u_1,u_2,\boldsymbol{\omega}^n)=U_{1,2}\big(y_1^{n-1}(u_1,u_2,\boldsymbol{\omega}^{n-1}),\,y_2^{n-1}(u_1,u_2,\boldsymbol{\omega}^{n-1}),\,\omega_n\big), \quad n\ge 1.$$

We shall show that $y_{1,2}^n$ satisfy (2.2) for all $n\ge 0$.

Step 3. Let us introduce a probability space $(\Omega^\infty, \mathcal{F}^\infty, \mathbb{P}^\infty)$ as the countable product of $(\Omega, \mathcal{F}, \mathbb{P})$ and note that the random variables $y_{1,2}^n$, $n \geq 0$, can be extended to Ω^∞ by the natural formula

$$y_{1,2}^n(u_1, u_2, \boldsymbol{\omega}^\infty) = y_{1,2}^n(u_1, u_2, \boldsymbol{\omega}^n), \quad \boldsymbol{\omega}^\infty = (\boldsymbol{\omega}^n, \omega_{n+1}, \omega_{n+2}, \ldots).$$

Thus, without loss of generality, we can assume that they are defined on the same probability space Ω^∞. To simplify notation, we write $(\Omega, \mathcal{F}, \mathbb{P})$ instead of $(\Omega^\infty, \mathcal{F}^\infty, \mathbb{P}^\infty)$.

For any non-negative integers r and n, we define the events

$$Q_{n,r} = \left\{\omega \in \Omega : d_r \leq \|y_1^n(\omega) - y_2^n(\omega)\| < d_{r-1}\right\},$$

where $d_r = 2^{-r} d_0$ for $r \geq 1$ and $d_{-1} = \infty$. Let us denote $p_{n,r} = \mathbb{P}(Q_{n,r})$ and set

$$\zeta_n = \sum_{r=0}^{\infty} 2^{-r} p_{n,r}.$$

We claim that

$$\zeta_n \leq \gamma^n, \quad n \geq 0, \tag{2.5}$$

where $\gamma < 1$ is a positive constant not depending on $u_1, u_2 \in \mathcal{A}$ and n.

Taking inequality (2.5) for granted, let us complete the proof of (2.2).

For any real number $s \geq 0$, we denote by $[s]$ its integer part. Let us choose $\alpha > 0$ so small that $\beta := 2^\alpha \gamma < 1$ and consider the event

$$R_n := \left\{\|y_1^n(\omega) - y_2^n(\omega)\| \geq d_{[\alpha n]}\right\} = \bigcup_{r=0}^{[\alpha n]} Q_{n,r}.$$

In view of (2.5), we have

$$\mathbb{P}(R_n) = \sum_{r=0}^{[\alpha n]} p_{n,r} \leq 2^{[\alpha n]} \sum_{r=0}^{[\alpha n]} 2^{-r} p_{n,r} \leq 2^{\alpha n} \zeta_n \leq (2^\alpha \gamma)^n = \beta^n.$$

Since $d_0 \leq 1/2$, we see that $d_{[\alpha n]} = 2^{-[\alpha n]} d_0 \leq 2^{-\alpha n}$. We have thus proved that

$$\mathbb{P}\left\{\|y_1^n(\omega) - y_2^n(\omega)\| \geq 2^{-\alpha n}\right\} \leq \beta^n.$$

This inequality implies (2.2) with $c' = \min\{\alpha \log 2, \log \beta^{-1}\}$ and $(\Omega', \mathcal{F}', \mathbb{P}') = (\Omega, \mathcal{F}, \mathbb{P})$.

Step 4. Thus, it remains to establish (2.5). Since $\zeta_0 \leq 1$, it is sufficient to show that $\zeta_n \leq \gamma\, \zeta_{n-1}$ for $n \geq 1$. We have

$$\begin{aligned}
\zeta_n &= \sum_{r=0}^{\infty} 2^{-r} \mathbb{P}(Q_{n,r}) \\
&= \sum_{r=0}^{\infty} 2^{-r} \sum_{m=0}^{\infty} p_{n-1,m} \mathbb{P}\left\{Q_{n,r} \mid Q_{n-1,m}\right\} \\
&\leq \sum_{m=0}^{\infty} p_{n-1,m} \left\{\sum_{r=0}^{m} \mathbb{P}\left\{Q_{n,r} \mid Q_{n-1,m}\right\} + 2^{-(m+1)} \sum_{r=m+1}^{\infty} \mathbb{P}\left\{Q_{n,r} \mid Q_{n-1,m}\right\}\right\}.
\end{aligned} \tag{2.6}$$

Let us estimate the two sums in r in the right-hand side of (2.6). In view of inequality (2.4) with $d \in [d_m, d_{m-1})$, for $m \geq 1$ we have

$$\sum_{r=0}^{m} \mathbb{P}\left\{Q_{n,r} \mid Q_{n-1,m}\right\} = \mathbb{P}\left\{\|y_1^n - y_2^n\| \geq d_m \mid Q_{n-1,m}\right\} \leq 2^{-m-2}, \quad (2.7)$$

$$\sum_{r=m+1}^{\infty} \mathbb{P}\left\{Q_{n,r} \mid Q_{n-1,m}\right\} = \mathbb{P}\left\{\|y_1^n - y_2^n\| < d_m \mid Q_{n-1,m}\right\} \leq 1. \quad (2.8)$$

We now consider the case $m = 0$. Inequality (2.3) implies that

$$\sigma_n := \mathbb{P}\{Q_{n,0} \mid Q_{n-1,0}\} \leq 1 - \theta.$$

Hence, denoting by $Q_{n,0}^c$ the complement of $Q_{n,0}$, we derive

$$\begin{aligned} \mathbb{P}\{Q_{n,0} \mid Q_{n-1,0}\} + 2^{-1}\sum_{r=1}^{\infty} \mathbb{P}\left\{Q_{n,r} \mid Q_{n-1,0}\right\} &= \sigma_n + 2^{-1}\mathbb{P}\{Q_{n,0}^c \mid Q_{n-1,0}\} \\ &= \sigma_n + (1-\sigma_n)/2 \leq 1 - \theta/2. \quad (2.9) \end{aligned}$$

Substitution of (2.7) – (2.9) into (2.6) results in

$$\zeta_n \leq (1-\theta/2)p_{n-1,0} + \frac{3}{4}\sum_{m=1}^{\infty} 2^{-m} p_{n-1,m} \leq \gamma\, \zeta_{n-1},$$

if we choose $\gamma = \max\{1-\theta/2, 3/4\} < 1$. The proof of Theorem 1.1 is complete.

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