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On the double porosity model of a single phase flow in random media

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Abstract. We consider the linearized equations of slightly compressible single fluid flow through a highly heterogeneous random porous medium, consisting of two types of material. Due to the high heterogeneity of the two materials the ratio of their permeability coefficients is of order ε^2, where ε is the characteristic scale of heterogeneities. Supposing that the matrix blocks set of the porous medium consists of random stationary inclusions, and assuming positive definiteness of the effective permeability tensor associated to the corresponding Neumann problem for the random fractures system, we obtain the homogenized problem for a random version of the double porosity model used in geohydrology. It includes as a particular case the periodic setting, already studied by homogenization theory methods (see, for example, [1,7]). The homogenized problem is obtained by using the stochastic two scale convergence in the mean, and by means of convergence results specially adapted to our a priori estimates and to the random geometry, which do not require extension of solutions to the matrix part.

Introduction

The question on how to deal with fractured rock domains has been investigated both by geohydrologists and by reservoir engineers in connection with simple or multiphase flow.

More recently, fractured rock domains received increasing attention in connection with the problem of geological isolation of radioactive waste, where, in addition to the transport of mass of fluid phases in single or multiphase flow, the issues of heat transport and mass transport of components have to be addressed.

Of special interest are reservoirs composed of fractured porous rocks in which the blocks surrounded by the network of fractures are porous. The permeability of such blocks is often rather low, but the porosity and hence the storage capacity is very high. The “double porosity” model for fractured porous rock domain was first introduced in the literature by Barenblatt in [2] and at present a large number of articles on these subjects exists in the literature of several scientific and engineering disciplines, including geology, hydrology, petroleum reservoir engineering and environmental engineering; see, for instance, [18, 19,3].

According to the Barenblatt’s dual porosity model we consider a large number of matrix blocks containing most of the stored fluid together with a system of high-conductivity fissures. In this model, the fracture width is considerably greater than the characteristic dimension of the pores and the permeability K^* of the fissure system considerably exceeds the permeability k of the individual blocks of porous media. At the same time, the fissures occupy a smaller volume than the pores, so the ratio of the volume of the fissures to the total volume is smaller than the porosity of any individual block of porous media.

To obtain the double porosity model, the fracture system's local properties are averaged over a volume containing both the fractures and a matrix. The so-called dual-porosity model for a porous medium consists of an equivalent coarse-grained porous medium in which the fissures play the role of "pores" and the blocks of porous media play the role of "grains".

Since flow in the fractures is much more rapid than that inside the matrix, the fluid does not flow directly from one matrix block to another and, finally, only the flow inside fractures combined with the matrix-fractures exchange is possible. The porous-rock matrix system plays the role of a global source term macroscopically distributed over the entire equivalent coarse-grained porous medium.

If we denote by ε the adimensionalized size of a typical block of porous media, then in order to have the same characteristic time scale for a parabolic evolution in one block and for the flow through the entire system of fractures, it is necessary to assume a ratio of permeability (or transmissibility) in the blocks and in the fissures to be of order ε^2.

This time ratio, ε^2, is exactly the one leading to the dual-porosity model. For instance, in enhanced oil recovery, at a time $t \ll 1$, a large fraction of the oil reserves is recovered from the fractures; then at time $t \sim \mathrm{O}(1)$, the exchange between porous blocks and fissures as described in [2] begins. It should also be noted that this ε^2 time scaling is done in the engineering literature, as, for instance, in [20,13], but is motivated by introducing a geometric factor of transmissibility. If one takes the ratio of the two permeabilities of order one, then by the usual theory of homogenization the limit model will be as, for instance, in [4,16] a single porosity model. If the ratio is smaller than that of order ε^2, then there is no contribution from the blocks to the global continuity system of equations in the limit model, which then corresponds to the homogenization of the only system of fissures.

The main goal of this work is to provide a rigorous mathematical justification of the dual porosity model for a randomly fractured porous medium. Such a mathematical study has already been done but only for periodically fractured media, for single phase flow in [1] and for two phase flow in [6]. For the sake of simplicity and in order to avoid the technical problems associated to the possible loss of ellipticity in the two phase flow model, we consider a weakly compressible single phase flow described by the parabolic equations (7)–(13) below. The unknown variables in this model will be the density of fluid in the blocks and the density of fluid in the fissures, coupled via the fluxes across the interfaces.

The microscopic model describing the exchange between the fractures system and the porous blocks is introduced in the first section. This section includes also the probabilistic description of the fractured media and basic a priori estimates.

In the next section we adapt two-scale convergence in the mean techniques to the problem under consideration.

Results of the article were announced in the note [7].

Section 3 is devoted to the convergence results. Our approach involves two different auxiliary stochastic problems. The first one is related to the flow in random fractures with Neumann condition at the interface. The second one is a stochastic parabolic equation defined in the matrix blocks.

Finally, in Section 4, we provide several examples of the random double porosity model in such random structures as disperse and generalized disperse media, perforated blocks structure, Voronoi tessellation models.

1. The ε-problem and a priori estimates

We start with a precise formulation of our microscopic problem (the ε-problem):

Let (Ω, Ξ, μ) be a probability space, and assume that a dynamical system $\mathcal{T}$ with n-dimensional time is given on Ω, i.e., a family of invertible measurable maps $\mathcal{T}(x): \Omega \to \Omega$, $x \in \mathbb{R}^n$, such that

(1) $\mathcal{T}(0) = \mathrm{Id}$ on Ω and $\mathcal{T}(x_1 + x_2) = \mathcal{T}(x_1)\mathcal{T}(x_2)$ for all $x_1, x_2 \in \mathbb{R}^n$;
(2) $\forall x \in \mathbb{R}^n$ and $\forall E \in \Xi$,

$$\mu\big(\mathcal{T}(x)^{-1}(E)\big) = \mu(E) \quad \text{(endomorphism property)}.$$

(3) $\forall E \in \Xi$ the set $\{(x, \omega) \in \mathbb{R}^n \times \Omega\colon \mathcal{T}(x)\omega \in E\}$ is an element of the σ-algebra $\mathcal{L} \times \Xi$ on $\mathbb{R}^n \times \Omega$, where $\mathcal{L}$ is the usual Lebesgue σ-algebra on $\mathbb{R}^n$.

With the measurable dynamics introduced above we associate a n-parameters group of unitary operators on $L^2(\Omega) \equiv L^2(\Omega, \Xi, \mu)$, as follows

$$\big(U(x)f\big)(\omega) = f\big(\mathcal{T}(x)\omega\big), \quad f \in L^1(\Omega).$$

We suppose that $L^2(\Omega)$ is separable and that the dynamical system $\{\mathcal{T}(x)\}$ is ergodic.

At the next step, we use a fixed measurable set $\mathcal{F} \in \Xi$ such that $\mu(\mathcal{F}) > 0$ and $\mu(\Omega \setminus \mathcal{F}) > 0$, and to define random fractures system $F(\omega) \subset \mathbb{R}^n$, $\omega \in \Omega$, obtained from $\mathcal{F}$ by setting

$$F(\omega) = \big\{x \in \mathbb{R}^n\colon\ \mathcal{T}(x)\omega \in \mathcal{F}\big\}. \tag{1}$$

In what follows we suppose that $F(\omega)$ is open and connected a.s. (for almost all $\omega \in \Omega$).

The random matrix blocks set $M(\omega)$ is constructed in a complementary way by setting

$$\mathcal{M} = \Omega \setminus \mathcal{F}, \qquad M(\omega) = \mathbb{R}^n \setminus F(\omega). \tag{2}$$

In connection with the random set $M(\omega)$ we introduce a homothetic structure $M_\varepsilon(\omega)$, $\omega \in \Omega$, by

$$M_\varepsilon(\omega) = \big\{x \in \mathbb{R}^n\colon\ \varepsilon^{-1}x \in M(\omega)\big\}; \tag{3}$$

further assumptions on the random structure will be given in Section 3.

Let G be a smooth bounded domain in $\mathbb{R}^n$. After having chosen our random structure in $\mathbb{R}^n$, we set

$$G_1^\varepsilon = \big\{x \in G\colon\ \mathrm{dist}(x, \partial G) \geqslant \varepsilon\big\}. \tag{4}$$

Now it is possible to introduce the random fracture system in G by

$$G_f^\varepsilon(\omega) = G \setminus \overline{M_\varepsilon(\omega) \cap G_1^\varepsilon}. \tag{5}$$

Then, the random matrix block part of G is defined as the complement of $G_f^\varepsilon(\omega)$ in G:

$$G_m^\varepsilon(\omega) = G \setminus \overline{G_f^\varepsilon(\omega)}. \tag{6}$$

After having defined the random geometry, we write for a typical realization, the equations of mass conservation combined with Darcy's law and the equation of state that includes gravity effects, for a slightly compressible fluid:

$$\Phi^* \frac{\partial \rho^\varepsilon}{\partial t} - \operatorname{div}\left\{ \frac{K^*}{\lambda c} \left[\nabla \rho^\varepsilon + c\rho_0 (2\rho^\varepsilon - \rho_0) g \right] \right\} = f \quad \text{in } G_f^\varepsilon \times]0,T[, \tag{7}$$

$$\varphi^\varepsilon \frac{\partial \sigma^\varepsilon}{\partial t} - \varepsilon \operatorname{div}\left\{ \frac{k^\varepsilon}{\lambda c} \left[\varepsilon \nabla \sigma^\varepsilon + c\sigma_0^\varepsilon (2\sigma^\varepsilon - \sigma_0^\varepsilon) g \right] \right\} = f \quad \text{in } G_m^\varepsilon \times]0,T[, \tag{8}$$

$$\frac{K^*}{\lambda c} \left[\nabla \rho^\varepsilon + c\rho_0 (2\rho^\varepsilon - \rho_0) g \right] \cdot \nu = \varepsilon \frac{k^\varepsilon}{\lambda c} \left[\varepsilon \nabla \sigma^\varepsilon + c\sigma_0^\varepsilon (2\sigma^\varepsilon - \sigma_0^\varepsilon) g \right] \cdot \nu \quad \text{on } \partial G_m^\varepsilon \times]0,T[, \tag{9}$$

$$\rho^\varepsilon(x,0) = \rho_{\text{in}} \quad \text{in } G_f^\varepsilon, \tag{10}$$

$$\sigma^\varepsilon(x,0) = \rho_{\text{in}} \quad \text{in } G_m^\varepsilon, \tag{11}$$

$$\sigma^\varepsilon = \rho^\varepsilon \quad \text{on } \partial G_m^\varepsilon \times]0,T[, \tag{12}$$

$$\frac{K^*}{\lambda c} \left[\nabla \rho^\varepsilon + cg\rho_0 (2\rho^\varepsilon - \rho_0) \right] \cdot \nu = 0 \quad \text{on } \partial G \times]0,T[. \tag{13}$$

In the above equations, g is the gravitational constant vector, $f(x,t)$ represents external force, ρ_0 and $\sigma_0^\varepsilon(x,\omega) = \sigma_0(\mathcal{T}(x/\varepsilon)\omega)$ are given reference densities, and ρ_{in} is the specified initial density. $\Phi^*(x)$ and $K^*(x)$ denote the porosity and the scalar permeability of the fractures set,

$$\varphi^\varepsilon(x,\omega) = \varphi\left(\mathcal{T}\left(\frac{x}{\varepsilon}\right)\right)\omega) \quad \text{and} \quad k^\varepsilon(x,\omega) = k\left(\mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right)$$

denote, respectively, the matrix block porosity and permeability, the latter being a symmetric tensor. All above quantities are assumed smooth, uniformly bounded and positive-definite. Finally, λ is the fluid viscosity and c is a constant compressibility.

Owing to the transmission conditions (9) and (12), one can rewrite the above equations using a globally defined density function ϑ^ε,

$$\vartheta^\varepsilon = \begin{cases} \rho^\varepsilon & \text{in } \overline{G_f^\varepsilon}(\omega) \times]0,T[, \\ \sigma^\varepsilon & \text{in } G_m^\varepsilon(\omega) \times]0,T[\end{cases} \tag{14}$$

and globally defined coefficients:

$$\begin{cases} \alpha^\varepsilon(x,\omega) = \chi_{G_f^\varepsilon(\omega)}(x)\Phi^* + \chi_{G_m^\varepsilon(\omega)}(x)\varphi^\varepsilon(x,\omega), \\ \beta^\varepsilon(x,\omega) = cg\left\{ \rho_0 \chi_{G_f^\varepsilon(\omega)}(x) + \dfrac{1}{\varepsilon}\sigma_0^\varepsilon(x,\omega)\chi_{G_m^\varepsilon(\omega)}(x) \right\}, \\ \kappa^\varepsilon(x,\omega) = \dfrac{1}{\lambda c}\{K^* \chi_{G_f^\varepsilon(\omega)}(x) + \varepsilon^2 k^\varepsilon(x,\omega)\chi_{G_m^\varepsilon(\omega)}(x)\}, \\ \vartheta_0^\varepsilon(x,\omega) = \rho_0 \chi_{G_f^\varepsilon(\omega)}(x) + \sigma_0^\varepsilon(x,\omega)\chi_{G_m^\varepsilon(\omega)}(x). \end{cases} \tag{15}$$

Then the variational formulation of (7)–(13) reads as follows:

Find $\vartheta^\varepsilon \in W(0,T) \equiv \{z \in L^2(0,T;H^1(G))\colon\ \partial z/\partial t \in L^2(0,T;(H^1(G))')\}$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_G \alpha^\varepsilon(x,\omega)\vartheta^\varepsilon(t,x,\omega)\psi(x)\,\mathrm{d}x\right)$$
$$+\int_G \kappa^\varepsilon(x,\omega)\big[\nabla\vartheta^\varepsilon(t,x,\omega)+\beta^\varepsilon(x,\omega)\big(2\vartheta^\varepsilon(t,x,\omega)-\vartheta_0^\varepsilon(x,\omega)\big)\big]\nabla\psi(x)\,\mathrm{d}x$$
$$=\int_G f(t,x)\psi(x)\,\mathrm{d}x \tag{16}$$

for any $\psi \in H^1(G)$, *and*

$$\vartheta^\varepsilon(0,x)=\rho_{\mathrm{in}}(x). \tag{17}$$

Suppose

$$f \in L^2\big(]0,T[\times G\big) \quad \text{and} \quad \rho_{\mathrm{in}} \in H^1(G). \tag{18}$$

Then using the linear parabolic theory we deduce immediately that problem (16), (17) is uniquely solvable for all $\varepsilon > 0$ almost surely in ω. Furthermore, ϑ^ε is a measurable function of ω.

A priori estimates are now straightforward.

Proposition 1.1. *Let all above assumptions hold true. Then for all* $\varepsilon > 0$ *we have a.s. in* ω:

$$\|\vartheta^\varepsilon\|_{L^\infty(0,T;L^2(G))} \leqslant C, \tag{19}$$
$$\|\partial_t\vartheta^\varepsilon\|_{L^2(0,T;(L^2(G))^n)} + \|\nabla\rho^\varepsilon\|_{L^\infty(0,T;(L^2(G_f^\varepsilon(\omega)))^n)} \leqslant C, \tag{20}$$
$$\|\nabla\sigma^\varepsilon\|_{L^\infty(0,T;(L^2(G_m^\varepsilon(\omega)))^n)} \leqslant \frac{C}{\varepsilon}, \tag{21}$$

where C *is an universal constant independent of* ε *and of* ω.

2. The adapted stochastic two-scale convergence results

Before giving our convergence results, we recall the definition and some properties of the stochastic two-scale convergence in the mean (see [8] for more details).

Let D_j denotes the infinitesimal generator in $L^2(\Omega)$ of the one-parameter group of translations in x_j, with $\mathcal{D}_j$ its respective domain of definition in $L^2(\Omega)$, i.e., for $f \in \mathcal{D}_j$

$$(D_jf)(\omega)=\frac{\partial}{\partial x_j}\big(U(x)f\big)(\omega)|_{x=0}. \tag{22}$$

Then $\{\sqrt{-1}\,D_j,\ j=1,\dots,n\}$ are closed, densely-defined and self-adjoint operators which commute pairwise on $\mathcal{D}(\Omega)=\bigcap_{j=1}^n\mathcal{D}_j$. Equipped with the inner product

$$(f,g)_{\mathcal{D}(\Omega)}=(f,g)_{L^2(\Omega)}+\sum_{j=1}^n(D_jf,D_jg)_{L^2(\Omega)} \tag{23}$$

$\mathcal{D}(\Omega)$ becomes a Hilbert space.

On the base of (22) we may define the stochastic gradient $\{\nabla_\omega f\}$, divergence $\{\mathrm{div}_\omega f\}$ and curl$\{\mathrm{curl}_\omega f\}$, as follows

$$\begin{cases} \nabla_\omega f = (D_1 f, \dots, D_n f), \\ \mathrm{div}_\omega g = \sum_j D_j g_j, \\ \mathrm{curl}_\omega g = D_i g_j - D_j g_i, \quad i \neq j. \end{cases} \tag{24}$$

Moreover, we will use the following spaces:

$$\mathcal{V}^2_{\mathrm{pot}}(\Omega) = \{ f \in L^2_{\mathrm{pot}}(\Omega),\ \mathbb{E}\{f\} = 0 \}, \tag{25}$$

$$\mathcal{V}^2_{\mathrm{sol}}(\Omega) = \{ f \in L^2_{\mathrm{sol}}(\Omega),\ \mathbb{E}\{f\} = 0 \}, \tag{26}$$

where $L^2_{\mathrm{pot}}(\Omega)$ (respectively $L^2_{\mathrm{sol}}(\Omega)$) is the set of all $f \in (L^2(\Omega))^n$ such that almost all realizations $f(\mathcal{T}(x)\omega)$ are potential (respectively solenoidal) in $\mathbb{R}^n$; for more details we refer to Jikov, Kozlov and Oleinik [12].

Next, we say that an element $\psi \in L^2(G \times \Omega)$ is admissible if the function

$$\psi_{\mathcal{T}} : (x, \omega) \to \psi\big(x, \mathcal{T}(x)\omega\big), \quad (x, \omega) \in G \times \Omega,$$

defines an element of $L^2(G \times \Omega)$.

For example, as was shown in [8], functions from $C(\overline{G}; L^\infty(\Omega))$ and from $L^2(G; B(\Omega))$ are admissible. In addition, every finite linear combination of functions of the form

$$(x, \omega) \to f(x) g(\omega), \quad (x, \omega) \in G \times \Omega,\ f \in L^2(G),\ g \in L^2(\Omega),$$

is also admissible.

We may now recall the definition of the stochastic two-scale convergence in the mean from Bourgeat, Mikelić and Wright [8].

Definition 2.1. A sequence $\{u^\varepsilon\}$ of functions from $L^2(G \times \Omega)$ is said to converge stochastically two-scale in the mean (s.2-s.m.) towards $u \in L^2(G \times \Omega)$ if for any admissible $\psi \in L^2(G \times \Omega)$ we have

$$\lim_{\varepsilon \to 0} \int_{G \times \Omega} u^\varepsilon(x, \omega) \psi\left(x, \mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right) \mathrm{d}x \, \mathrm{d}\mu = \int_{G \times \Omega} u(x, \omega) \psi(x, \omega) \, \mathrm{d}x \, \mathrm{d}\mu. \tag{27}$$

After obtaining the a priori estimates (19)–(21) a possible way to proceed could be to extend ρ^ε from $G^\varepsilon_f(\omega)$ to G, and to use the functions obtained to pass to the limit, as $\varepsilon \to 0$, by virtue of Theorem 3.7 from Bourgeat, Mikelić and Wright [8].

Here we prefer another approach, motivated by the results on the homogenization of Neumann problem in perforated domains from Jikov, Kozlov and Oleinik [12]. First, we state and prove the following convergence result:

Proposition 2.2. *Let $\{u^\varepsilon\} \subset H^1(G)$ be such a sequence that*

$$\begin{cases} \|u^\varepsilon\|_{L^2(G)} \leqslant C, \\ \|\nabla u^\varepsilon\|_{L^2(G_f^\varepsilon(\omega))} \leqslant C, \\ \|\nabla u^\varepsilon\|_{L^2(G_m^\varepsilon(\omega))} \leqslant \dfrac{C}{\varepsilon}. \end{cases} \tag{28}$$

Suppose that the set of all functions $\psi \in \mathcal{D}(\Omega)$ such that $\psi = 0$ on $\mathcal{M}$, is dense in $L^2(\Omega \setminus \mathcal{M})$, and the set of all functions $\psi \in \mathcal{D}(\Omega)$, being zero on $\Omega \setminus \mathcal{M}$, is dense in $L^2(\mathcal{M})$. Let X be the closure of the space $\mathcal{V}^2_{\mathrm{pot}}(\Omega)$ in $L^2(\Omega \setminus \mathcal{M})^n$. Suppose, furthermore, that the tensor $\mathcal{A}_N^0$ associated to the homogenized Neumann problem and defined by

$$\xi \cdot \mathcal{A}_N^0 \xi = \inf_{v \in X} \int_{\Omega \setminus \mathcal{M}} |\xi + v|^2 \, \mathrm{d}\mu, \quad \xi \in \mathbb{R}^n, \tag{29}$$

is positive definite. Then there exist functions $u \in H^1(G)$, $v \in L^2(G; \mathcal{D}(\Omega))$, $v = 0$ on $\Omega \setminus \mathcal{M}$, and $u_1 \in L^2(G;\, X)$, $u_1 = 0$ on $\mathcal{M}$, such that, up to a subsequence,

$$u^\varepsilon \xrightarrow{\text{s.2-s.m.}} u(x) + \chi_{\mathcal{M}}(\omega) v(x, \omega), \tag{30}$$

$$\chi_{G_f^\varepsilon(\omega)} \nabla u^\varepsilon \xrightarrow{\text{s.2-s.m.}} \chi_{\Omega \setminus \mathcal{M}} \left[\nabla_x u(x) + u_1(x, \omega)\right], \tag{31}$$

$$\varepsilon \chi_{G_m^\varepsilon(\omega)} \nabla u^\varepsilon \xrightarrow{\text{s.2-s.m.}} \chi_{\mathcal{M}}(\omega) \nabla_\omega v(x, \omega), \tag{32}$$

Proof. We generalize to the random case the construction developed for the periodic case in Fasano, Mikelić and Primicerio [10, Proposition 2.2].

Using the above a priori estimates and the stochastic two-scale in the mean compactness theorem from Bourgeat, Mikelić and Wright [8], we conclude that, after taking a proper subsequence, the sequences $\{u^\varepsilon\}$, $\{\chi_{G_f^\varepsilon(\omega)} \nabla u^\varepsilon\}$ and $\{\varepsilon \chi_{G_m^\varepsilon(\omega)} \nabla u^\varepsilon\}$ have stochastic two-scale limits. We have then:

- $u^\varepsilon \xrightarrow{\text{s.2-s.m.}} u_0(x, \omega)$,
- $\chi_{G_f^\varepsilon(\omega)} \nabla u^\varepsilon \xrightarrow{\text{s.2-s.m.}} \xi_0(x, \omega)$,
- $\varepsilon \chi_{G_m^\varepsilon(\omega)} \nabla u^\varepsilon \xrightarrow{\text{s.2-s.m.}} \xi_0(x, \omega) z_0(x, \omega)$.

We should find relations between u_0, ξ_0 and z_0. At the first step we take $g(x, \omega) = g_1(\omega) g_2(x)$, where $g_1 \in L^2(\mathcal{M})$, $g_1 = 0$ on $\Omega \setminus \mathcal{M}$, and $g_2 \in C_0^\infty(G)$. Obviously g is an admissible function and we get

$$0 = \int_\Omega \int_G \chi_{G_f^\varepsilon(\omega)} \nabla u^\varepsilon g\left(x, \mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right) \mathrm{d}x \, \mathrm{d}\mu \to \int_\Omega \int_G \xi_0(x, \omega) g_1(\omega) g_2(x) \, \mathrm{d}x \, \mathrm{d}\mu$$

and, thus, $\xi_0 = 0$ on $G \times \mathcal{M}$. Similarly, $z_0 = 0$ on $G \times (\Omega \setminus \mathcal{M})$ and we obtain

$$\begin{cases} \xi_0(x, \omega) = \chi_{\Omega \setminus \mathcal{M}}(\omega) \xi_0(x, \omega), \\ z_0(x, \omega) = \chi_{\mathcal{M}}(\omega) z_0(x, \omega). \end{cases} \tag{33}$$

Let now $h \in C_0^\infty(G)$, $g \in \mathcal{D}(\Omega)$ and $\psi \in (\mathcal{D}(\Omega))^n$. Suppose g and ψ vanishing on $\mathcal{M}$. Then

$$\lim_{\varepsilon\to 0}\int_\Omega\int_G u^\varepsilon(x)h(x)g\left(\mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right)\mathrm{d}x\,\mathrm{d}\mu = \int_\Omega\int_G u_0(x,\omega)h(x)g(\omega)\,\mathrm{d}x\,\mathrm{d}\mu \tag{34}$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega\int_G \nabla u^\varepsilon(x)h(x)\psi\left(\mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right)\mathrm{d}x\,\mathrm{d}\mu = \int_\Omega\int_G \xi_0(x,\omega)h(x)\psi(\omega)\,\mathrm{d}x\,\mathrm{d}\mu. \tag{35}$$

On the other hand

$$\begin{aligned}
&\varepsilon\int_\Omega\int_G \chi_{G_f^\varepsilon(\omega)}\nabla u^\varepsilon(x)h(x)\psi\left(\mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right)\\
&\quad= -\int_\Omega\int_G \chi_{G_f^\varepsilon(\omega)}u^\varepsilon\left[\operatorname{div}_\omega\psi\left(\mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right)h(x)+\varepsilon\nabla_x h(x)\psi\left(\mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right)\right]\\
&\quad\to -\int_\Omega\int_G \chi_{\Omega\setminus\mathcal{M}}(\omega)u_0(x,\omega)h(x)\operatorname{div}_\omega\psi(\omega)\\
&\quad= -\int_\Omega \chi_{\Omega\setminus\mathcal{M}}(\omega)\left\{\int_G u_0(x,\omega)h(x)\,\mathrm{d}x\right\}\operatorname{div}_\omega\psi(\omega)\,\mathrm{d}\mu = 0,
\end{aligned} \tag{36}$$

as $\varepsilon \to 0$.

If we set $H(\omega) = \int_G u_0(x,\omega)h(x)\,\mathrm{d}x$, then $H \in L^2(\Omega)$ and

$$\int_\Omega \chi_{\Omega\setminus\mathcal{M}}(\omega)H(\omega)\operatorname{div}_\omega\psi(\omega)\,\mathrm{d}\mu = 0$$

for all $\psi \in (\mathcal{D}(\Omega))^n$, $\psi = 0$ on $\mathcal{M}$. After taking only j-th components to be nonzero, we get

$$\int_\Omega \chi_{\Omega\setminus\mathcal{M}}(\omega)H(\omega)D_j\zeta(\omega)\,\mathrm{d}\mu = 0 \tag{37}$$

for all $\zeta \in \mathcal{D}(\Omega)$, $\zeta = 0$ on $\mathcal{M}$, and for any $j \in \{1,\dots,n\}$. Taking into account the ergodicity of the dynamical system and connectivity of the fractures, by the same arguments as in Bourgeat, Mikelić and Wright [8], we conclude that

$$H(\omega) = \int_G u_0(x,\omega)h(x)\,\mathrm{d}x$$

is constant (a.s.) on $\Omega \setminus \mathcal{M}$. Therefore,

$$u_0(x,\omega) = u(x) \quad \text{a.e. on } G\times(\Omega\setminus\mathcal{M}). \tag{38}$$

We proceed by supposing in addition that $\operatorname{div}_\omega \psi = 0$ in Ω. Then

$$\int_\Omega\int_G \chi_{G_f^\varepsilon(\omega)}\nabla u^\varepsilon h(x)\psi\left(\mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right)\mathrm{d}x\,\mathrm{d}\mu - \int_\Omega\int_G \chi_{G_f^\varepsilon(\omega)}u^\varepsilon\nabla_x h(x)\psi\left(\mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right)\mathrm{d}x\,\mathrm{d}\mu$$

$$\to \int_\Omega\int_G \xi_0(x,\omega)\psi(\omega)h(x)\,\mathrm{d}x\,\mathrm{d}\mu = -\int_G u(x)\mathrm{div}_x\left\{\int_\Omega \chi_{\Omega\setminus\mathcal{M}}(\omega)h(x)\psi(\omega)\,\mathrm{d}\mu\right\}, \tag{39}$$

as $\varepsilon \to 0$; the fact that $\psi = 0$ in $\mathcal{M}$ has also been used here.

For any $\vartheta \in (L^2(G))^n$ we define $w \in L^2(G;X)$ to be a unique solution to problem (29) with $\xi = (\mathcal{A}_N^0)^{-1}\vartheta(x)$. Then the function $q(x,\omega)$ defined by

$$q = \chi_{\Omega\setminus\mathcal{M}}\{(\mathcal{A}_N^0)^{-1}\vartheta(x) + w(x,\omega)\}, \tag{40}$$

satisfies the relation $\mathbb{E}\{q\} = \vartheta(x)$, and $q \in L^2(G; L^2_{\mathrm{sol}}(\Omega))$. Owing to the density arguments (see Bourgeat, Mikelić and Wright [8] for more details) we get

$$\int_\Omega\int_G \xi_0(x,\omega)q(x,\omega)\,\mathrm{d}x\,\mathrm{d}\mu = \int_G \nabla_x u(x)\vartheta(x)\,\mathrm{d}x$$

for any $\vartheta \in (L^2(G))^n$. Therefore, $\nabla_x u \in (L^2(G))^n$ and $u \in H^1(G)$.

Furthermore, we have

$$0 = \varepsilon\int_{G_f^\varepsilon(\omega)}\left\{\frac{\partial u^\varepsilon}{\partial x_i}\frac{\partial}{\partial x_j}g\left(x,\mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right) - \frac{\partial u^\varepsilon}{\partial x_j}\frac{\partial}{\partial x_i}g\left(x,\mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right)\right\}\mathrm{d}x$$

for all $g \in C_0^\infty(G;\mathcal{D}(\Omega))$, $g = 0$ on $\mathcal{M}$. Passing to the limit, as $\varepsilon \to 0$, gives

$$0 = \int_G\int_\Omega \chi_{\Omega\setminus\mathcal{M}}(\omega)\{\xi_{0i}(x,\omega)D_j g(x,\omega) - \xi_{0j}(x,\omega)D_i g(x,\omega)\}\,\mathrm{d}x\,\mathrm{d}\mu$$

and we conclude that

$$\xi_0(x,\omega) - \nabla_x u(x) \in L^2(G;X)^n.$$

It remains to identify z_0. First, it follows from Theorem 3.7 from Bourgeat, Mikelić and Wright [8] that

$$\varepsilon\nabla u^\varepsilon \xrightarrow{\text{s.2-s.m.}} \nabla_\omega u_0(x,\omega)$$

and

$$\varepsilon\chi_{G_m^\varepsilon(\omega)}\nabla u^\varepsilon \xrightarrow{\text{s.2-s.m.}} \chi_{\mathcal{M}}(\omega)\nabla_\omega u_0(x,\omega).$$

Therefore, $z_0 = \nabla_\omega u_0(x,\omega)$ (a.e.) on $G\times\mathcal{M}$, and we obtain

$$u^\varepsilon \xrightarrow{\text{s.2-s.m.}} \chi_{\Omega\setminus\mathcal{M}}(\omega)u(x) + \chi_{\mathcal{M}}(\omega)u_0(x,\omega) = u(x) + \chi_{\mathcal{M}}(\omega)v(x,\omega), \tag{41}$$

where $v(x,\omega) = u_0(x,\omega) - u(x)$ on $G\times\Omega$. It should be noticed that $\nabla_\omega v = \nabla_\omega u_0 = z_0$ on $G\times\mathcal{M}$.

Our last step is to prove that $v \in \mathcal{D}(\Omega)$ for a.e. $x \in G$. By passing to the limit, as $\varepsilon \to 0$, in $\varepsilon \nabla u^\varepsilon$ we obtain in $G \times \Omega$:

$$\int_G \int_\Omega \chi_{\mathcal{M}}(\omega) \nabla_\omega v \psi(x,\omega)\,\mathrm{d}x\,\mathrm{d}\mu = -\int_G \int_\Omega \chi_{\mathcal{M}}(\omega) v(x,\omega)\,\mathrm{div}_\omega\,\psi(x,\omega) \tag{42}$$

for all $\psi \in L^2(G;\mathcal{D}(\Omega))^n$. Consequently,

$$\nabla_\omega\{\chi_{\mathcal{M}}(\omega) v(x,\omega)\} = \chi_{\mathcal{M}}(\omega) \nabla_\omega v(x,\omega)$$

a.e. in $G \times \mathcal{M}$ and $\chi_{\mathcal{M}}(\omega) v(x,\omega) \in \mathcal{D}(\Omega)$. Proposition is now proved. □

Remark 2.3. It should be noted that $\mathcal{A}_N^0$ is always positive definite in the periodic case if the fracture part is connected. Sufficient conditions for positive-definiteness of $\mathcal{A}_N^0$ in the random case are given in Jikov, Kozlov and Oleinik [12]. We discuss this question in the last section of this paper.

Remark 2.4. In the case of disperse media, one can prove easily that the functions from $\mathcal{D}(\Omega)$, being zero on $\mathcal{M}$, are dense in $L^2(\Omega \setminus \mathcal{M})$. Indeed, by the definition (see [12] and Example 1 in Section 4), $M(\omega)$ consists a.s. of closed components diffeomorphic to a ball and having a piecewise smooth boundary. They have no interior points in common, and their diameters belong to a fixed interval $[t_1, t_2]$, $0 < t_1 < t_2 < +\infty$. For each component K we denote its δ-neighborhood $\{x \in \mathbb{R}^n\colon \mathrm{dist}(x,K) \leqslant \delta\}$ by K_δ. Let $f \in L^2(\Omega \setminus \mathcal{M})$. Then by the Fubini theorem and the ergodicity, $f(\mathcal{T}(x)\omega)\chi_{F(\omega)} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, for any regularizing sequence $\rho_\delta = \delta^{-n}\rho(\cdot/\delta)$ with $\rho \in C_0^\infty(\{|x| < 1\})$, $\int \rho(x)\,\mathrm{d}x = 1$, we have $(\rho_\delta * [f\chi_{\mathbb{R}^n \setminus \bigcup K_\delta}](\mathcal{T}(\cdot)\omega) \to f(\mathcal{T}(\cdot)\omega)\chi_{F(\omega)}$, as $\delta \to 0$, a.s. on Ω, and $\rho_\delta * (f\chi_{\mathbb{R}^n \setminus \bigcup K_\delta})$ is a C_0^∞-function equal to zero on $M(\omega)$. The ergodicity now implies the fact that the functions from $\mathcal{D}(\Omega)$, equal to zero on $\mathcal{M}$, are dense in $L^2(\Omega \setminus \mathcal{M})$.

3. Auxiliary problems and convergence result

The peculiarity of the double-porosity models is the presence of two kinds of auxiliary problems. An auxiliary problem of the first kind is used to compute the effective permeability; it turns out to be connected with the Neumann problem for the elliptic part of the corresponding equations in fractures, and reads:

Find $v_\eta \in X = \{\text{closure of } \mathcal{V}^2_{\mathrm{pot}}(\Omega) \text{ in } L^2(\Omega \setminus \mathcal{M})\}$ such that

$$\mathbb{E}\{\zeta\chi_{\Omega\setminus\mathcal{M}}(\eta + v_\eta)\} = \int_{\Omega\setminus\mathcal{M}} \zeta(\omega)\bigl(\eta + v_\eta(\omega)\bigr)\,\mathrm{d}\mu = 0 \tag{43}$$

for all $\zeta \in \mathcal{V}^2_{\mathrm{pot}}(\Omega)$.

Proposition 3.1 (Jikov et al. [12]). *The above problem* (43) *has a unique solution. Moreover, the corresponding tensor $\mathcal{A}_N^0$ is constant, positive and uniquely defined by*

$$\mathcal{A}_N^0 \eta = \mathbb{E}\{\chi_{\Omega\setminus\mathcal{M}}(\eta + v_\eta)\}, \quad \eta \in \mathbb{R}^n. \tag{44}$$

Remark 3.2. The problem of non-degeneracy of $\mathcal{A}_N^0$ is discussed for instance in Jikov et al. [12]. One possible way to prove non-degeneracy is to verify the extension property for the realizations of a random medium; for example, in the particular case of random spherical structure in $\mathbb{R}^3$ the matrix $\mathcal{A}_N^0$ is always positive definite. For some other random models the proof of non-degeneracy of $\mathcal{A}_N^0$ relies on the percolation channels technique like in [14].

The second auxiliary problem is used to compute the source term, and connected with the corresponding equation in the matrix blocks. A general formulation of this problem reads

Find $\zeta \in L^2(0,T;Z)$, $Z = \{z \in \mathcal{D}(\Omega)\colon\ z = 0 \text{ on } \Omega \setminus \mathcal{M}\}$, *such that* $\partial\zeta/\partial t \in L^2(0,T;Z')$ *and*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varphi(\omega)\chi_{\mathcal{M}}(\omega)\zeta(t,\omega)\xi(\omega)\,\mathrm{d}\mu + \frac{1}{\lambda c}\int_\Omega k(\omega)\chi_{\mathcal{M}}(\omega)\nabla_\omega\zeta(t,\omega)\nabla_\omega\xi(\omega)\,\mathrm{d}\mu \\
&\quad + \int_\Omega \frac{g}{\lambda}k(\omega)\sigma_0(\omega)\chi_{\mathcal{M}}\bigl(2\zeta(t,\omega) - \sigma_0(\omega)\bigr)\nabla_\omega\xi(\omega)\,\mathrm{d}\mu = \gamma(t)\int_{\mathcal{M}}\xi(\omega)\,\mathrm{d}\mu, \quad \forall\xi \in Z, && (45)\\
&\zeta(0,\omega) = \rho_{\mathrm{in}}. && (46)
\end{aligned}$$

Problem (45), (46) can be studied by the classical parabolic variational theory methods with the choice of the spaces $V = Z$, $H = L^2(\mathcal{M})$. Under the hypothesis of Proposition 2.2, we have $Z \subset L^2(\mathcal{M})$ with a dense and continuous embedding. The corresponding bilinear form

$$a(\zeta,\xi) = \frac{1}{\lambda c}\int_{\mathcal{M}} k(\omega)\nabla_\omega\zeta\nabla_\omega\xi\,\mathrm{d}\mu + \frac{2g}{\lambda}\int_{\mathcal{M}} k(\omega)\sigma_0(\omega)\zeta\nabla_\omega\xi\,\mathrm{d}\mu \tag{47}$$

is continuous on Z and the related quadratic form satisfies the estimate

$$\begin{aligned}
a(\xi,\xi) &= \frac{1}{\lambda c}\int_{\mathcal{M}} k(\omega)|\nabla_\omega\xi|^2\,\mathrm{d}\mu + \frac{g}{\lambda}\int_{\mathcal{M}} k(\omega)\sigma_0(\omega)\nabla_\omega|\xi|^2 \\
&\geqslant \frac{1}{2\lambda c}\int_{\mathcal{M}} k(\omega)|\nabla_\omega\xi|^2\,\mathrm{d}\mu - \frac{2g^2c}{\lambda}\int_{\mathcal{M}} k(\omega)\sigma_0^2\xi^2\,\mathrm{d}\mu.
\end{aligned}$$

Therefore, $a(\cdot,\cdot)$ is (V,H)-coercive and for $\theta \in L^2(0,T)$ there exists a unique $\zeta \in W(0,T) = \{z \in L^2(0,T;Z)\colon\ \partial z/\partial t \in L^2(0,T;Z')\}$ that satisfies (45), (46). Furthermore, $\zeta \in C([0,T];L^2(\mathcal{M}))$.

Now we obtain the following stochastic two-scale compactness result for the solutions to problem (16), (17).

Proposition 3.3. *Suppose* (4)–(6) *and* (18), *and assume that the tensor* $\mathcal{A}_N^0$ *defined by* (29), *is positive definite. Let* $\{\vartheta^\varepsilon\}_{\varepsilon>0}$ *satisfy* (16), (17). *Then there exist* $\vartheta \in L^2(0,T;H^1(G))$, $\partial_t\vartheta \in L^2(]0,T[\times G)$, $v \in L^2(]0,T[\times G;\mathcal{D}(\Omega))$, $v = 0$ *on* $\mathcal{F}$, *and* $\vartheta_1 \in L^2(G\times]0,T[;X)$, $\vartheta_1 = 0$ *on* $\mathcal{M}$, *such that, up to a subsequence,*

$$\vartheta^\varepsilon \xrightarrow{\text{s.2-s.m.}} \vartheta(x,t) + \chi_{\mathcal{M}}(\omega)v(x,t,\omega), \tag{48}$$

$$\chi_{G_f^\varepsilon(\omega)}\nabla\vartheta^\varepsilon \xrightarrow{\text{s.2-s.m.}} \chi_{\Omega\setminus\mathcal{M}}(\omega)\bigl[\nabla_x\vartheta(x,t) + \vartheta_1(x,t,\omega)\bigr], \tag{49}$$

$$\varepsilon\chi_{G_m^\varepsilon(\omega)}\nabla\vartheta^\varepsilon \xrightarrow{\text{s.2-s.m.}} \chi_{\mathcal{M}}(\omega)\nabla_\omega v(x,t,\omega), \tag{50}$$

$$\partial_t\vartheta^\varepsilon \xrightarrow{\text{s.2-s.m.}} \partial_t\vartheta(x,t) + \chi_{\mathcal{M}}(\omega)\partial_t v(x,t,\omega). \tag{51}$$

Proof. This is an immediate consequence of the a priori estimates (19)–(21), of the non-degeneracy of $\mathcal{A}_N^0$ and of Proposition 2.2. We note that t is a parameter.

The following corollaries are to describe some properties of the components ϑ, v and ϑ_1 of the stochastic two-scale limit above. □

Corollary 3.4. *Let ϑ, ϑ_1 be the limits defined from* (48)–(50); *then the variational identity*

$$\int_{\Omega\setminus\mathcal{M}} K^*\{\nabla_x\vartheta(x,t)+\vartheta_1(x,t,\omega)+cg\rho_0(2\vartheta-\rho_0)\}\xi\,\mathrm{d}\mu=0 \tag{52}$$

holds true for any $\xi\in\mathcal{V}^2_{\mathrm{pot}}(\Omega)$, a.e. on $G\times[0,T]$.

Proof. Let $\psi=\psi(x,t)\in C_0^\infty(G\times[0,T[)$, and suppose $\xi\in\mathcal{D}(\Omega)$. We write (16), (17) in the following equivalent form:

$$\begin{aligned}
&-\int_0^T\!\!\int_G \alpha^\varepsilon\vartheta^\varepsilon\varepsilon\xi\left(\mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right)\frac{\partial\psi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t-\int_G\alpha^\varepsilon\rho_{\mathrm{in}}\varepsilon\xi\left(\mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right)\psi(x,0)\,\mathrm{d}x\\
&\quad+\int_0^T\!\!\int_G\kappa^\varepsilon\left[\nabla\vartheta^\varepsilon+\beta^\varepsilon(2\vartheta^\varepsilon-\vartheta_0^\varepsilon)\right]\left\{\nabla_\omega\xi\left(\mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right)\psi+\varepsilon\xi\nabla\psi\right\}\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^T\!\!\int_G f\varepsilon\xi\psi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{53}$$

After applying the s.2-s.m. convergence results and (48)–(50), we obtain the equation

$$\int_0^T\!\!\int_G\int_{\Omega\setminus\mathcal{M}}K^*\{\nabla_x\vartheta(x,t)+\vartheta_1(x,t,\omega)+cg\rho_0\bigl(2\vartheta(x,t)-\rho_0\bigr)\}\nabla_\omega\xi(\omega)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\mu=0.$$

Now (52) is straightforward. □

Corollary 3.5. *Let $\delta^j\in X$ be the solutions of* (43) *for $\eta=e_j$, $j=1,\dots,n$. Then*

$$K^*\vartheta_1(x,t,\omega)=\sum_j K^*\{\nabla_x\vartheta(x,t)+cg\rho_0\bigl(2\vartheta(x,t)-\rho_0\bigr)\}e_j\delta^j(\omega), \tag{54}$$

and moreover the effective flux is given by:

$$\begin{aligned}
&\mathcal{A}_N^0K^*\{\nabla_x\vartheta(x,t)+cg\rho_0\bigl(2\vartheta(x,t)-\rho_0\bigr)\}\\
&\quad=\mathbb{E}\{\chi_{\Omega\setminus\mathcal{M}}(\omega)K^*\bigl(\nabla_x\vartheta(x,t)+\vartheta_1(x,t,\omega)+cg\rho_0\bigl(2\vartheta(x,t)-\rho_0\bigr)\bigr)\}.
\end{aligned}\tag{55}$$

Finally, ϑ_1 is given explicitly by (54) *in terms of $\vartheta(x,t)$.*

Proof. This is the direct consequence of the linearity of problem (52) and of the non-degeneracy of $\mathcal{A}_N^0$. □

Finally, we are going to derive the variational formulation of the homogenized problem describing the global behaviour. To this end we choose proper admissible test functions and pass to the limit, as $\varepsilon \to 0$.

Lemma 3.6. *Let v be defined by* (48)–(50); *then it satisfies the variational equation*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{M}} \varphi(\omega)\{\vartheta(x,t)+v(x,t,\omega)\}\xi(\omega)\,\mathrm{d}\mu \\
&\quad+\frac{1}{\lambda c}\int_{\mathcal{M}} k(\omega)\big[\nabla_\omega v(x,t,\omega)+c\sigma_0(\omega)\big(2\vartheta(x,t)+2v(x,t,\omega)-\sigma_0(\omega)\big)g\big]\nabla_\omega\xi(\omega)\,\mathrm{d}\mu \\
&\quad= f(x,t)\int_{\mathcal{M}}\xi(\omega)\,\mathrm{d}\mu \quad \forall\xi\in\mathcal{D}(\Omega),\ \xi=0 \text{ on } \Omega\setminus\mathcal{M},
\end{aligned}\tag{56}$$

i.e., $\forall\xi\in Z$, *in the sense of distributions on* $]0,T[$ *and a.e. on* G.

$$v(x,0,\omega)=0,\qquad \vartheta(x,0)=\rho_{\mathrm{in}}.\tag{57}$$

Proof. Suppose $\psi=\psi(x,t)\in C_0^\infty(G\times[0,T[)$ and $\xi\in\mathcal{D}(\Omega)$, $\xi=0$ a.s. on $\Omega\setminus\mathcal{M}$. We substitute a test function of the form $\xi\psi$ in Eq. (53). Then (53) reads

$$\begin{aligned}
&-\int_0^T\!\!\int_G \alpha^\varepsilon\vartheta^\varepsilon\xi\left(\mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right)\frac{\partial\psi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t-\int_G\alpha^\varepsilon\rho_{\mathrm{in}}\xi\left(\mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right)\psi(x,0)\,\mathrm{d}x \\
&\quad+\int_0^T\!\!\int_G\kappa^\varepsilon\left[\nabla\vartheta^\varepsilon+\beta^\varepsilon(2\vartheta^\varepsilon-\vartheta_0^\varepsilon)\right]\left\{\frac{1}{\varepsilon}\nabla_\omega\xi\left(\mathcal{T}\left(\frac{x}{\varepsilon}\right)\omega\right)\psi+\xi\nabla\psi\right\}\mathrm{d}x\,\mathrm{d}t=\int_0^T\!\!\int_G f\xi\psi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{58}$$

Passing to the limit in ε in the latter formula gives

$$\begin{aligned}
&-\int_0^T\!\!\int_G\int_{\mathcal{M}}\varphi(\omega)\big(\vartheta(x,t)+v(x,t,\omega)\big)\frac{\partial\psi}{\partial t}\xi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\mu-\int_G\int_{\mathcal{M}}\varphi(\omega)\rho_{\mathrm{in}}\xi(\omega)\psi(x,0)\,\mathrm{d}x\,\mathrm{d}\mu \\
&\quad+\int_0^T\!\!\int_G\int_{\mathcal{M}}\frac{k(\omega)}{\lambda c}\{\nabla_\omega v(x,t,\omega)+gc(2\vartheta+2v-\sigma_0)\sigma_0\}\nabla_\omega\xi(\omega)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\mu \\
&\quad=\int_0^T\!\!\int_G\int_{\mathcal{M}}f(x,t)\xi(\omega)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\mu
\end{aligned}\tag{59}$$

and (56) easily follows. Then, from (17) and (48) taking into account the relation $v\chi_{\mathcal{F}}=0$, we get

$$v(x,0,\omega)+\vartheta(x,0)=\rho_{\mathrm{in}}(x).$$

Thus, $v(x,0,\omega)$ does not depend on ω and (57) follows. $\square$

Remark 3.7. It follows from (56) that the function $v(x,t,\omega)$ is determined as soon as $\vartheta(x,t)$ is known.

Proposition 3.8. *Let ϑ and v be defined by* (48)–(50); *then ϑ satisfies the equation*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_G\left\{\int_\Omega\left(\Phi^*\chi_{\Omega\setminus\mathcal{M}}(\omega)+\varphi(\omega)\chi_{\mathcal{M}}(\omega)\right)\mathrm{d}\mu\,\vartheta(x,t)+\int_{\mathcal{M}}\varphi(\omega)v(x,t,\omega)\,\mathrm{d}\mu\right\}\psi(x)\,\mathrm{d}x\\
&\qquad+\frac{1}{\lambda c}\int_G\mathcal{A}_N^0K^*\{\nabla_x\vartheta+cg\rho_0(2\vartheta-\rho_0)\}\nabla_x\psi(x)\,\mathrm{d}x\\
&\quad=\int_G f(x,t)\psi(x)\,\mathrm{d}x,\quad\forall\psi\in H^1(G),
\end{aligned}\tag{60}$$

with the initial condition.

$$\vartheta(x,0)=\rho_{\mathrm{in}}(x).\tag{61}$$

Proof. Using $\psi\in C^\infty(\overline{G}\times[0,T])$, $\psi(T)=0$, as test function in (58) and taking into account (48)–(50), (55) and the properties of the s.2-s.m. convergence, we obtain finally (60) and the relation

$$\begin{aligned}
&\vartheta(x,0)\int_\Omega\left(\Phi^*\chi_{\Omega\setminus\mathcal{M}}(\omega)+\varphi(\omega)\chi_{\mathcal{M}}(\omega)\right)\mathrm{d}\mu+\int_{\mathcal{M}}\varphi(\omega)v(x,0,\omega)\,\mathrm{d}\mu\\
&\quad=\rho_{\mathrm{in}}\int_\Omega\left(\Phi^*\chi_{\Omega\setminus\mathcal{M}}(\omega)+\varphi(\omega)\chi_{\mathcal{M}}(\omega)\right)\mathrm{d}\mu,
\end{aligned}\tag{62}$$

which, in turn, implies (61). □

Remark 3.9. Eq. (60) can also be rewritten as follows:

$$\begin{aligned}
&\mathbb{E}\{\Phi^*\chi_{\Omega\setminus\mathcal{M}}(\omega)+\varphi(\omega)\chi_{\mathcal{M}}(\omega)\}\frac{\partial}{\partial t}\vartheta(x,t)+\frac{\partial}{\partial t}\mathbb{E}\{\chi_{\mathcal{M}}\varphi(\omega)v(x,t,\omega)\}\\
&\quad-\operatorname{div}\left\{\frac{1}{\lambda c}\mathcal{A}_N^0K^*\left[\nabla\vartheta+c\rho_0(2\vartheta-\rho_0)g\right]\right\}=f\quad\text{in } G\times]0,T[,\\
&\mathcal{A}_N^0\left[\nabla\vartheta+c\rho_0(2\vartheta-\rho_0)g\right]\cdot\nu=0\quad\text{on }\partial G\times]0,T[,\\
&\vartheta(x,0)=\rho_{\mathrm{in}}(x)\quad\text{on } G.
\end{aligned}$$

Remark 3.10. The tensor $\mathcal{A}_N^0$ defined by (29), characterizes the relation between the initial fractures permeability K^* and the efficient permeability tensor in (60), depending on the geometry of the fractures system. This tensor is the rigorous version of the so-called "tortuosity factor" widely used in the engineering literature.

The result of the limiting process is summarized in the following proposition.

Proposition 3.11. *Let $\vartheta\in L^2(0,T;H^1(G))$, $\partial_t\vartheta\in L^2(]0,T[\times G)$, and $v\in L^2(]0,T[\times G;Z)$, $\partial_tv\in L^2(]0,T[\times G\times\mathcal{M})$, be defined by* (48)–(51). *Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_G\left[\int_\Omega\left[\Phi^*\chi_{\Omega\setminus\mathcal{M}}+\varphi\chi_{\mathcal{M}}\right]\mathrm{d}\mu\,\vartheta(x,t)+\int_{\mathcal{M}}\varphi(\omega)v(x,t,\omega)\,\mathrm{d}\mu\right]\psi_1(x)\,\mathrm{d}x\right.$$

$$
\begin{aligned}
&+\int_G\int_{\mathcal{M}}\big(\vartheta(x,t)+v(x,t,\omega)\big)\varphi(\omega)\xi(\omega)\psi_2(x)\,\mathrm{d}\mu\,\mathrm{d}x\Big\}\\
&+\frac{1}{\lambda c}\int_G\mathcal{A}_N^0K^*\big[\nabla_x\vartheta(x,t)+cg\rho_0\big(2\vartheta(x,t)-\rho_0\big)\big]\nabla_x\psi_1(x)\,\mathrm{d}x\\
&+\int_G\int_{\mathcal{M}}\frac{k(\omega)}{\lambda c}\big[\nabla_\omega v(x,t,\omega)+c\sigma_0(\omega)\big(2\vartheta(x,t)+2v(x,t,\omega)-\sigma_0(\omega)\big)g\big]\nabla_\omega\xi(\omega)\psi_2\,\mathrm{d}\mu\,\mathrm{d}x\\
&=\int_G\int_{\mathcal{M}}f(x,t)\xi(\omega)\psi_2(x)\,\mathrm{d}\mu\,\mathrm{d}x+\int_G f(x,t)\psi_1(x)\,\mathrm{d}x\quad \textit{a.e. on }]0,T[,\\
&\forall\psi_1,\psi_2\in H^1(G)\ \textit{and}\ \forall\xi\in Z,
\end{aligned}\tag{63}
$$

$$
v(x,0,\omega)=0\quad \textit{in } L^2(G;Z),\qquad \vartheta(x,0)=\rho_{\mathrm{in}}\quad \textit{in } H^1(G).\tag{64}
$$

Problem (63), (64) *has a unique solution.*

Proof. It is enough to prove uniqueness. Let $\{\vartheta, v\}$ be a solution for the homogeneous problem. Then we have

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\{\varPhi^*\chi_{\Omega\setminus\mathcal{M}}(\omega)\}\int_G\vartheta^2(x,t)\,\mathrm{d}x+\int_G\int_{\mathcal{M}}\frac{\varphi(\omega)}{2}\big(\vartheta(x,t)+v(x,t,\omega)\big)^2\,\mathrm{d}\mu\,\mathrm{d}x\\
&+\frac{1}{\lambda c}\int_0^t\int_G\mathcal{A}_N^0K^*\nabla_x\vartheta(x,\tau)\nabla_x\vartheta(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\\
&+\frac{1}{\lambda c}\int_0^t\int_G\int_{\mathcal{M}}k(\omega)\nabla_\omega v(x,\tau,\omega)\nabla_\omega v(x,\tau,\omega)\,\mathrm{d}\mu\,\mathrm{d}x\\
&+\frac{2\rho_0 g}{\lambda}\int_0^t\int_G\mathcal{A}_N^0K^*\vartheta(x,\tau)\nabla_x\vartheta(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\\
&+\frac{2g}{\lambda}\int_0^t\int_G\int_{\mathcal{M}}k(\omega)\big(\vartheta(x,\tau)+v(x,\tau,\omega)\big)\nabla_\omega v(x,\tau,\omega)\,\mathrm{d}\mu\,\mathrm{d}x=0.
\end{aligned}\tag{65}
$$

Now we apply Gronwall's inequality and get $\vartheta=0$ and $v=0$. □

4. Examples of random structures

In this last section we deal with the models of random structures commonly used in various applications.

Example 1. We start by considering the classical disperse media model defined in [12]. Let S be a piecewise smooth bounded domain, we say that a random ergodic two-component medium is a disperse medium if a.s. one of the component (fractures system) is connected and unbounded and the other one (matrix blocks) consists of bounded connected closed sets $S_j(\omega)$ having no points in common, and being obtained from S by a successful dilatation, rotation and shift. Moreover, we assume that the dilatation coefficient $\theta_j(\omega)$ belongs to a nonrandom interval $[\theta_\star,\theta^\star]$, $0<\theta_\star\leqslant\theta^\star<\infty$. In what follows the symbol $\mathcal{H}_j(\omega)$ stands for the linear transformation that maps S onto $S_j(\omega)$.

In the example below the extension condition is assumed to hold (see [12] for details), and for the sake of simplicity we assume the coefficients k^ε and φ^ε in (8) to be constant and σ_0^ε to be equal to zero.

Under the above assumptions the solution of the second auxiliary problem (45), (46) can be computed by means of the Green function $Q(y,z,t)$ for the operator $\varphi\partial/\partial t - (k/(\lambda c))\Delta_y$ in the cylinder $S \times S \times (0,\infty)$ with homogeneous Dirichlet boundary conditions. Namely, if we denote by $\tilde{v}(x,t,y,\omega)$ a realization $v(x,t,\mathcal{T}(y)\omega)$ of $v(x,t,\omega)$, then the auxiliary problem (56), (57) reads a.s.

$$\varphi\left(\frac{\partial}{\partial t}\vartheta(x,t) + \frac{\partial}{\partial t}\tilde{v}(x,t,y,\omega)\right) - \frac{k}{\lambda c}\Delta_y \tilde{v}(x,t,y,\omega) = f(x,t), \quad (y,t) \in \bigcup_j S_j(\omega) \times (0,T), \tag{66}$$

$$\tilde{v}|_{t=0} = 0, \qquad \tilde{v}|_{\mathbb{R}^N \setminus \bigcup_j S_j} = 0. \tag{67}$$

Let $Q(y,z,t)$ be the above Green function in $S \times S \times (0,\infty)$, i.e., the function $Q(y,z,t-\tau)$ satisfying for a.e. $z \in S$

$$\begin{aligned} &\varphi\frac{\partial}{\partial t}Q(y,z,t-\tau) - \frac{k}{\lambda c}\Delta_y Q(y,z,t-\tau) = \delta(y-z)\delta(t-\tau) \quad \text{in } S \times (0,\infty), \\ &Q|_{t=\tau} = 0, \qquad Q(y,z,t-\tau) = 0 \quad \text{on } \partial S \times (0,\infty). \end{aligned} \tag{68}$$

It is well known (see, e.g., [15]) that

$$\left|\frac{\partial^r}{\partial t^r}D_y^s Q\right| \leqslant C(t-\tau)^{-(n+2r+s)/2}\exp\left\{-c_0\frac{|y-z|^2}{t-\tau}\right\} \tag{69}$$

for $2r + |s| = 1, 2$ and these derivatives are Hölder continuous in the arguments z and τ.

Clearly, in any cylinder $S_j(\omega) \times S_j(\omega) \times (0,T)$, the corresponding Green function satisfies the relation

$$Q_j(y,z,t) = \theta_j^{-N} Q\left(H_j^{-1}(y), H_j^{-1}(z), \frac{t}{\theta_j^2}\right) \tag{70}$$

and thus for $y \in S_j$ we have

$$\begin{aligned} \tilde{v}(x,t,y,\omega) = &\int_0^t \int_{S_j} \theta_j^{-N} Q\left(H_j^{-1}(y), H_j^{-1}(z), \frac{t-s}{\theta_j^2}\right)\left(f(x,s) - \varphi\frac{\partial}{\partial s}\vartheta(x,s)\right) \mathrm{d}z\,\mathrm{d}s \\ &\in W_2^{2,1}\big(S_j \times (0,T)\big). \end{aligned} \tag{71}$$

We note that x is just a parameter, entering through f and $\partial\vartheta/\partial t$.

Now, from the definition of $\tilde{v}$, we get

$$v(x,t,\omega) = \tilde{v}(x,t,0,\omega). \tag{72}$$

In order to characterize the limit operator we introduce the "distribution" function for the dilatation coefficient:

$$F(\theta) = \lim_{r\to\infty}\left\{\frac{\#j\colon\ S_j(\omega) \subset B_r,\ \theta_j \leqslant \theta}{|B_r|}\right\},$$

where # stands for the number of elements in a set, B_r is the ball of radius r centered at the origin and $|B_r|$ its volume. The latter limit does exist by the Birkhoff theorem, and the function $F(\theta)$ becomes the distribution function of a dilatation coefficient after proper normalization.

Applying once more the Birkhoff theorem to the solution of problem (66) and taking into account (70)–(72), we get

$$\begin{aligned}
\int_{\mathcal{M}} v(x,t,\omega)\mu(\mathrm{d}\omega) &= \lim_{r\to\infty}\left\{\frac{1}{|B_r|}\int_{B_r}\tilde{v}(x,t,y,\omega)\,\mathrm{d}y\right\}\\
&= \lim_{r\to\infty}\left\{\frac{1}{|B_r|}\sum_{S_j\subset B_r}\int_0^t\int_{S_j}\int_{S_j} Q_j(y,z,t-s)\Big(f(x,s)-\varphi\frac{\partial}{\partial s}\vartheta(x,s)\Big)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}z\right\}\\
&= \int_0^t\Big(f(x,s)-\varphi\frac{\partial}{\partial s}\vartheta(x,s)\Big)\int_{\theta_\star}^{\theta^\star}\mathrm{d}F(\theta)\int_{\theta S}\int_{\theta S}\theta^{-N}Q\Big(\frac{y}{\theta},\frac{z}{\theta},\frac{t-s}{\theta^2}\Big)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}z\\
&= \int_0^t\Big(f(x,s)-\varphi\frac{\partial}{\partial s}\vartheta(x,s)\Big)\kappa_F(t-s)\,\mathrm{d}s,
\end{aligned}$$

where we have used the notation

$$\kappa_F(s) = \int_{\theta_\star}^{\theta^\star}\mathrm{d}F(\theta)\int_{\theta S}\int_{\theta S}\theta^{-N}Q\Big(\frac{y}{\theta},\frac{z}{\theta},\frac{s}{\theta^2}\Big)\,\mathrm{d}y\,\mathrm{d}z. \tag{73}$$

Finally, with (73) the limit problem (60), (61), (56), (57) reads

$$\begin{aligned}
&\big[\mu(\Omega\setminus\mathcal{M})\Phi^* + \mu(\mathcal{M})\varphi\big]\frac{\partial}{\partial t}\vartheta(x,t) - \frac{1}{\lambda c}\,\mathrm{div}_x\big(\mathcal{A}_N^0 K^*\nabla_x\vartheta(x,t)\big)\\
&\quad = f(x,t) - \frac{\partial}{\partial t}\Big(\int_0^t\Big(f(x,s)-\varphi\frac{\partial}{\partial s}\vartheta(x,s)\Big)\kappa_F(t-s)\,\mathrm{d}s\Big),\quad (x,t)\in G\times\,]0,T[, \qquad (74)\\
&\vartheta(x,0) = \rho_{\mathrm{in}},\qquad \mathcal{A}_N^0\nabla\vartheta(x,t)\cdot\nu = 0\quad \text{on } \partial G\times(0,T),
\end{aligned}$$

$$\begin{aligned}
&\varphi\Big(\frac{\partial}{\partial t}\vartheta(x,t)+\frac{\partial}{\partial t}\tilde{v}(x,t,y,\omega)\Big) - \frac{k}{\lambda c}\Delta_y\tilde{v}(x,t,y,\omega) = f(x,t),\quad (y,t)\in\bigcup_j S_j(\omega)\times(0,T),\\
&\tilde{v}(x,0,y,\omega) = 0,\quad \tilde{v}(x,t,y,\omega) = 0,\quad y\in\mathbb{R}^n\setminus\bigcup_j S_j(\omega).
\end{aligned} \tag{75}$$

Example 2 (*Generalized "disperse" media***).** In case of classical disperse media each inclusion can be obtained from a sample set S by a linear diffeomorphism, and we generalize this construction as follows. Let S be the same as in the preceding example, and assume $\mathcal{H}(\theta_1,\dots,\theta_k)$ is a k-parametrized family of diffeomorphisms mapping a fixed neighborhood S_δ of the sample set S into $\mathbb{R}^n$, such that

$$\left|\frac{\partial\mathcal{H}_i}{\partial x_j}\right| \leqslant c < \infty,\quad i,j = 1,\dots,N,\qquad \det\left|\frac{\partial\mathcal{H}}{\partial x}\right| \geqslant c_1 > 0;$$

moreover, let $\theta\to\mathcal{H}(\theta)$ be continuous and injective.

A random medium is said to be generalized disperse medium if a.s. the matrix part consists of bounded connected closed components having no points in common; each such a component $S_j(\omega)$ can be obtained from the sample set S by a successful diffeomorphism from the family $\{\mathcal{H}(\theta)\}$.

Now, if we denote by $F(\mathrm{d}\theta)$ the following measure in $\mathbb{R}^n$:

$$\int_{\mathcal{O}} 1\,\mathrm{d}F(\theta) = \lim_{r\to\infty}\left\{\frac{\#j\colon\ S_j(\omega)\subset B_r,\ S_j=\mathcal{H}(\theta)(S),\ \theta\in\mathcal{O}}{|B_r|}\right\}$$

for any Borel set $\mathcal{O}\subset\mathbb{R}^n$, then this measure F is well-defined due to the Birkhoff theorem. Denote by $Q(t,y,z,\theta)$ the Green function of the operator $\varphi\partial/\partial t-(k/(\lambda c))\Delta_y$ stated in the cylinder $\mathcal{H}(\theta)(S)\times(0,\infty)$ with homogeneous Dirichlet boundary conditions. In the same way as above, one can prove that the limit problem (60), (61) has the form (74), (75) with $\kappa_F(s)$ given by

$$\kappa_F(s) = \int_{\mathbb{R}^k}\mathrm{d}F(\theta)\int_{\mathcal{H}(\theta)(S)}\int_{\mathcal{H}(\theta)(S)} Q(s,y,z,\theta)\,\mathrm{d}y\,\mathrm{d}z.$$

Example 3 (*Coated perforation***).** Consider the same construction as in Example 1, and assume that the sample set is now $S\setminus S^1$, where S is like in Example 1, and S^1 is a smooth open subdomain of S compactly embedded to S. Thus, the fractures system consists a.s. of two parts, one of them is connected unbounded component $\mathbb{R}^n\setminus\bigcup_j S_j$ and the other one is the union $\bigcup_j S_j^1$ of bounded components (inclusions) situated inside all the matrix blocks.

Denote by $\mathcal{M}'$ the subset of Ω that corresponds to the whole system of inclusions S_j:

$$\bigcup_j S_j(\omega) = \{y\in\mathbb{R}^n\colon\ \mathcal{T}(y)\omega\in\mathcal{M}'\}.$$

Evidently, $\mathcal{M}'$ is the union $\mathcal{M}$ and the subset $\mathcal{F}'$ related to all the bounded inclusions S_j^1. Although in this fractured porous medium model the fractures system is not connected and thus one of our assumptions is violated, still, after a slight modification, the technique developed in Sections 1–3 applies and we obtain the following assertion.

Lemma 4.1. *Assume the extension condition for $\mathcal{M}'\subset\Omega$ to be satisfied. Then,* (48), (50) *and* (49) *hold in $\mathcal{M}'$ and $\Omega\setminus\mathcal{M}'$ respectively*; *the function $\vartheta(x,t)$ satisfies the equation obtained from* (60) *by replacing $\mathcal{M}$ by $\mathcal{M}'$ everywhere, and the function $\tilde{v}(x,t,y,\omega)=v(x,t,\mathcal{T}(y)\omega)$ is a solution to the following auxiliary problem*

$$\begin{aligned}
&\varphi\frac{\partial}{\partial t}\{\vartheta(x,t)+\tilde{v}(x,t,y)\} = \frac{k}{\lambda c}\Delta_y\tilde{v}(x,t,y)+f(x,t), \quad y\in S_j\setminus S_j^1,\\
&\nabla_y\tilde{v}=0,\quad y\in S_j^1,\\
&\varphi\frac{\partial}{\partial t}\big(\tilde{v}(x,t,y)+\vartheta(x,t)\big) = \frac{1}{|S_j^1|}\int_{\partial S_j^1}\frac{k}{\lambda c}\left(\frac{\partial\tilde{v}}{\partial n_y}\right)^{\mathrm{out}}\mathrm{d}s_y+f(x,t),\quad y\in S_j^1,\\
&\tilde{v}(x,0,y)=0,\qquad \tilde{v}|_{\mathbb{R}^n\setminus\bigcup_i S_i}=0,
\end{aligned}\tag{76}$$

where n_y is outer normal to S_j^1, and symbol $(\cdot)^{\mathrm{out}}$ indicates the limit of the corresponding function from the complement of S_j^1.

Remark 4.1. Let

$$V = \{ z \in H^1(S_j) \mid z \text{ is an unknown constant with respect to } y \text{ on } S_j^1,\ z = 0 \text{ on } \partial S_j \}$$

and let H be the closure of V in $L^2(S_j)$. Then the problem (76) is well-posed, and the corresponding variational formulation is:

find a function $\tilde{v} \in L^2(0,T;V) \cap C([0,T];H)$, for a.e. $x \in G$, such that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\varphi \int_{S_j} \tilde{v}(x,t,y)\zeta(y)\,\mathrm{d}y + \frac{k}{\lambda c}\int_0^t \int_{S_j \setminus S_j^1} \nabla_y \tilde{v}(x,t,y)\nabla_y \zeta(y)\,\mathrm{d}y \\ &\quad = \int_{S_j} \left\{ f(x,t) - \varphi \frac{\partial}{\partial t}\vartheta(x,t) \right\} \zeta(y)\,\mathrm{d}y, \quad \forall \zeta \in V, \\ &\tilde{v}(x,0,y) = 0. \end{aligned} \tag{77}$$

This problem is similar to that arising when homogenizing the hard inclusions, and for more details we refer to [12].

Remark 4.2. If we denote by $Q(y,z,t)$ the "Green function" of problem (76) posed on a sample set S, i.e., the Schwartz kernel of the corresponding semigroup, then the relation (70) holds and the limit problem can be represented like in (74).

We proceed with examples involving random Voronoi tessellation structures.

Example 4 (*Voronoi tessellations***).** We begin by defining a spatial process (or random point set). Denote by Γ the space of all locally finite subsets in $\mathbb{R}^n$; for any bounded Borel set $B \subset \mathbb{R}^n$, we define the mapping $N_B : \Gamma \to \{0,1,2,\ldots\}$ as follows

$$N_B(\gamma) = \#(\gamma \cap B), \quad \gamma \in \Gamma,$$

where # stands for the number of elements in a set, then we equip Γ with the minimal σ-algebra of its subsets that makes N_B measurable for any bounded Borel set B in $\mathbb{R}^n$, and denote this σ-algebra by $\mathcal{G}$. Let (Ω, Ξ, μ) be a probability space.

Then, any $(\Xi, \mathcal{G})$-measurable mapping $\Phi : \Omega \to \Gamma$ is called spatial process in $\mathbb{R}^n$.

In what follows we assume the spatial process $\Phi = \{x_i\}_{i=1}^{\infty}$ to be stationary, i.e., to have invariant distribution with respect to all translations in $\mathbb{R}^n$. Thus the distribution of Φ coincides with that of $\Phi + y = \{x + y\colon\ x \in \Phi\}$ for any $y \in \mathbb{R}^n$.

Similarly, Φ is said to be isotropic if its distribution is invariant under any rotation about the origin in $\mathbb{R}^n$.

It can be shown (see [17], for instance) that the stationarity implies, for any bounded Borel set B, the relation

$$\Lambda(B) \stackrel{\text{def}}{=} \mathbb{E}\{N_B(\Phi)\} = \tau |B|$$

with some constant τ, $0 \leqslant \tau \leqslant \infty$, called the intensity of Φ.

Given a random point set $\Phi = \{x_i\}_{i=1}^{\infty}$, each x_i generates a random cell $C(x_i|\Phi)$ (called Voronoi cell) in the following way

$$C(x_i|\Phi) = \{z \in \mathbb{R}^n\colon\ |z - x_i| \leqslant |z - x_j| \text{ for all } x_j \in \Phi\}. \tag{78}$$

Next, we recall the important statement proved, for example, in [17].

Proposition 4.3. *Let Φ be a stationary spatial process, and assume $\Phi \neq \emptyset$ a.s. Then a.s. all the Voronoi cells $C(x_i|\Phi)$ are bounded convex polytopes.*

Detailed information on point processes and Voronoi tessellations can be found in [17,9].

Denote by $Z(x_i|\Phi)$ the centers of gravity of the Voronoi cells $C(x_i|\Phi)$. Then, given a small positive α, we introduce $C^\alpha(x_i|\Phi)$ to be the homothetic dilatation of $C(x_i|\Phi)$ with respect to $Z(x_i|\Phi)$ with coefficient $1-\alpha$. Let Π be the space of convex polytopes in $\mathbb{R}^n$ endowed with a proper σ-algebra $\mathcal{U}$. Then for any bounded Borel set B in $\mathbb{R}^n$ such that $|B| > 0$, the distribution on $(\Pi, \mathcal{U})$ defined by

$$P_n^\alpha(U) = \mathbb{E}\left\{ \sum_{x_i \in \Phi} (\mathbf{1}_{\{Z(x_i|\Phi) \in B\}}\, \mathbf{1}_{\{C^\alpha(x_i|\Phi) - Z(x_i|\Phi) \in U\}}) \right\} \Big/ (\tau |B|), \quad U \in \mathcal{U},$$

is said to be a Palm measure of $C^\alpha(x_i|\Phi)$; here and in what follows we assume that $0 < \tau < \infty$. It can be shown that P_n^α does not depend on B, thus P_n^α is well-defined probability measure on $(\Pi, \mathcal{U})$ that characterizes the distribution of the typical cell.

We proceed by introducing the random double-porosity model generated by Voronoi tessellation. Given a stationary spatial process Φ in $\mathbb{R}^n$, we consider a random set defined by

$$\mathbf{C}^\alpha(\omega) = \bigcup_{x_i \in \Phi} C^\alpha\big(x_i(\omega)|\Phi(\omega)\big)$$

and may identify $\mathbf{C}^\alpha(\omega)$ with a random matrix blocks set $M(\omega)$. Then, $\mathcal{M} = \{\omega\colon\ 0 \in \mathbf{C}^\alpha(\omega)\}$. As usually, we assume ergodicity, and denote by $Q_p(y,z,t)$ the Green function of the operator $\varphi\partial/\partial t - (k/(\lambda c))\Delta_y$ in the cylinder $p \times (0, +\infty)$ with the homogeneous Dirichlet boundary condition, where p is a convex polytope in $\mathbb{R}^n$.

The statement below describes the double porosity model in case of a general spatial process Φ.

Theorem 4.4. *Let all the above assumptions hold, and assume, moreover, that $\mathcal{A}_N^0$, the homogenized Neumann tensor defined in* (29)*, is positive definite. Then the limit problem* (60), (61) *becomes*

$$\begin{aligned}
&[\mu(\Omega \setminus \mathcal{M})\Phi^* + \mu(\mathcal{M})\varphi] \frac{\partial}{\partial t}\vartheta(x,t) - \frac{1}{\lambda c}\operatorname{div}_x\big(\mathcal{A}_N^0 K^* \nabla_x \vartheta(x,t)\big) \\
&\quad = f(x,t) - \frac{\partial}{\partial t}\left(\int_0^t \left(f(x,s) - \varphi\frac{\partial}{\partial s}\vartheta(x,s)\right)\kappa_P^\alpha(t-s)\,\mathrm{d}s\right), \quad (x,t) \in G \times\,]0,T[,
\end{aligned} \tag{79}$$

$$\varphi\left(\frac{\partial}{\partial t}\vartheta(x,t) + \frac{\partial}{\partial t}\tilde{v}(x,t,y,\omega)\right) - \frac{k}{\lambda c}\Delta_y \tilde{v}(x,t,y,\omega) = f(x,t), \quad y \in \bigcup_j p_j(\omega), \tag{80}$$

$$\tilde{v}(x,t,y,\omega) = 0 \quad \textit{on} \ \bigcup_j \partial p_j(\omega)$$

with $\kappa_P^\alpha(s) = \lambda \int_\Pi P_n^\alpha(\mathrm{d}p) \int_p \int_p Q_p(y,z,s)\,\mathrm{d}y\,\mathrm{d}z$.

An important particular case of a Voronoi tessellation widely used in hydrogeology, is the so-called Poisson–Voronoi tessellation.

Definition 4.5. A Voronoi tessellation is called the Poisson–Voronoi diagram if for any bounded Borel set B:

– $N_B(\varPhi)$ has Poisson distribution with mean $\tau|B|$, $\tau > 0$;
– given the event $N_B(\varPhi) = k$, all the $x_1, x_2, \dots, x_k$ are independent and uniformly distributed in B.

The Poisson–Voronoi diagram is ergodic for any $\tau > 0$. It follows, for instance, from the mixing properties obtained in [11].

Unfortunately, the positive definiteness of $\mathcal{A}_N^0$ in case of the Poisson–Voronoi diagram is still an open question, and we should modify slightly our fractured double porosity model to achieve the non-degeneracy of the tensor $\mathcal{A}_N^0$. Namely, using the notation $r(x_i|\varPhi)$ for the diameter of the largest ball contained in $C(x_i|\varPhi)$, and $d(x_i|\varPhi)$ for the diameter of $C(x_i|\varPhi)$, we define a matrix blocks set as follows

$$\mathbf{C}^{\alpha,\delta}(\omega) = \bigcup_{\substack{x_i\in\varPhi,\ r(x_i|\varPhi)\geqslant\delta \\ d(x_i|\varPhi)\leqslant\delta^{-1}}} C^\alpha(x_i|\varPhi), \tag{81}$$

δ is a small positive number. By the definition (81), all the very big components as well as very small ones have been removed from the random set, and it is then easy to show that $\mathbf{C}^{\alpha,\delta}(\omega)$ possess almost surely the following properties:

(1) $\mathbf{C}^{\alpha,\delta}(\omega)$ consists of isolated convex components;
(2) the diameter of each component does not exceed δ^{-1};
(3) the distance between any two components is not less than $\alpha\delta$.

The above properties (1)–(3) ensure the non-degeneracy of $\mathcal{A}_N^0 > 0$; one can easily show this by means of the variational formula for $\mathcal{A}_N^0$ (see [12]). The ergodicity is trivially inherited. Letting now, for any bounded Borel set B of positive measure and $U \in \varPi$,

$$P_n^{\alpha,\delta}(U) = \frac{\mathbb{E}\{\sum_{x_i\in\varPhi}(\mathbf{1}_{\{Z(x_i|\varPhi)\in B\}}\,\mathbf{1}_{\{r(x_i|\varPhi)\geqslant\delta,\ d(x_i|\varPhi)\leqslant\delta^{-1}\}}\,\mathbf{1}_{\{C^\alpha(x_i|\varPhi)-Z(x_i|\varPhi)\in U\}})\}}{\tau(\delta|B|_n)},$$

with

$$\tau(\delta) = \frac{\mathbb{E}(\#\{x_i \in B\colon\ r(x_i|\varPhi) \geqslant \delta,\ d(x_i|\varPhi) \leqslant \delta^{-1}\})}{|B|},$$

we arrive at the following statement.

Theorem 4.6. *Let the random matrix blocks set be given by* $C^{\alpha,\delta}(\omega)$*. Then the limit equations* (60), (61) *have the form* (79), (80) *with*

$$\kappa_P^\alpha(s) = \tau(\delta) \int_\Pi P_n^{\alpha,\delta}(\mathrm{d}p) \int_p \int_p Q_p(y,z,s)\,\mathrm{d}y\,\mathrm{d}z.$$

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