Asymptotic Analysis 21 (1999) 303–315
IOS Press

Estimates in probability of the residual between the random and the homogenized solutions of one-dimensional second-order operator

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Abstract. The work is giving estimations of the discrepancy between solutions of the initial and the homogenized problems for a one-dimensional second-order elliptic operators with random coefficients satisfying strong or uniform mixing conditions. We obtain several sharp estimates in terms of the corresponding mixing coefficient.

Keywords: Stochastic homogenization, random operators, moderate deviations

1. Introduction

In the mathematical literature there are now many papers devoted to homogenization of random operators with coefficients being stationary random field (see, for instance, [3] and references therein) and of operators posed in randomly perforated domain (see [2,3]). But all these results are mainly giving the convergence of the solutions towards the solution of the limit (or homogenized) equation, without estimate of the residual. The first successful attempt to give such an estimate is the work of Yurinski [6], where the expectation of some norm of the residual for the divergence form second-order elliptic random operator is estimated by a positive power of a small parameter ε that characterizes the microscopic length scale. This power of ε depends only on the dimension of the space, the ellipticity constant and on some characteristics of the mixing conditions; but this power is implicit and could not be computed explicitly. Later, similar problems have been studied for symmetric elliptic systems [5]; in this case the residual is estimated by some negative power of $|\log\varepsilon|$, which could not, once more, be computed explicitly. The aim of our paper is to investigate in the one-dimensional case the probabilistic property of the residual. We assume that the coefficients of the operator is a stationary random field satisfying strong or uniform mixing condition. The first two sections deal with the case when the corresponding mixing coefficient decays like a negative power $-\alpha$ of a distance. Namely, in the first part, we suppose that $\alpha > 1$, i.e., that the random variables $a(x,\cdot)$ and $a(x+d,\cdot)$ are weakly dependent for large d. This allows to apply the central limit theorem. In the second part we study the case when the mixing properties of the random field are not so good, i.e., when $\alpha \leqslant 1$. Finally, in the third part, using large deviation type estimates and assuming that the mixing coefficient decays like the exponent of some power of the distance, we get more precise bounds in probability for the fields with such “good” mixing properties. It should be noted that we do not state the large (or moderate) deviation principle itself; we prove, in fact, only upper bounds of moderate deviation type, which are sufficient for our purposes. In this way we obtain estimates of the discrepancy under much weaker conditions than those required for large or moderate deviation principle.

2. Statement of the problem

We consider the following differential equation in the interval $]0,1[$ with the source term $f(x) \in L^2(0,1)$:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x} a\left(\omega, \frac{x}{\varepsilon}\right) \frac{\mathrm{d}}{\mathrm{d}x} u^\varepsilon = f(x), \\ u^\varepsilon(0) = 0, \quad u^\varepsilon(1) = 1, \end{cases} \tag{1}$$

where $a(\omega, y)$ is a stationary ergodic random process, $0 < c_1 < a(\omega, y) < c_2 < \infty$.

It is well known (see, for instance, [3]) that under these conditions $u^\varepsilon(x)$ converges as $\varepsilon \to 0$ in $H^1(0,1)$ for a.e. ω to $u_0(x)$, a purely deterministic solution of the so-called homogenized problem

$$\begin{cases} \bar{a} \frac{\mathrm{d}^2}{\mathrm{d}x^2} u^0(x) = f(x), \\ u^0(0) = 0, \quad u^0(1) = 1, \end{cases} \tag{2}$$

with $\bar{a} = (\mathbf{E}\{1/a(\omega,0)\})^{-1}$, where $\mathbf{E}\{\cdot\}$ denotes the mathematical expectation. Moreover, due to the only one dimension of the space, the solution of (1) could be computed explicitly:

$$u^\varepsilon(x) = \int_0^x \frac{F(s)}{a(\omega, s/\varepsilon)} \,\mathrm{d}s - \left[\left(\int_0^1 \frac{F(s)}{a(\omega, s/\varepsilon)} \,\mathrm{d}s - 1 \right) \bigg/ \int_0^1 \frac{\mathrm{d}s}{a(\omega, s/\varepsilon)} \right] \int_0^x \frac{\mathrm{d}s}{a(\omega, s/\varepsilon)}, \tag{3}$$

where we used the notation $F(s) = \int_0^s f(z)\,\mathrm{d}z$. Let us denote

$$A(\omega, s/\varepsilon) = \frac{1}{a(\omega, s/\varepsilon)} - \mathbf{E}\left\{ \frac{1}{a(\omega, 0)} \right\}, \quad \varepsilon > 0. \tag{4}$$

Our aim is now to find the limiting distribution of $(u^\varepsilon - u^0)$. For this, we should estimate the limiting distribution of all the terms forming u^ε in (3). To this end, we introduce the σ-algebras $\mathcal{F}_t = \sigma\{a(\omega, t)\}$, $\mathcal{F}_{\leqslant t} = \sigma\{a(\omega, x),\ x \leqslant t\}$ and $\mathcal{F}_{\geqslant t} = \sigma\{a(\omega, x),\ x \geqslant t\}$ and define the following mixing conditions:

Definition of mixing conditions

Following, respectively, [6], [1] and [4] we give the following definitions:

Definition 2.1. A family of σ-algebras $\mathcal{F}_t$, $0 \leqslant t \leqslant \infty$, defined in a probability space $(\Omega, \mathcal{F}, P)$, satisfies a uniformly strong mixing condition with coefficient $\varphi(d) = cd^{-\alpha}$ if the inequality

$$\left| \mathbf{E}\{\xi\eta\} - \mathbf{E}\{\xi\}\mathbf{E}\{\eta\} \right| \leqslant \varphi(d) \mathbf{E}\{\xi^2\}^{1/2} \mathbf{E}\{\eta^2\}^{1/2} \tag{5}$$

holds for any random variables ξ and η measurable with respect to $\mathcal{F}_{\leqslant t}$ and to $\mathcal{F}_{\geqslant t+d}$, respectively.

A family $\mathcal{F}_t$ is said to satisfy the strong mixing condition with coefficient $\phi(d) = cd^{-\alpha}$ if

$$\left|\mathbf{E}\{\xi\eta\} - \mathbf{E}\{\xi\}\mathbf{E}\{\eta\}\right| \leqslant \phi(d) \tag{6}$$

for any $\mathcal{F}_{\leqslant t}$-measurable ξ, $|\xi| \leqslant 1$, and $\mathcal{F}_{\geqslant t+d}$-measurable η, $|\eta| \leqslant 1$. A random process X_t is said to satisfy the uniformly strong (respectively, strong) mixing condition if the corresponding family of σ-algebras $\mathcal{F}_t = \sigma(X_t)$ satisfies the (uniformly) strong mixing condition.

Remark 1. It should be noted that the above mixing conditions are sometimes defined in a slightly different form as, for instance, in [3, Section 9.2], where:

– a uniformly strong mixing condition with coefficient $\varphi(d)$ holds if the inequality

$$\left|\mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B)\right| \leqslant \varphi(d)\mathbf{P}(A) \tag{5'}$$

is valid for any $\mathcal{F}_{\leqslant t}$-measurable event A and for any $\mathcal{F}_{\geqslant t+d}$-measurable event B;

– a strong mixing condition with coefficient $\phi(s)$ is satisfied if

$$\left|\mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B)\right| \leqslant \phi(d). \tag{6'}$$

These conditions (5′) and (6′) are close but not equivalent to (5) and (6), respectively. However, all the statements formulated below are valid both under (5) and (5′) (respectively, (6) and (6′)), so in what follows we will identify these conditions. The coefficient $\varphi(d)$ in (5) is sometimes called "maximal correlation coefficient".

Moreover, it is clear that condition (6) is weaker than (5).

3. Mixing condition with $\alpha > 1$

Now, under the assumption that $\alpha > 1$ in (5′)–(6′), we estimate the distribution of u^ε in (3); namely, we investigate the distribution of the difference $u^\varepsilon - u^0$. We start with the term $\int_0^x A(\omega, s/\varepsilon)\,\mathrm{d}s$ and obtain the first lemma.

Lemma 3.1. *Let the process $a(\omega, s)$ satisfy either the uniformly strong mixing condition with a coefficient $\varphi(d) = cd^{-\alpha}$, $\alpha > 1$, or the strong mixing condition with a coefficient $\phi(d) = cd^{-\alpha}$, $\alpha > 2$, and let $A(\omega, s)$ be defined by* (4)*. Then, the distribution of the process*

$$M_x^\varepsilon(\omega) = \frac{1}{\sqrt{\varepsilon}} \int_0^x A\left(\omega, \frac{s}{\varepsilon}\right) F(s)\,\mathrm{d}s,$$

as an element of the space $C(0,1)$, converges in law, as $\varepsilon \to 0$, to the distribution of the Gaussian martingale

$$M_x^0(\omega) = \int_0^x \sigma F(s)\,\mathrm{d}w_s,$$

where w_s is the standard Wiener process and σ, the variance, is defined as follows:

$$\sigma^2 = \mathbf{E}\left\{ \int_0^\infty A(\omega,0)A(\omega,s)\,\mathrm{d}s \right\}.$$

Proof. This statement is a direct consequence of Theorem 9.6.2 and Lemmas 9.6.2 and 9.6.3 in [4]. □

Similarly, the distribution of the process

$$Y_x^\varepsilon(\omega) = \frac{1}{\sqrt{\varepsilon}} \int_0^x A\left(\omega, \frac{s}{\varepsilon}\right) \mathrm{d}s$$

in (3) converges in law towards the distribution of the random process σw_x. Moreover, the joint distribution of M_x^ε and Y_x^ε converges as $\varepsilon \to 0$ to the joint distribution of M_x^0 and σw_x with the same Wiener process.

Clearly, the random variables

$$\frac{1}{\sqrt{\varepsilon}} \int_0^1 A\left(\omega, \frac{x}{\varepsilon}\right) F(s)\,\mathrm{d}s \quad \text{and} \quad \frac{1}{\sqrt{\varepsilon}} \int_0^1 A\left(\omega, \frac{x}{\varepsilon}\right) \mathrm{d}s$$

converge in law, respectively, to $M_1^0(\omega)$ and to σw_1.

By the definitions of $A(\omega,t)$, M_x^ε and Y_x^ε, we have

$$\int_0^x \frac{F(s)\,\mathrm{d}s}{a(\omega, s/\varepsilon)} = \int_0^x \mathbf{E}\left\{\frac{1}{a(\omega,0)}\right\} F(s)\,\mathrm{d}s + \sqrt{\varepsilon} M_x^\varepsilon,$$

$$\int_0^x \frac{\mathrm{d}s}{a(\omega, s/\varepsilon)} = \mathbf{E}\left\{\frac{1}{a(\omega,0)}\right\} x + \sqrt{\varepsilon} Y_x^\varepsilon.$$

Substituting these two last relations in (3) and applying Lemma 3.1, we get finally the following result:

Theorem 3.1. *The normalized difference* $(1/\sqrt{\varepsilon})(u^\varepsilon - u^0)$ *converges in law in* $C(0,1)$, *as* $\varepsilon \to 0$, *as folllows*:

$$\frac{1}{\sqrt{\varepsilon}}\left(u^\varepsilon - u^0\right) \xrightarrow{d} M_x^0 + xM_1^0 - \sigma x \int_0^1 \mathbf{E}\left\{\frac{1}{a(\omega,0)}\right\} F(s)\,\mathrm{d}s\, w_1 + \sigma\left(\int_0^1 F(s)\,\mathrm{d}s - 1\right) w_x. \tag{7}$$

Relation (7) implies that the typical deviation of u^ε from u^0 is of order $\sqrt{\varepsilon}$ and that for any $\delta > 0$ the inequality

$$\mathbf{P}\left\{\frac{1}{\sqrt{\varepsilon}} \left\|u^\varepsilon - u^0\right\|_{C(0,1)} \geqslant \frac{1}{\delta}\right\} \leqslant \exp\left(-\frac{c}{\delta^2}\right) \tag{8}$$

holds for all sufficiently small ε, with the constant c depending only on σ.

4. Mixing condition with small α

Now, in order to consider the case when the power α in (5′) is less than 1 (respectively, $\alpha \leqslant 2$ in (6′)), we rescale the space argument $y = x/\varepsilon^\beta$ and denote by $\mathcal{A}^\varepsilon(x)$ the rescaled process $\mathcal{A}^\varepsilon(x) = A(x/\varepsilon^\beta)$ and by $\mathcal{F}_t^\varepsilon$ the corresponding σ-algebra $\sigma(\mathcal{A}^\varepsilon(t))$. Then, for the process $\mathcal{A}^\varepsilon(x)$, the mixing conditions (5′) and (6′) become

$$\bigl|\mathbf{E}\{\xi\eta\} - \mathbf{E}\{\xi\}\mathbf{E}\{\eta\}\bigr| \leqslant c\varepsilon^{\alpha\beta} d^{-\alpha} \mathbf{E}\{\xi^2\}^{1/2}\mathbf{E}\{\eta^2\}^{1/2} \tag{9}$$

for any $\mathcal{F}^\varepsilon_{\leqslant t}$-measurable ξ and $\mathcal{F}^\varepsilon_{\geqslant t+d}$-measurable η, and, respectively,

$$\bigl|\mathbf{E}\{\xi\eta\} - \mathbf{E}\{\xi\}\mathbf{E}\{\eta\}\bigr| \leqslant c\varepsilon^{\alpha\beta} d^{-\alpha} \tag{10}$$

for any $\mathcal{F}^\varepsilon_{\leqslant t}$-measurable ξ, $|\xi| \leqslant 1$, and $\mathcal{F}^\varepsilon_{\geqslant t+d}$-measurable η, $|\eta| \leqslant 1$.

Now, choosing $\beta = 1/\alpha - 1 + \delta$ with δ arbitrary small positive number, we get

$$\begin{aligned} \frac{1}{\varepsilon}\mathbf{E}\biggl\{\int_0^1 \mathcal{A}^\varepsilon(0)\mathcal{A}^\varepsilon\Bigl(\frac{s}{\varepsilon}\Bigr)\,\mathrm{d}s\biggr\} &= \frac{1}{\varepsilon}\mathbf{E}\biggl\{\int_0^1 A(0)A\Bigl(\frac{t}{\varepsilon^{1+\beta}}\Bigr)\,\mathrm{d}t\biggr\} = \varepsilon^\beta \mathbf{E}\biggl\{\int_0^{\varepsilon^{-(1+\beta)}} A(0)A(s)\,\mathrm{d}s\biggr\} \\ &\leqslant \varepsilon^\beta \int_0^{\varepsilon^{-(1+\beta)}} s^{-\alpha}\,\mathrm{d}s \leqslant \varepsilon^{\alpha\delta} \end{aligned} \tag{11}$$

and, hence,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbf{E}\biggl\{\int_0^1 \mathcal{A}^\varepsilon(0)\mathcal{A}^\varepsilon\Bigl(\frac{s}{\varepsilon}\Bigr)\,\mathrm{d}s\biggr\} = 0. \tag{12}$$

To prove the next assertion, one can use (12) and a very simplified version of the proof of Theorem 9.6.2 and Lemmas 9.6.2 and 9.6.3 in [4] (see, also, the proof of Lemma 4.2 below).

Lemma 4.1. *Under either the uniform strong mixing condition* (5′) *with* $\phi(d) = cd^{-\alpha}$, $\alpha \leqslant 1$, *or the strong mixing condition* (6′) *with* $\phi(d) = cd^{-\alpha}$, $\alpha \leqslant 2$, *the family of processes*

$$\eta_x^\varepsilon(\omega) = \frac{1}{\sqrt{\varepsilon}}\int_0^x F(s)\mathcal{A}^\varepsilon\Bigl(\frac{s}{\varepsilon}\Bigr)\,\mathrm{d}s = \frac{1}{\sqrt{\varepsilon}}\int_0^x F(s)A\Bigl(\frac{s}{\varepsilon^{1+\beta}}\Bigr)\,\mathrm{d}s$$

converges in probability in the space $C(0,1)$, *as* $\varepsilon \to 0$, *to the process* $\eta_x(\omega) \equiv 0$.

Remark 2. The statement of Lemma 4.1 could be read as follows: the family of measures Q^ε generated in the space $C(0,1)$ by the process $\eta_x^\varepsilon(\omega)$ converges weakly, as $\varepsilon \to 0$, to a δ-type measure concentrated on the function $\eta_x \equiv 0$.

By Lemma 4.1, introducing a new small parameter $\varepsilon = \mu^{1+\beta}$, with β defined as above, in the same way as in Lemma 3.1 we find

$$\lim_{\varepsilon\to 0}\mathbf{P}\biggl\{\max_x\biggl|\frac{1}{\varepsilon^{\alpha/2-\delta}}\int_0^x F(s)A\Bigl(\frac{s}{\varepsilon}\Bigr)\,\mathrm{d}s\biggr| > C\biggr\}$$

$$
\begin{aligned}
&= \lim_{\mu\to 0} \mathbf{P}\left\{ \max_x \left| \frac{1}{\mu^{(\alpha/2-\delta)(\beta+1)}} \int_0^x F(s)A\left(\frac{s}{\mu^{1+\beta}}\right) \mathrm{d}s \right| > C \right\} \\
&= \lim_{\mu\to 0} \mathbf{P}\left\{ \max_x \left| \frac{1}{\mu^{1/2-\delta(1/\alpha-\alpha/2)-\delta^2}} \int_0^x F(s)A\left(\frac{s}{\mu^{1/\alpha+\delta}}\right) \mathrm{d}s \right| > C \right\} = 0 \qquad (13)
\end{aligned}
$$

for any $C > 0$ and any small positive number δ.

Using representation (3) for u^ε, a similar representation for u^0 and applying the same arguments as in Theorem 3.1, we deduce from (13) the following convergence result.

Theorem 4.1. *For any $C > 0$ and any small positive δ, the relation*

$$
\lim_{\varepsilon\to 0} \mathbf{P}\left\{ \max_x |u^\varepsilon - u^0| > C\varepsilon^{\alpha/2-\delta} \right\} = \lim_{\varepsilon\to 0} \mathbf{P}\left\{ \max_x \left| \frac{u^\varepsilon - u^0}{\varepsilon^{\alpha/2-\delta}} \right| > C \right\} = 0 \qquad (14)
$$

holds.

Our next step is to estimate the expectation of $\max_x |u^\varepsilon(x) - u^0(x)| = \|u^\varepsilon - u^0\|_{C(0,1)}$ for small ε. For this aim we prove the following lemma.

Lemma 4.2. *If the uniform strong mixing condition* (5′) *with $\varphi(d) = cd^{-\alpha}$ is satisfied, then, according to the range of α, we have*

$$
\mathbf{E}\left\{ \max_x \left(u^\varepsilon(x) - u^0(x)\right)^2 \right\} \leqslant \begin{cases} c(\alpha)\varepsilon & \text{for } \alpha > 1, \\ c\varepsilon|\ln\varepsilon| & \text{for } \alpha = 1, \\ c(\alpha)\varepsilon^\alpha & \text{for } \alpha < 1. \end{cases} \qquad (15)
$$

If the strong mixing condition (6′) *with $\phi(d) = cd^{-\alpha}$ is satisfied, then*

$$
\mathbf{E}\left\{ \max_x \left(u^\varepsilon(x) - u^0(x)\right)^2 \right\} \leqslant \begin{cases} c(\alpha)\varepsilon & \text{for } \alpha > 2, \\ c(\alpha)\varepsilon^{\alpha/2} & \text{for } \alpha \leqslant 2. \end{cases} \qquad (16)
$$

Proof. We consider the case when (5′) holds with $\varphi(d) = cd^{-\alpha}$ and $\alpha > 1$. The other cases can be studied in the same way.

First of all, let us represent the integral $\int_0^1 F(s)A(s/\varepsilon)\,\mathrm{d}s$ in the following form:

$$
\int_0^1 F(s)A\left(\frac{s}{\varepsilon}\right) \mathrm{d}s = \varepsilon \int_0^{1/\varepsilon} F(\varepsilon s)A(s)\,\mathrm{d}s = \varepsilon \sum_{k=0}^{1/(2\sqrt{\varepsilon})} \eta_{2k} + \varepsilon \sum_{k=0}^{1/(2\sqrt{\varepsilon})} \eta_{2k+1},
$$

where

$$
\eta_k = \int_{k/\sqrt{\varepsilon}}^{(k+1)/\sqrt{\varepsilon}} F(\varepsilon s)A(s)\,\mathrm{d}s, \qquad (17)
$$

and define the processes $\xi'_n = \sum_{k=0}^n \eta_{2k}$ and $\xi''_n = \sum_{k=0}^n \eta_{2k+1}$.

Due to the uniform boundedness of $F(s)$ and $A(\omega, s)$ and the fact that the length of the integration intervals in (17) is $1/\sqrt{\varepsilon}$, in order to obtain the first estimates in (15), it suffices to prove that

$$\mathbf{E}\Big\{\varepsilon^2 \max_{n\leqslant 1/2\sqrt{\varepsilon}} \big({\xi'_n}^2 + {\xi''_n}^2\big)\Big\} \leqslant c\varepsilon. \tag{18}$$

To this end we define a random variable $\zeta_k = \mathbf{E}\{\eta_{2k}|\mathcal{F}_{\leqslant(2k-1)/\sqrt{\varepsilon}}\}$, where $\mathbf{E}\{\cdot|\mathcal{H}\}$ stands for the conditional expectation with respect to the σ-algebra $\mathcal{H}$, and note that in view of $\mathcal{F}_{\leqslant(2k-1)/\sqrt{\varepsilon}}$-measurability of ζ_k, condition (5′) with $d = 1/\sqrt{\varepsilon}$ implies the inequality

$$\begin{aligned}\mathbf{E}\{\zeta_k^2\} &= \mathbf{E}\big\{\big(\mathbf{E}\{\eta_{2k}|\mathcal{F}_{\leqslant(2k-1)/\sqrt{\varepsilon}}\}\big)^2\big\} = \mathbf{E}\big(\mathbf{E}\{\zeta_k\eta_{2k}|\mathcal{F}_{\leqslant(2k-1)/\sqrt{\varepsilon}}\}\big)\\ &= \mathbf{E}\{\zeta_k\eta_{2k}\} \leqslant \varphi\left(\frac{1}{\sqrt{\varepsilon}}\right)\big(\mathbf{E}\{\zeta_k^2\}\big)^{1/2}\big(\mathbf{E}\{\eta_{2k}^2\}\big)^{1/2}.\end{aligned}$$

Therefore,

$$\mathbf{E}\{\zeta_k^2\} \leqslant \varphi^2\left(\frac{1}{\sqrt{\varepsilon}}\right)\mathbf{E}\{\eta_k^2\} \leqslant c^2\varepsilon^\alpha\mathbf{E}\{\eta_k^2\}. \tag{19}$$

The next step is to estimate the expectation of η_{2k}^2 and ${\xi'_n}^2$. Denote by $R(s)$ the correlation function $\mathbf{E}\{A(\omega,0)A(\omega,s)\}$ of the process $A(\omega, s)$. Clearly, under condition (5′) or (6′), $R(s) \leqslant c|s|^{-\alpha}$, and we find

$$\begin{aligned}\mathbf{E}\{\eta_{2k}^2\} &= \mathbf{E}\bigg\{\bigg(\int_{(2k)/\sqrt{\varepsilon}}^{(2k+1)/\sqrt{\varepsilon}} F(\varepsilon s)A(s)\,\mathrm{d}s\bigg)^2\bigg\}\\ &= \int_0^{1/\sqrt{\varepsilon}}\int_0^{1/\sqrt{\varepsilon}} F\left(\varepsilon\left(s+\frac{2k}{\sqrt{\varepsilon}}\right)\right)F\left(\varepsilon\left(t+\frac{2k}{\sqrt{\varepsilon}}\right)\right)R(t-s)\,\mathrm{d}s\,\mathrm{d}t\\ &\leqslant c\int_0^{1/\sqrt{\varepsilon}}\int_0^{1/\sqrt{\varepsilon}} R(t-s)\,\mathrm{d}s\,\mathrm{d}t \leqslant \frac{c}{\sqrt{\varepsilon}},\end{aligned} \tag{20}$$

where the inequality $\alpha > 1$ has also been used. Hence,

$$\mathbf{E}\big\{(\varepsilon\eta_{2k})^2\big\} \leqslant c\varepsilon^{3/2}. \tag{21}$$

Similarly,

$$\mathbf{E}\big\{\big(\varepsilon\xi'_n\big)^2\big\} \leqslant c\varepsilon, \quad n = 0, 1, 2, \dots, \frac{1}{2\sqrt{\varepsilon}}. \tag{22}$$

From (19) and (21),

$$\mathbf{E}\big\{(\varepsilon\zeta_k)^2\big\} \leqslant c\varepsilon^{\alpha+3/2}. \tag{23}$$

Summing the last inequalities over $k \leqslant 1/(2\sqrt{\varepsilon})$, we obtain

$$\mathbf{E}\left\{\left(\sum_{k=1}^{1/(2\sqrt{\varepsilon})} \varepsilon\zeta_k\right)^2\right\} \leqslant \frac{1}{2\sqrt{\varepsilon}}\mathbf{E}\left\{\sum_{k=1}^{1/(2\sqrt{\varepsilon})} (\varepsilon\zeta_k)^2\right\} \leqslant \frac{1}{4\varepsilon}c\varepsilon^{\alpha+3/2} = c\varepsilon^{\alpha+1/2}. \tag{24}$$

Similarly,

$$\mathbf{E}\left\{\left(\sum_{k=1}^{n} \varepsilon\zeta_k\right)^2\right\} \leqslant c\varepsilon^{\alpha+1/2}, \quad 0 \leqslant n \leqslant \frac{1}{2\sqrt{\varepsilon}}, \tag{25}$$

where c does not depend on n.

Now, estimating $\{\max_{n\leqslant 1/(2\sqrt{\varepsilon})}(\sum_{k=1}^n \varepsilon\zeta_k)^2\}$ by the sum $\{\sum_{n=0}^{1/(2\sqrt{\varepsilon})}(\sum_{k=1}^n \varepsilon\zeta_k)^2\}$, we deduce from (25) the following relation:

$$\mathbf{E}\left\{\max_{n\leqslant 1/(2\sqrt{\varepsilon})}\left(\sum_{k=1}^{n} \varepsilon\zeta_k\right)^2\right\} \leqslant c\varepsilon^{\alpha}. \tag{26}$$

By the definition of ζ_k, the process $\mathcal{N}_n = \sum_{k=1}^n \varepsilon(\eta_{2k} - \zeta_k)$ is a martingale. Thus, by the Doob inequality for martingales [4, Section 1.9], from (22) and (26) we have

$$\begin{aligned}
\mathbf{E}\left\{\max_{n\leqslant 1/(2\sqrt{\varepsilon})}\left(\sum_{k=1}^{n} \varepsilon(\eta_{2k} - \zeta_k)\right)^2\right\} &\leqslant 4\mathbf{E}\left\{\left(\sum_{k=1}^{1/(2\sqrt{\varepsilon})} \varepsilon(\eta_{2k} - \zeta_k)\right)^2\right\} \\
&\leqslant 8\left[\mathbf{E}\left\{\left(\sum_{k=1}^{1/(2\sqrt{\varepsilon})} \varepsilon\eta_{2k}\right)^2\right\} + \mathbf{E}\left\{\left(\sum_{k=1}^{1/(2\sqrt{\varepsilon})} \varepsilon\zeta_k\right)^2\right\}\right] \leqslant c\big(\varepsilon + \varepsilon^{\alpha}\big).
\end{aligned}$$

Finally, in view of (26), the last inequality implies

$$\mathbf{E}\left\{\max_{n\leqslant 1/(2\sqrt{\varepsilon})}\big(\varepsilon\xi'_n\big)^2\right\} \leqslant 2\mathbf{E}\left\{\max_{n\leqslant 1/(2\sqrt{\varepsilon})}\left(\sum_{k=1}^{n} \varepsilon(\eta_{2k} - \zeta_k)\right)^2 + \max_{n\leqslant 1/(2\sqrt{\varepsilon})}\left(\sum_{k=1}^{n} \varepsilon\zeta_k\right)^2\right\} \leqslant c\varepsilon.$$

The estimate for ξ''_n can be obtained in the same way, and then the lemma is proved. □

Now, from Lemma 4.2, applying the Chebyshev inequality, we get the upper bound for the probabilities $\mathbf{P}\{\max_x |u^\varepsilon(x) - u^0(x)| > \varepsilon^\gamma\}$, $0 < \gamma < 1/2$:

Corollary 4.1. *Under the uniform strong mixing condition* (5′) *with* $\varphi(d) = cd^{-\alpha}$, *depending on the value of* α, *we have*

$$\mathbf{P}\Big\{\max_x\big|u^\varepsilon(x) - u^0(x)\big| > \varepsilon^\gamma\Big\} \leqslant \begin{cases} c(\alpha)\varepsilon^{1-2\gamma} & \textit{if } \alpha > 1,\ \forall\gamma < 1/2, \\ c\varepsilon^{1-2\gamma}|\ln\varepsilon| & \textit{if } \alpha = 1,\ \forall\gamma < 1/2, \\ c(\alpha)\varepsilon^{\alpha-2\gamma} & \textit{if } \alpha < 1,\ \forall\gamma < \alpha/2. \end{cases}$$

Under the strong mixing condition (6′) *with* $\phi(d) = cd^{-\alpha}$*, depending on the value of* α*, we have*

$$\mathbf{P}\Big\{ \max_x |u^\varepsilon(x) - u^0(x)| > \varepsilon^\gamma \Big\} \leqslant \begin{cases} c(\alpha)\varepsilon^{1-2\gamma} & \text{if } \alpha > 2,\ \forall \gamma < 1/2, \\ c(\alpha)\varepsilon^{\alpha/2-2\gamma} & \text{if } \alpha \leqslant 2,\ \forall \gamma < \alpha/4. \end{cases}$$

5. Exponential estimates and moderate deviation principle

The estimates of the previous sections are applicable in rather general cases, but they are not always sharp enough, specially when the mixing coefficient decays rapidly. In that case more precise estimates can be obtained. In order to characterize a proper class of processes, below we define the moderate exponential mixing conditions M^κ which are a slightly weakened version of the conditions F^κ in [1, Section 7.7].

Definition 5.1. We say that a family of processes ζ_t^ε satisfies the moderate exponential mixing conditions M^κ if

$$\limsup_{\varepsilon\to 0} \varepsilon^{1-2\kappa} \ln\Big[\max_{0\leqslant x\leqslant 1} \mathbf{E}\Big\{ \exp\Big| \varepsilon^{\kappa-1} \int_0^x \zeta_t^\varepsilon \, dt \Big| \Big\} \Big] < \infty. \tag{27}$$

For the reader convenience, we recall the definition of moderate deviation conditions F^κ (see [1, Section 7.7]).

Definition 5.2. The moderate deviation condition F^κ is satisfied if there is a continuous positive function $\mathcal{C}(s)$ such that for any step function $\psi(s)$ the relation

$$\lim_{\varepsilon\to 0} \varepsilon^{1-2\kappa} \ln \mathbf{E}\Big\{ \exp\Big(\varepsilon^{\kappa-1} \int_0^x \psi(s)\zeta_t^\varepsilon \, dt \Big) \Big\} = \frac{1}{2} \int_0^1 \mathcal{C}(s)\psi^2(s) \, ds \tag{28}$$

holds, and if there exists a positive function $\theta(t)$, $\lim_{t\to 0} \theta(t) = 0$, and $t_0 > 0$ such that

$$\limsup_{\varepsilon\to 0} \max_{\substack{\varepsilon\leqslant t<t_0 \\ 0\leqslant h\leqslant 1-t}} \varepsilon^{1-2\kappa} \ln\Big| \mathbf{E}\Big\{ \exp\Big(\pm\frac{\varepsilon^{\kappa-1}}{\theta(t)} \int_h^{h+t} \zeta_t^\varepsilon \, dt \Big) \Big\} \Big| < \infty. \tag{29}$$

The next lemma will be a basic tool in obtaining exponential estimates of the probabilities $\mathbf{P}\{ \max_{0\leqslant x\leqslant 1} |u^\varepsilon(x) - u^0(x)| \geqslant c\varepsilon^\kappa \}$.

Lemma 5.1. *Let a process* ζ_t^ε *satisfy the condition* M^κ, $0 < \kappa < 1/2$*. Then, there are* $c_0 > 0$ *and* $c_1 > 0$ *such that the estimate*

$$\mathbf{P}\Big\{ \max_{0\leqslant t\leqslant 1} \Big| \int_0^t \zeta_s^\varepsilon \, ds \Big| \geqslant c_0\varepsilon^\kappa \Big\} \leqslant \exp\big(-c_1\varepsilon^{2\kappa-1}\big) \tag{30}$$

is valid for all sufficiently small ε*.*

Proof. Let us first remark that the condition M^κ implies, for all sufficiently small ε, the uniform in $x \in (0,1)$ inequality

$$\mathbf{E}\left\{ \exp\left|\varepsilon^{\kappa-1}\int_0^x \zeta_t^\varepsilon \,\mathrm{d}t\right|\right\} < \exp\left(C\varepsilon^{2\kappa-1}\right). \tag{31}$$

Let us denote $\xi_t^\varepsilon = \int_0^t \zeta_s^\varepsilon \,\mathrm{d}s$; now, by the exponential Chebyshev inequality and (31), we get

$$\begin{aligned}\mathbf{P}\left\{|\xi_t^\varepsilon| \geqslant c_0\varepsilon^\kappa\right\} &= \mathbf{P}\left\{\frac{|\xi_t^\varepsilon|}{\varepsilon^{1-\kappa}} \geqslant c_0\varepsilon^{2\kappa-1}\right\} = \mathbf{P}\left\{\exp\left(\frac{|\xi_t^\varepsilon|}{\varepsilon^{1-\kappa}}\right) \geqslant \exp\left(c_0\varepsilon^{2\kappa-1}\right)\right\}\\ &\leqslant \exp\left(-c_0\varepsilon^{2\kappa-1}\right)\mathbf{E}\left\{\exp\left(\frac{|\xi_t^\varepsilon|}{\varepsilon^{1-\kappa}}\right)\right\} \leqslant \exp\left((C-c_0)\varepsilon^{2\kappa-1}\right)\end{aligned}$$

uniformly in $x \in (0,1)$, for all sufficiently small ε. Under proper choice of c_0 this implies the statement of Lemma 5.1. □

Applying this last lemma to each term on the right-hand side of (3), we obtain the following

Corollary 5.1. *Let both $A(s/\varepsilon)$ and $F(s)A(s/\varepsilon)$ satisfy the condition M^κ for some κ, $0 < \kappa < 1/2$. Then, there are constants $c_0 > 0$ and $c_1 > 0$ such that the inequality*

$$\mathbf{P}\left\{\max_{0\leqslant x\leqslant 1}\left|u^\varepsilon(x) - u^0(x)\right| \geqslant c_0\varepsilon^\kappa\right\} \leqslant \exp\left(-c_1\varepsilon^{2\kappa-1}\right) \tag{32}$$

holds for all sufficiently small ε.

Proof. If we set $x_k = k\varepsilon^\kappa$, $k = 1, 2, \ldots, 1/\varepsilon^\kappa$, then, in view of the uniform boundedness of $A(s)$ and $F(s)$, we get

$$\max_{0\leqslant x\leqslant 1}\left|\int_0^x F(s)A\left(\frac{s}{\varepsilon}\right)\mathrm{d}s\right| \leqslant \max_k\left|\int_0^{x_k} F(s)A\left(\frac{s}{\varepsilon}\right)\mathrm{d}s\right| + c\varepsilon^\kappa.$$

By Lemma 5.1, using the evident inequality

$$\mathbf{P}\left\{\max_k\left|\int_0^{x_k} F(s)A\left(\frac{s}{\varepsilon}\right)\mathrm{d}s\right| \geqslant c_0\varepsilon^\kappa\right\} \leqslant \sum_k \mathbf{P}\left\{\left|\int_0^{x_k} F(s)A\left(\frac{s}{\varepsilon}\right)\mathrm{d}s\right| \geqslant c_0\varepsilon^\kappa\right\},$$

we obtain the desired estimate. □

Remark 3. It should be noted that under a moderate deviation condition F^κ, not only the estimate of Lemma 5.1 could be obtained, but also the exact logarithmic asymptotics for the probability that $\varepsilon^\kappa\xi_t^\varepsilon = \varepsilon^\kappa\int_0^x \zeta_s^\varepsilon \,\mathrm{d}s$ belongs to a small neighbourhood of any absolutely continuous function. Namely, for any absolutely continuous function $g(\cdot)$ and any $\delta > 0$ we have for all sufficiently small ε:

$$\exp\left(-\frac{S(g(\cdot)) + \delta}{\varepsilon^{1-2\kappa}}\right) \leqslant \mathbf{P}\left\{\max_{0\leqslant x\leqslant 1}\left|\varepsilon^\kappa\xi_t^\varepsilon - g(t)\right| \leqslant \delta\right\} \leqslant \exp\left(-\frac{S(g(\cdot)) - \delta}{\varepsilon^{1-2\kappa}}\right);$$

here the rate function $S(g(\cdot))$ associated to the process $\varepsilon^\kappa \xi_t^\varepsilon$ is defined by the formula (see [4, Section 7.7, Theorem 7.1])

$$S\big(g(\cdot)\big) = \int_0^1 \mathcal{C}^{-1}(s)\big(\dot g(s)\big)^2 \, \mathrm{d}s.$$

6. Exponential estimates. Examples

Most of the results of previous sections are based on assumptions that, in general, could not be easily verified. In this section we give several sufficient conditions that provide a moderate exponential mixing M^κ and we derive some consequences from Lemma 5.1.

Proposition 6.1. *Suppose that the process $a(\omega, s)$ satisfies condition* (5) *or* (6) *and that the corresponding coefficient $\varphi(d)$* (*or $\phi(d)$*) *satisfies the estimate*

$$\varphi(d) \leqslant \exp\big(-cd^s\big) \tag{33}$$

for some $s > 1$ and $c > 0$. Then, for all κ, $1/(1+s) < \kappa < 1/2$, both the processes $A(s/\varepsilon)$ and $F(s)A(s/\varepsilon)$ satisfy the moderate mixing condition M^κ.

Proof. We rewrite the integral $\varepsilon^{\kappa-1}\int_0^1 F(s)A(s/\varepsilon)\,\mathrm{d}s$ in the following form:

$$\varepsilon^{\kappa-1}\int_0^1 F(s)A(s/\varepsilon)\,\mathrm{d}s = \varepsilon^\kappa \int_0^{1/\varepsilon} F(\varepsilon s)A(s)\,\mathrm{d}s = \sum_{k=0}^{\varepsilon^{\kappa-\delta-1}/2} \eta_{2k} + \sum_{k=0}^{\varepsilon^{\kappa-\delta-1}/2} \eta_{2k+1} = \xi' + \xi'',$$

where

$$\eta_k = \varepsilon^\kappa \int_{k\varepsilon^{\delta-\kappa}}^{(k+1)\varepsilon^{\delta-\kappa}} F(\varepsilon s)A(s)\,\mathrm{d}s.$$

Denoting $N(\varepsilon) = (\varepsilon^{\kappa-\delta-1}/2)$, from (5′) and the relation $\kappa > 1/(1+s)$ we have

$$\begin{aligned}
&\Big|\mathbf{E}\big\{\exp(2\xi')\big\} - \mathbf{E}\big\{\exp(2\eta_0)\big\}\mathbf{E}\big\{\exp(2\eta_2 + 2\eta_4 + \cdots + 2\eta_{2N(\varepsilon)})\big\}\Big| \\
&\quad = \Big|\mathbf{E}\big\{\exp(2\eta_0)\big(\exp(2\eta_2 + 2\eta_4 + \cdots + 2\eta_{2N(\varepsilon)}) - \mathbf{E}\big\{\exp(2\eta_2 + 2\eta_4 + \cdots + 2\eta_{2N(\varepsilon)})\big\}\big)\big\}\Big| \\
&\quad \leqslant \varphi\big(\varepsilon^{\delta-\kappa}\big)\big(\mathbf{E}\big\{\exp(4\eta_0)\big\}\big)^{1/2}\big(\mathbf{E}\big\{\exp(4\eta_2 + 4\eta_4 + \cdots + 4\eta_{2N(\varepsilon)})\big\}\big)^{1/2} \\
&\quad \leqslant C\exp\big(-c\varepsilon^{s(\delta-\kappa)}\big)\exp\big(c'\varepsilon^{\kappa-1}\big) \leqslant C\exp\big(c'\varepsilon^{\kappa-1} - c\varepsilon^{s(\delta-\kappa)}\big) \leqslant C\exp\big(-(c/2)\varepsilon^{s(\delta-\kappa)}\big)
\end{aligned}$$

for all sufficiently small ε.

Iterating this inequality, we find

$$\Big|\mathbf{E}\big\{\exp(2\xi')\big\} - \prod_{k=0}^{N(\varepsilon)} \mathbf{E}\big\{\exp(2\eta_k)\big\}\Big| \leqslant C\exp\big(-(c/2)\varepsilon^{s(\delta-\kappa)}\big). \tag{34}$$

Then, taking into account the exponential decay of $\varphi(d)$, in the same way as in Lemma 4.2, we obtain

$$\mathbf{E}\{2\eta_k\} = 0, \qquad \mathbf{E}\{(2\eta_k)^2\} \leqslant c\varepsilon^{\kappa+\delta}, \qquad \mathbf{E}\{(2\eta_k)^n\} \leqslant c_n\varepsilon^{n(\kappa+\delta)/2}, \quad n = 3, 4, \ldots. \tag{35}$$

Thus, taking sufficiently large number of terms in the Taylor expansion

$$\exp(2\eta_k) = 1 + 2\eta_k + (2\eta_k)^2/2 + \cdots,$$

we have

$$\ln\bigl(\mathbf{E}\{\exp(2\eta_k)\}\bigr) \leqslant c\varepsilon^{\kappa+\delta}.$$

Summing up over $k \leqslant (\varepsilon^{\kappa-\delta-1}/2)$ gives

$$\varepsilon^{1-2\kappa} \ln\left(\prod_{k=0}^{N(\varepsilon)} \mathbf{E}\{\exp(2\eta_{2k})\}\right) \leqslant \varepsilon^{1-2\kappa} N(\varepsilon) c\varepsilon^{\kappa+\delta} \leqslant c. \tag{36}$$

Similar estimate can be obtained for the odd terms. Finally, by (34), (36) and the Cauchy–Bunyakovski inequality, we get

$$\varepsilon^{1-2\kappa} \ln \mathbf{E}\{\exp(\xi' + \xi'')\} \leqslant \varepsilon^{1-2\kappa}\frac{1}{2}\bigl(\ln \mathbf{E}\{\exp(2\xi')\} + \ln \mathbf{E}\{\exp(2\xi'')\}\bigr) \leqslant c.$$

The case of strong mixing condition can be studied in the same way and proposition is proved. □

In fact, the above arguments can be used to consider the case of smaller power s in (33) as well. Indeed, after introducing a process $\mathcal{A}^\varepsilon(t) = F(t)A(t/\varepsilon^\beta)$, one can prove the following assertion exactly in the same way as Proposition 6.1.

Proposition 6.2. *Suppose that the process $a(s,\omega)$ satisfies condition* (5′) *or* (6′) *and that estimate* (33) *holds for some $s > 0$ and $c > 0$. Then for any $\beta > (1+s)/(2s)$ and for any κ, $\max(0, 1 - s\beta/(1+s)) < \kappa < 1/2$, both processes $A(s/\varepsilon^\beta)$ and $F(s)A(s/\varepsilon^\beta)$ satisfy the moderate exponential mixing condition M^κ.*

Now, using the new small parameter $\mu = \varepsilon^{1/\beta}$ we deduce from the last proposition the following

Proposition 6.3. *Suppose that the process $a(s,\omega)$ satisfies condition* (5) *or* (6) *and that estimate* (33) *holds for some $s > 0$ and $c > 0$. Then, for any $\beta > (1+s)/(2s)$ and any κ, $\max(0, 1/\beta - s/(1+s)) < \kappa < 1/(2\beta)$, there are positive constants c_0 and c_1 such that*

$$\mathbf{P}\left\{\max_{0\leqslant t\leqslant 1}\left|\int_0^t F(s)A\left(\frac{s}{\mu}\right)\mathrm{d}s\right| \geqslant c_0\mu^\kappa\right\} \leqslant \exp\bigl(-c_1\mu^{2\kappa-(1/\beta)}\bigr). \tag{37}$$

Applying now Propositions 6.1–6.3 to each term on the right-hand side of (3), we deduce the following assertion:

Theorem 6.1. *Let the process* $a(t,\omega)$ *satisfy condition* (5) *or* (6) *with the coefficient* $\varphi(d)$ $(\phi(d))$ *obeying estimate* (33) *for some* $s > 0$ *and* $c > 0$. *If* $s > 1$, *then for any* $\kappa \in (1/(s+1), 1/2)$ *there are* $c_0 > 0$ *and* $c_1 > 0$ *such that for all sufficiently small* ε,

$$\mathbf{P}\Big\{ \max_{0\leqslant x\leqslant 1} \big|u^\varepsilon(x) - u^0(x)\big| \geqslant c_0\varepsilon^\kappa \Big\} \leqslant \exp\big(c_1\varepsilon^{1-2\kappa}\big);$$

and if $0 < s \leqslant 1$, *then for each* $\beta > (1+s)/(2s)$ *and each* κ, $\max(0, 1/\beta - s/(1+s)) < \kappa < 1/(2\beta)$, *there are strictly positive constants* c_0 *and* c_1 *such that for any sufficiently small* ε,

$$\mathbf{P}\Big\{ \max_{0\leqslant x\leqslant 1} \big|u^\varepsilon(x) - u^0(x)\big| \geqslant c_0\varepsilon^\kappa \Big\} \leqslant \exp\big(-c_1\varepsilon^{2\kappa-(1/\beta)}\big).$$

Remark 4. If the distribution of $a(t,\omega)$ has a finite correlation length L, then for any $d > L$ estimate (33) trivially holds for any $s > 0$.

Acknowledgement

This work was done during the stay of A. Piatnitski at the University Jean Monnet (Saint-Etienne) whose support is gratefully acknowledged.

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