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On homogenization of networks and junctions

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Abstract. In the paper we propose a new approach to the homogenization theory on periodic wire-networks and junctions, based on singular measures on these structures. We characterize the Sobolev spaces on such constructions and describe the fields of potential and solenoidal (divergence free) vector-function. Then we compare the effective coefficients obtained for the singular structures and the classical effective coefficients for thin constructions with vanishing thickness, and show that the corresponding diagram is commutative.

0. Introduction

In the paper we develop a new approach to the homogenization problems stated on periodic networks and junctions.

The method elaborated in this work provides convenient tools for studying rod-constructions, skeletal and lattice structures and other thin constructions. The investigation of such models is important to researchers working with cellular materials (lightweight materials) such as honeycombs, foams, wood, cancellous bone, corks. Other modern engineering applications are space antennas, solar panels, civil engineering technologies and many others. Concerning methods for attacking such problems in a classical engineering way we refer to [10].

The classical homogenization techniques (see, for example, [3–5,8,9,16,30]) involve resolving auxiliary PDE problems which makes the homogenization procedure quite complicated from the numerical point of view.

The classical homogenization method and the classical engineering approach have recently been compared for some interesting problems in [19,31] (see, also, [20]).

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In contrast to the standard homogenization technique, our approach inspired by the ideas from [12–15], enables to deal not only with the classical fine-scale structures but also with the problems stated on infinitely thin constructions whose description involves singular measures. In the present work we develop the measure approach for networks and junctions. This method allows us to reduce essentially the computations in various applications. On the other hand, it requires a delicate analysis of Sobolev spaces with nonabsolutely continuous measures. In the first part of this work we provide rigorous definitions of such spaces, investigate their properties and describe important functional classes such as the fields of potential and solenoidal (divergence free) vectors.

The importance of practical applications stimulated mathematical research in the area. There are several recent works devoted to the homogenization of thin structures and other singular media. We quote here the works [1,2,7,8,17,18,21–27].

An interesting attempt to simplify the homogenization process for thin rod-structures was undertaken in [28,29], where the author replaced the equations in the interior parts of rods by one-dimensional equations stated on the respective segments.

The last section of the paper is devoted to the homogenization problems on singular structures. We consider the limit of the effective coefficients obtained for thin structures by the classical homogenization method, as the thickness vanishes, and show that our method gives the same values of the effective coefficients.

For simplicity in this work we only consider 2D constructions involving straight segments and regular junctions. The techniques developed here also apply in the case of curved multidimensional structures.

1. Sobolev spaces on singular sets

Let Ω be a domain in $\mathbb{R}^2$, and suppose that μ is a Borel finite positive (for example, probability) measure on Ω. The space $L_2(\Omega, \mathrm{d}\mu)$ is defined in a usual way, the corresponding norm is

$$\|u\|^2 = \int_\Omega |u(x)|^2 \,\mathrm{d}\mu.$$

We introduce the space $H^1(\Omega, \mathrm{d}\mu)$ as follows:

Definition 1. A function $u(x)$ belongs to $H^1(\Omega, \mathrm{d}\mu)$, if there exist a sequence $\{u_n\}$, $u_n \in C^\infty(\overline{\Omega})$, and $z \in (L_2(\Omega, \mathrm{d}\mu))^2$ such that

$$u_n \to u \quad \text{in } L_2(\Omega, \mathrm{d}\mu) \tag{1}$$

and

$$\nabla u_n \to z \quad \text{in } \big(L_2(\Omega, \mathrm{d}\mu)\big)^2. \tag{2}$$

We say that z is a gradient of u and denote it by ∇u.

Remark 1. In the above definition the strong convergence in (1) and (2) can be replaced by the weak convergence in the same spaces. In what follows we verify the weak convergence.

In general, the gradient of $H^1(\Omega, \mathrm{d}\mu)$-function is not unique (see, for instance, Proposition 1). We say that a function z is a gradient of zero if there exists a sequence $u_n \in C^\infty(\overline{\Omega})$ such that $u_n \to 0$ and $\nabla u_n \to z$, as $n \to \infty$, in $L_2(\Omega, \mathrm{d}\mu)$, and denote the set of gradients of zero by $\Gamma(0)$. It is easy to see that $\Gamma(0)$ is a closed subspace of $(L_2(\Omega, \mathrm{d}\mu))^2$. The gradient of a $H^1(\Omega, \mathrm{d}\mu)$-function is defined as the corresponding equivalence class.

1.1. Segments

Let $I = \{x \mid a \leqslant x_1 \leqslant b;\ x_2 = 0\}$ be a segment in $\mathbb{R}^2$, and suppose that a bounded domain Ω contains I. For any sufficiently small $\delta > 0$ consider the bar $I_\delta := \{x \mid a < x_1 < b;\ -\delta < x_2 < \delta\} \subset \Omega$ (see Fig. 1).

Denote by μ_δ the probability measure in Ω, concentrated and uniformly distributed on I_δ:

$$\mu_\delta(\mathrm{d}x) = \frac{\mathbf{1}_{x \in I_\delta}}{\delta(b-a)}\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

It is easy to see that the family μ_δ converges weakly, as $\delta \to 0$, to a singular probability measure μ concentrated on the segment I and uniformly distributed on it. In terms of distributions this measure μ can be represented as follows $\mu(\mathrm{d}x) = \frac{1}{b-a}\,\mathrm{d}x_1 \times \delta(x_2)$, where $\delta(z)$ stands for the Dirac mass at zero. Consider a family of smooth functions u_δ subject to the bound

$$\int_\Omega \left(u_\delta^2 + |\nabla u_\delta|^2\right) \mathrm{d}\mu_\delta \leqslant C.$$

Then there are functions $u_0 \in L_2(\Omega, \mathrm{d}\mu)$ and $z = (z_1, z_2) \in (L_2(\Omega, \mathrm{d}\mu))^2$ such that

$$u_\delta \rightharpoonup u_0, \quad \nabla u_\delta \rightharpoonup z \quad \text{weakly as } \delta \to 0 \tag{3}$$

(see [13]). The latter convergence is defined as follows: for any functions $\varphi \in C_0^\infty(\mathbb{R}^2)$, $\psi \in (C_0^\infty(\mathbb{R}^2))^2$

$$\int_\Omega u_\delta \varphi\,\mathrm{d}\mu_\delta \to \int_\Omega u_0 \varphi\,\mathrm{d}\mu, \tag{4}$$

$$\int_\Omega (\nabla u_\delta, \psi)\,\mathrm{d}\mu_\delta \to \int_\Omega (z, \psi)\,\mathrm{d}\mu, \tag{5}$$

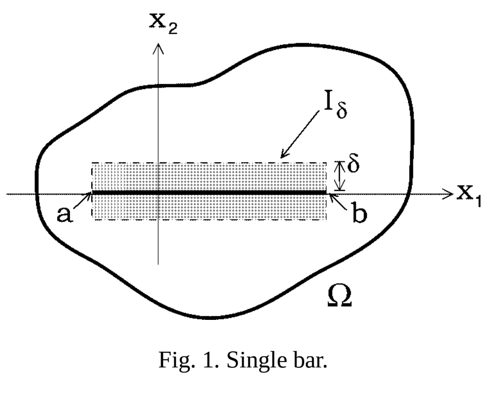

Fig. 1. Single bar.

as $\delta \to 0$.

Let us recall that, according to Definition 1, a function u is an element of $H^1(\Omega, \mathrm{d}\mu)$ if there are a sequence of functions $u_n \in C^\infty(\overline{\Omega})$ and $z \in (L_2(\Omega, \mathrm{d}\mu))^2$ such that (1), (2) hold true.

Remark 2. Note that $\mu(\Omega \setminus I) = 0$. Therefore all the functions taking the same values on the segment I, coincide as elements of $L_2(\Omega, \mathrm{d}\mu)$. Thus due to (1) and (2), an element of the space $H^1(\Omega, \mathrm{d}\mu)$ is uniquely defined by the respective element of the space $H^1([a,b])$. Later on we will identify these spaces.

Proposition 1. *For the measure μ introduced above, the gradient of a function $u \in H^1(\Omega, \mathrm{d}\mu)$ is not unique.*

Proof. Let us show that for an arbitrary function $u = u(x_1)$, $u(x_1) \in H^1([a,b])$, considered as a function of two variables (x_1, x_2), the corresponding gradient has the form:

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, w(x_1)\right), \tag{6}$$

where w is an arbitrary function from $L_2(\Omega, \mathrm{d}\mu)$. Indeed, setting

$$u_n(x_1, x_2) \equiv u(x_1) + x_2 w(x_1)$$

and smoothening u and w if necessary, we obtain the convergence $u_n \to u$ strongly in $L_2(\Omega, \mathrm{d}\mu)$, as $n \to +\infty$. Moreover,

$$\left.\frac{\partial u_n}{\partial x_1}\right|_{x_2=0} \to \frac{\partial u}{\partial x_1}, \qquad \left.\frac{\partial u_n}{\partial x_2}\right|_{x_2=0} \to w(x_1).$$

By Definition 1 $(\frac{\partial u}{\partial x_1}(x_1, x_2), w(x_1))$ is a gradient of u and $u \in H^1(\Omega, \mathrm{d}\mu)$. This completes the proof. □

Lemma 2. *The function u_0 defined in* (3) *belongs to $H^1(\Omega, \mathrm{d}\mu)$. Moreover, for u_0 and z from* (3) *the following relation holds*:

$$z = \nabla u_0. \tag{7}$$

Proof. An analysis of the proof of Proposition 1 shows that (7) follows from the relation

$$z_1 = \frac{\partial u_0}{\partial x_1}. \tag{8}$$

To obtain this relation we consider the family u_δ used in (3) and denote

$$\overline{u}_\delta(x_1) = \frac{1}{2\delta}\int_{-\delta}^{\delta} u_\delta \, \mathrm{d}x_2.$$

Consider a $C_0^\infty(\Omega)$-function φ, that depends only on x_1 in a neighbourhood of the segment I. From (4), we get

$$\int_\Omega u_\delta \varphi(x)\,\mathrm{d}\mu_\delta \equiv \frac{1}{2\delta(b-a)} \int_a^b \int_{-\delta}^{\delta} u_\delta \varphi(x)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \frac{1}{b-a}\int_a^b \overline{u}_\delta(x_1)\varphi(x_1)\,\mathrm{d}x_1$$
$$\to \frac{1}{b-a}\int_a^b u_0\varphi\,\mathrm{d}x_1.$$

Therefore $\overline{u}_\delta \rightharpoonup u_0$ in $L_2([a,b])$. Now,

$$\int_\Omega \frac{\partial u_\delta}{\partial x_1}\varphi(x)\,\mathrm{d}\mu_\delta \equiv \frac{1}{2\delta(a-b)} \int_a^b \int_{-\delta}^{\delta} \frac{\partial u_\delta}{\partial x_1}\varphi(x)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \frac{1}{b-a}\int_a^b \frac{\partial \overline{u}_\delta}{\partial x_1}\varphi(x_1)\,\mathrm{d}x_1$$
$$\to \frac{1}{b-a}\int_a^b z_1\varphi\,\mathrm{d}x_1.$$

On the other hand if we assume in addition that $\varphi = 0$ in the vicinity of the end-points of the segment, then

$$\int_\Omega \frac{\partial u_\delta}{\partial x_1}\varphi(x)\,\mathrm{d}\mu_\delta = -\int_\Omega u_\delta \frac{\partial \varphi(x)}{\partial x_1}\,\mathrm{d}\mu_\delta = -\frac{1}{b-a}\int_a^b \overline{u}_\delta(x_1)\frac{\partial\varphi(x_1)}{\partial x_1}\,\mathrm{d}x_1$$
$$\to -\frac{1}{b-a}\int_a^b u_0 \frac{\partial \varphi(x_1)}{\partial x_1}\,\mathrm{d}x_1.$$

This means that $z_1 = \partial u_0/\partial x_1$. The lemma is proved. $\square$

1.2. Cross

Let Ω be a bounded domain in $\mathbb{R}^2$, and denote by $X_\delta \subset \Omega$ the union of the crossing bars $\{x \mid a_1 < x_1 < b_1;\ -\delta < x_2 < \delta\} \cup \{x \mid -\delta < x_1 < \delta;\ a_2 < x_2 < b_2\}$ with $a_1 < 0 < b_1$ and $a_2 < 0 < b_2$ (see Fig. 2).

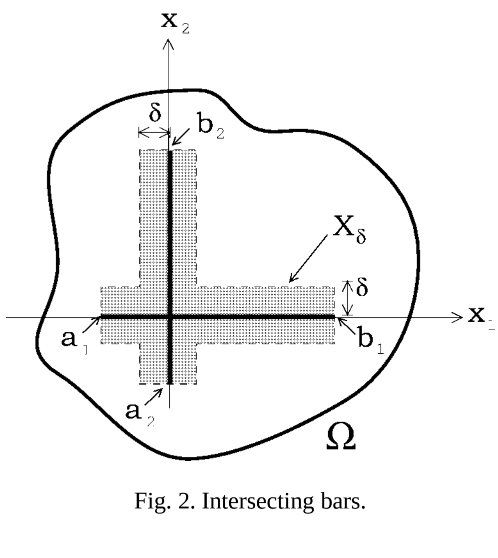

Fig. 2. Intersecting bars.

In this subsection the notation μ_δ is used for a family of probability measures in Ω, supported by the cross-bar X_δ and uniformly distributed on it. The weak limit of this family, as $\delta \to 0$, is a singular probability measure μ uniformly distributed on the cross $X := \{x \mid a_1 < x_1 < b_1;\ x_2 = 0\} \cup \{x \mid x_1 = 0;\, a_2 < x_2 < b_2\}$. Consider a family of smooth functions u_δ in Ω, subject to the bound

$$\int_\Omega \bigl(u_\delta^2 + |\nabla u_\delta|^2\bigr)\, \mathrm{d}\mu_\delta \leqslant C.$$

Then there are functions $u_0 \in L_2(\Omega, \mathrm{d}\mu)$ and $z = (z_1, z_2) \in (L_2(\Omega, \mathrm{d}\mu))^2$ such that

$$u_\delta \rightharpoonup u_0, \quad \nabla u_\delta \rightharpoonup z \quad \text{weakly as } \delta \to 0. \tag{9}$$

The latter convergence is defined in (4), (5).

The following statement characterizes the Sobolev space $H^1(\Omega, \mathrm{d}\mu)$ for the measure μ defined above or for a slightly more general measure on X. Let $\mu = \mu_1 + \mu_2 + \mu_3 + \mu_4$, and assume that μ_1, μ_2, μ_3 and μ_4 are singular positive measures uniformly distributed on the segments $\{x \mid a_1 < x_1 < 0;\ x_2 = 0\}$, $\{x \mid 0 < x_1 < b_1;\ x_2 = 0\}$, $\{x \mid x_1 = 0;\ a_2 < x_2 < 0\}$ and $\{x \mid x_1 = 0;\ 0 < x_2 < b_2\}$, respectively. The following result holds.

Lemma 3. *The function u belongs to $H^1(\Omega, \mathrm{d}\mu)$, if*

$$u \in H^1(\Omega, \mathrm{d}\mu_1) \cap H^1(\Omega, \mathrm{d}\mu_2) \cap H^1(\Omega, \mathrm{d}\mu_3) \cap H^1(\Omega, \mathrm{d}\mu_4)$$

and $u|_X$ is continuous at the origin.

Proof. This statement easily follows from (8) and the properties of H^1-functions in the one-dimensional case. $\square$

Now introducing the sequence of smooth cut-off functions $\beta_m(x)$ such that $\beta_m(x) = 0$ in $\{x \mid |x| < 1/m\}$ and $\beta_m(x) = 1$ in $\{x \mid |x| > 2/m\}$ and applying the same arguments as in the proof of Lemma 2 we arrive at the following statement:

Lemma 4. *The function u_0 defined in* (9) *belongs to $H^1(\Omega, \mathrm{d}\mu)$. Moreover, for u_0 and z from* (9), *the following relation holds*:

$$z = \nabla u_0. \tag{10}$$

Remark 3. Using the approach proposed here, one can generalize these results to more complex "star"-structures and infinite periodic, quasi periodic and random wire structures (see Fig. 3).

1.2.1. Potential and solenoidal vectors

Suppose we are given a periodic network R_0 (see Fig. 4) and a periodic singular measure μ that satisfies the normalization condition $\mu(\square) = 1$, where the symbol $\square$ stands for the cell of periodicity. For simplicity we assume that all the end points of each segment in the network are the intersection points of two or more segments.

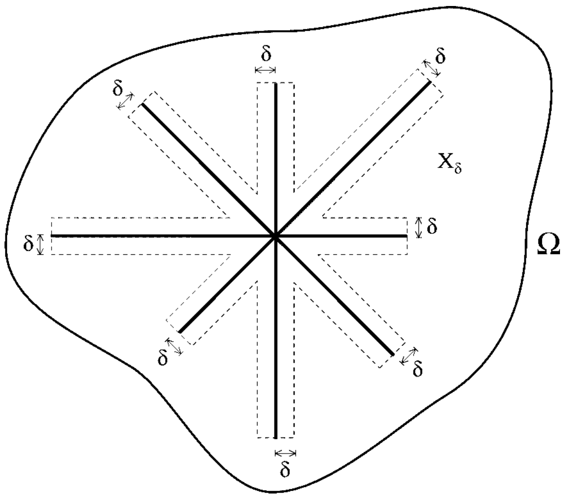

Fig. 3. Intersecting bars. "Star"-structure.

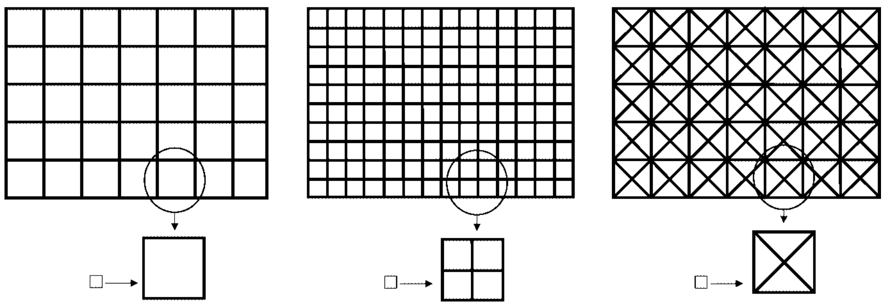

Fig. 4. Periodic network.

According to Definition 1, a function u is an element of $H^1(\square, \mathrm{d}\mu)$ if for some sequence of smooth □-periodic functions $\{u_n\}$, one has

$$u_n \to u \quad \text{in } L_2(\square, \mathrm{d}\mu), \qquad \nabla u_n \to z \quad \text{in } L_2(\square, \mathrm{d}\mu),$$

z is said to be a gradient of u.

Our next aim is to introduce, in case of singular measures, the subspaces of potential and solenoidal vector-functions and to study their properties.

Definition 2. A vector-function $v \in (L_2(\square, \mathrm{d}\mu))^2$ is said to be *potential* if it belongs to the closure of the following linear set:

$$\{w \mid w = \nabla\psi,\ \psi \in C^\infty_{\text{per}}(\square)\}$$

in the norm

$$\|w\| = \left(\int_\square w^2 \,\mathrm{d}\mu\right)^{1/2},$$

where the symbol $C^{\infty}_{\text{per}}(\square)$ stands for the space of $\square$-periodic elements of $C^{\infty}(\mathbb{R}^2)$. For the subspace of all the potential vector-functions we use the notation $L_2^{\text{pot}}(\square, \mathrm{d}\mu)$.

The following proposition shows an interesting property of the Lebesgue measure on a torus.

Proposition 5. *Let μ be a periodic measure. If for any $\varphi \in C^{\infty}_{\text{per}}(\square)$ the relation*

$$\int_{\square} \nabla\varphi \,\mathrm{d}\mu = 0, \tag{11}$$

holds, then the measure μ is the Lebesgue measure.

Proof. Consider the distribution $F \in D'(\square)$ defined by the relation

$$\langle F, \varphi\rangle = \int_{\square} \varphi \,\mathrm{d}\mu, \quad \varphi \in C^{\infty}_{\text{per}}(\square).$$

Then, by (11)

$$\langle \nabla F, \varphi\rangle = -\langle F, \nabla\varphi\rangle = 0,$$

where ∇F is understood in the sense of distributions. Therefore, $\nabla F = 0$ and $F = const$. This implies $\mu(\mathrm{d}x) = c\,\mathrm{d}x$. $\square$

Definition 3. A vector-function $p \in (L_2(\square, \mathrm{d}\mu))^2$ is said to be *solenoidal* (or *divergence free*) if

$$\int_{\square} p\nabla\psi \,\mathrm{d}\mu = 0 \tag{12}$$

for any function $\psi \in C^{\infty}_{\text{per}}(\square)$. We denote by $L_2^{\text{sol}}(\square, \mathrm{d}\mu)$ the subspace of all divergence free vectors.

Note that, in the case of network constructions, the solenoidal vector-functions are always tangential to the segments. Indeed, the normal component of potential vectors can be chosen arbitrarily (see Proposition 1) and consequently, the solenoidal vectors must be orthogonal to any normal vector.

Consider an arbitrary network construction R and a singular measure μ concentrated on R. Suppose μ is uniformly distributed on each segment of R, and let $I_1, \dots, I_k$ be the segments intersecting at the origin. Denote by $e_1, \dots, e_k$ the unit vectors directed along $I_1, \dots, I_k$, respectively (see Fig. 5), and by $\theta_1, \dots, \theta_k$ the densities of $\mathrm{d}\mu$ with respect to the standard Lebesgue measure on the corresponding segments.

The assertion below describes the structure of solenoidal vectors on R.

Lemma 6. *For each segment I_i, the restriction of a solenoidal vector p on I_i takes the form $\lambda_i e_i$, where λ_i is a constant. Moreover, we have*

$$\sum_{i=1}^{k} \theta_i\lambda_i = 0. \tag{13}$$

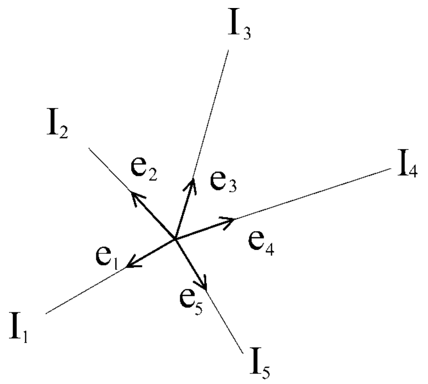

Fig. 5. Network node.

Remark 4. For more general networks involving segments with free end points, any solenoidal vector-function is necessary equal to zero at each such a segment.

Proof of Lemma 6. Consider the set of test-functions φ concentrated in a neighbourhood of the segment I_s for a fixed s. Using φ as a test-function in (15), we deduce that p is a constant vector on I_s.

In order to justify (13) it suffices to substitute in (12) a test-function supported by a small neighbourhood of the origin, and to integrate by parts along each segment. This completes the proof. □

Consider a sequence of usual potential or solenoidal vector-functions that converges in the sense of (4). The next theorem states that the limit is necessary a potential (solenoidal) vector-function.

Theorem 1. *If* $v_\delta \in L_2^{\mathrm{pot}}(\Box, \mathrm{d}\mu_\delta)$ *is a family of potential vectors such that*

$$v_\delta \rightharpoonup v \quad \textit{as } \delta \to 0$$

in the sense of (4)*, then* v *is a potential vector. If* $p_\delta \in L_2^{\mathrm{sol}}(\Box, \mathrm{d}\mu_\delta)$ *is a family of the solenoidal vectors such that*

$$p_\delta \rightharpoonup p \quad \textit{as } \delta \to 0$$

in the sense of (4)*, then* p *is a solenoidal vector.*

The proof of this theorem relies on the following lemma, widely used in the sequel:

Lemma 7. *If* $p \in L_2^{\mathrm{sol}}(\Box, \mathrm{d}\mu)$*, then there exists a sequence* $\widetilde{p}_\delta \in L_2^{\mathrm{sol}}(\Box, \mathrm{d}\mu_\delta)$ *such that* $\widetilde{p}_\delta \rightharpoonup p$ *and*

$$\int_\Box \widetilde{p}_\delta^{\,2}\,\mathrm{d}\mu_\delta \to \int_\Box p^2\,\mathrm{d}\mu \tag{14}$$

as $\delta \to 0$.

We call this property of solenoidal vectors *strong approximability*. Lemma 7 will be proved in Section 1.4 for the case of networks and junctions.

Proof of Theorem 1. If v_δ is potential, then by Lemma 7 we have

$$0 = \lim_{\delta\to 0} \int_\square v_\delta \widetilde{p}_\delta \,\mathrm{d}\mu_\delta = \int_\square vp \,\mathrm{d}\mu$$

for an arbitrary solenoidal vector p, and the first statement of Theorem 1 follows.

The second statement is almost obvious. By the definition (4) we conclude that

$$0 = \int_\square p_\delta \nabla\varphi \,\mathrm{d}\mu_\delta \to \int_\square p\nabla\varphi \,\mathrm{d}\mu \tag{15}$$

for any potential $\nabla\varphi$. Thus, p is solenoidal. The theorem is proved. □

1.3. Junctions

A detail study of junctions was done in [6]. In this subsection we deal with singular measures on junctions.

Let Ω be a bounded domain in $\mathbb{R}^2$, and denote by $R_\delta \subset \Omega$ the union of the square $Q = \{x \mid -1 < x_1 < 1;\ -1 < x_2 < 1\}$ and the bar $\Pi_\delta = \{x \mid 0 < x_1 < 2;\ -\delta < x_2 < \delta\}$ (see Fig. 6).

Denote by $\widetilde{\mu}_\delta$ a probability measure in Ω which is uniformly distributed on Π_δ and by μ_δ the sum of this measure and the Lebesque measure on the square Q. We suppose that $\mu_\delta(\Omega \setminus \overline{R_\delta}) = 0$. Let $\widetilde{\mu}$ be the weak limit of the family $\widetilde{\mu}_\delta$ as $\delta \to 0$, clearly $\widetilde{\mu}$ is a singular probability measure concentrated on the segment $I = \{x \mid 0 < x_1 < 2;\ x_2 = 0\}$. Denote by μ the sum of this measure and the usual Lebesque measure on the square Q. The measure μ is supported by the junction $R_0 := Q \cup I$.

For a general junction structure we call *junction's "body"* any 2D connected component of the structure. For example, Q is a junction's "body" of R_0.

Consider a family of smooth functions u_δ subject to the bound

$$\int_\Omega \left(u_\delta^2 + |\nabla u_\delta|^2\right) \mathrm{d}\mu_\delta \leqslant C.$$

Then there are functions $u_0 \in L_2(\Omega, \mathrm{d}\mu)$ and $z = (z_1, z_2) \in (L_2(\Omega, \mathrm{d}\mu))^2$ such that

$$u_\delta \rightharpoonup u_0, \quad \nabla u_\delta \rightharpoonup z \quad \text{weakly as } \delta \to 0. \tag{16}$$

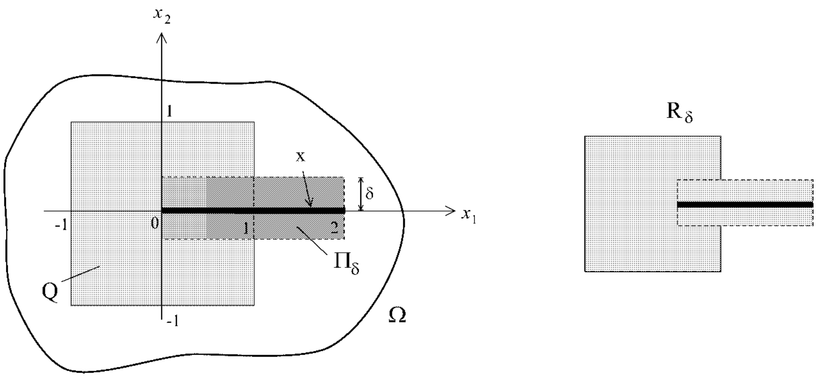

Fig. 6. Simple junction.

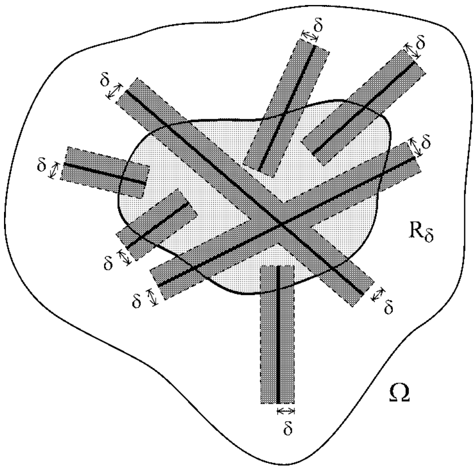

Fig. 7. General junction.

The latter convergence is defined by (4), (5).

The following statement characterizes the Sobolev space $H^1(\Omega, \mathrm{d}\mu)$.

Lemma 8. *The function* u *belongs to* $H^1(\Omega, \mathrm{d}\mu)$, *if* $u \in H^1(Q) \cap H^1(I, \mathrm{d}\widetilde{\mu})$ *and the restriction* $u|_{x_1 \in [0,1];\ x_2=0}$ *is an element of* $H^1([0,1])$.

Proof. We introduce the sequence of smooth cut-off functions β_m such that $\beta_m = 0$ in $[-\frac{1}{m}, 1 + \frac{1}{m}] \times [-\frac{1}{m}, \frac{1}{m}]$ and $\beta_m(x) = 1$ in the exterior of $[-\frac{2}{m}, 1 + \frac{2}{m}] \times [-\frac{2}{m}, \frac{2}{m}]$, and then repeat the reasoning from the previous subsection to get the result. □

In the same way as in the previous subsections, one can prove the following lemma:

Lemma 9. *The function* u_0 *defined in* (16) *belongs to* $H^1(\Omega, \mathrm{d}\mu)$. *Moreover, for* u_0 *and* z *from* (16), *the following relation holds*:

$$z = \nabla u_0. \tag{17}$$

Remark 5. Using the approach suggested here, one can generalize these results to more complex junctions (see, for instance, Fig. 7).

1.3.1. Potential and solenoidal vectors

Suppose we are given a periodic junction construction R_0 (see Fig. 8) and a periodic measure μ that satisfies the normalization condition $\mu(\Box) = 1$, where the symbol $\Box$ stands for the cell of periodicity. The solenoidal (divergence free) and potential vector-functions are defined here in the same way as in the case of networks (see Section 1.2.1).

In what follows we identify the cell of periodicity $\Box$ with the torus T^2. For simplicity we assume that any end point of a segment of R_0 is either an intersection point of two or more segments, or situated in a junction's "body" of R_0. We also assume that the cell of periodicity $\Box$ contains only one junction's "body" which is sufficiently regular.

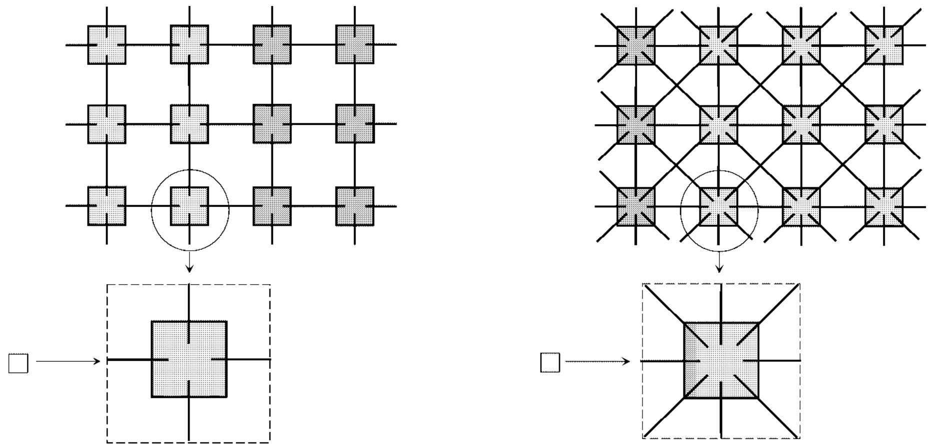

Fig. 8. Periodic junction-structure.

In order to describe the structure of solenoidal vector-functions on the junction R_0, we first denote all the segments of R_0 on T^2 by I_j, $j = 1, 2, \dots, m$, and the plane domain by Q. Let y_j be the induced coordinate on I_j. According to our assumptions, the measure μ admits the following representation

$$\mathrm{d}\mu = \mathrm{d}\mu_0 + \sum_{j=1}^{m} \mathrm{d}\mu_j,$$

where μ_0 is proportional to the Lebesgue measure on Q and μ_j, $j = 1, \dots, m$, are singular measures concentrated on I_j and proportional to the 1D Lebesgue measures on this segment:

$$\mu_0(A) = \theta_0 \int_{A\cap Q} 1\,\mathrm{d}x_1\,\mathrm{d}x_2, \quad \forall A \subset \Box,$$

$$\mu_j(A) = \theta_j \int_{A\cap I_j} 1\,\mathrm{d}y_j, \quad \forall A \subset \Box,\ j = 1, 2, \dots, m,$$

for some constants $\theta_0, \theta_1, \dots, \theta_m$.

The structure of a solenoidal vector-function on the junction construction is given by the following statement:

Theorem 2. *Each solenoidal vector-function $p \in L_2^{\mathrm{sol}}(\Box, \mathrm{d}\mu)$ can be represented as a sum*

$$p(x) = p^0(x) + \sum_{j=1}^{m} p^j(x),$$

where the vector-function p^0 defined in Q is such that $p_0 \in (L_2(\Box, \mathrm{d}\mu_0))^2$, and p^j defined on I_j, $j = 1, \dots, m$, is such that $p_j \in (L_2(\Box, \mathrm{d}\mu_0))^2$. A vector-function p^j, $j = 1, \dots, m$, is directed along the segment I_j; on the part of I_j located outside Q, p^j is a constant vector. The vector-function p^0 is a usual *solenoidal vector-function in the set* $Q_- = \overline{Q} \setminus \bigcup_{j=1}^m I_j$. *The derivative of* p^j *along the segment* I_j *compensates the jump of the normal components of* p^0 *at* I_j

$$\frac{\mathrm{d}}{\mathrm{d}y_j} p^j(x) = n^j \cdot p^0(x^+) - n^j \cdot p^0(x^-),$$

where n^j *is the normal to* I_j *and the symbols* $+$ *and* $-$ *indicate that the corresponding values should be taken on the opposite banks of the cut.*

If x *is an intersection point of the segments* $I_{s_1}, \dots, I_{s_l}$, *then the following relation holds*

$$\theta_{s_1} p^{s_1}(x) + \theta_{s_2} p^{s_2}(x) + \cdots + \theta_{s_l} p^{s_l}(x) = 0.$$

In particular, in any isolated end point, we have $p^j = 0$.

Proof. Consider one of the segments, say I_1, and assume without loss of generality that I_1 coincides with the interval $\{(x_1, x_2)\colon\ 0 \leqslant x_1 \leqslant 1,\ x_2 = 0\}$. This can always be achieved by means of a proper linear transformation. Then $y_1 = x_1$ and $\mathrm{d}\mu_1 = \theta_0\,\mathrm{d}x_1$.

Let φ be a C^∞-function with the support in a neighbourhood of I_1, such that $\varphi(0,0) = \varphi(1,0) = 0$. Integrating by parts we get

$$\begin{aligned}
0 &= \int_{T^2} \nabla\varphi(x) p(x)\,\mathrm{d}\mu = \theta_0 \int_Q \nabla\varphi(x) p_0(x)\,\mathrm{d}x_1\,\mathrm{d}x_2 + \theta_1 \int_0^1 \nabla\varphi(x) p(x)\,\mathrm{d}x_1 \\
&= \theta_0 \int_0^1 \big(p_2^0(x_1^+, 0) - p_2^0(x_1^-, 0)\big)\varphi(x_1, 0)\,\mathrm{d}x_1 - \theta_1 \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x_1} p_1^1(x_1)\varphi(x_1, 0)\,\mathrm{d}x_1 \\
&\quad + \theta_1 \int_0^1 p_2^1(x_1) \frac{\partial}{\partial x_2}\varphi(x_1, 0)\,\mathrm{d}x_1 \\
&= \int_0^1 \Big\{\theta_0\big(p_2^0(x_1^+, 0) - p_2^0(x_1^-, 0)\big) - \theta_1 \frac{\mathrm{d}}{\mathrm{d}x_1} p_1^1(x_1)\Big\}\varphi(x_1, 0)\,\mathrm{d}x_1 + \theta_1 \int_0^1 p_2^1(x_1)\frac{\partial}{\partial x_2}\varphi(x_1, 0)\,\mathrm{d}x_1.
\end{aligned}$$

In view of the arbitrariness of φ this yields

$$\theta_0\big(p_2^0(x_1^+, 0) - p_2^0(x_1^-, 0)\big) = \theta_1 \frac{\mathrm{d}}{\mathrm{d}x_1} p_1^1(x_1), \qquad p_2^1(x_1) = 0.$$

The other assertions of the theorem are obtained in a similar way. □

Remark 6. If the jump of the normal component of p^0 is equal to zero along each I_j, then we have a trivial “uncoupled” case of solenoidal vector-functions $p^0 \in L_2^{\mathrm{sol}}(T^2, \mathrm{d}\mu_0)$ and $p^j \in L_2^{\mathrm{sol}}(I_j, \mathrm{d}\mu_j)$.

Remark 7. The statements of the latter theorem remain valid for junction structures having finite number of elements and for junctions with more complex geometry. Indeed, in the proof we did not use the periodicity of R_0, all our arguments were local.

Proposition 10. *All the statements of Theorem* 1 *hold true in the case of junction structures.*

It remains to prove the strong approximability property (Lemma 7) for all the above cases.

1.4. Strong approximability

In this section we prove Lemma 7 for networks and junction structures.

Remark 8. It should be noted that the convergence introduced in Lemma 7 is equivalent to the *strong convergence* of the family $\widetilde{p}_\delta$ to p, which is defined as follows:

$$\int_\square v_\delta \widetilde{p}_\delta \, \mathrm{d}\mu_\delta \to \int_\square vp \, \mathrm{d}\mu$$

for any v_δ which converges weakly to v as $\delta \to 0$.

We give below the proof of strong approximability for different geometrical structures.

Proof of Lemma 7. (i) *Networks and rod-structures*. Here we borrow the notation and the constructions from Section 1.2 (see Fig. 9).

For the sake of convenience we introduce the following sets:

$$\begin{aligned}
&D_0 := (-\delta, \delta)^2,\\
&D_1 := \{x \mid a_1 < x_1 < -\delta,\ -\delta < x_2 < \delta\},\\
&D_2 := \{x \mid -\delta < x_1 < \delta,\ \delta < x_2 < b_2\},\\
&D_3 := \{x \mid \delta < x_1 < b_1,\ -\delta < x_2 < \delta\},\\
&D_4 := \{x \mid -\delta < x_1 < \delta,\ a_2 < x_2 < -\delta\},
\end{aligned}$$

and denote $S_i := \partial D_0 \cap \partial D_i$, $i = 1, \ldots, 4$.

Let p be an arbitrary periodic solenoidal vector from $L_2^{\mathrm{sol}}(\square, \mathrm{d}\mu)$. Taking into account the structure of the solenoidal vector on crosses (see Lemma 6), we construct a family of vector-functions $\widetilde{p}_\delta$ as follows:

- In the domains D_i, $i = 1, 2, 3, 4$, we set $\widetilde{p}_\delta = \lambda_i e_i$.

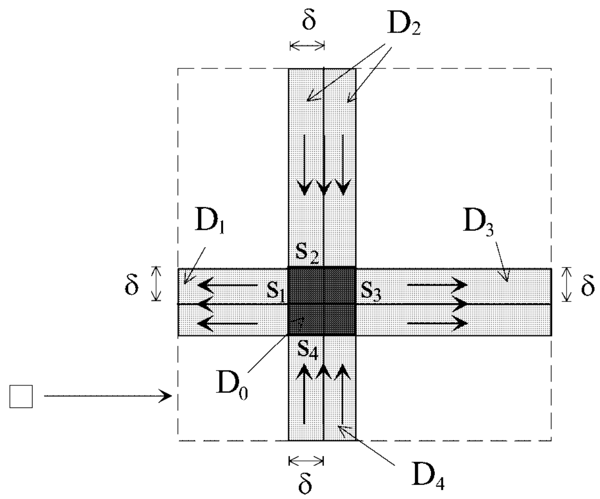

Fig. 9. Cell of periodicity of lattice.

- In the domain D_0 we set $\widetilde{p}_\delta = \nabla\varphi$, where φ is a solution of the Neumann problem

$$\begin{cases} \Delta\varphi = 0 & \text{in } D_0, \\ \dfrac{\partial\varphi}{\partial e_i} = \lambda_i & \text{on } S_i,\ i = 1, \ldots, 4; \end{cases} \tag{18}$$

  the compatibility condition for this problem is satisfied due to (13).

The family $\widetilde{p}_\delta$ has been constructed to converge weakly to p in the sense of (4). Let us prove (14), i.e.,

$$\int_\square \widetilde{p}_\delta^2 \, \mathrm{d}\mu_\delta \to \int_\square p^2 \, \mathrm{d}\mu \quad \text{as } \delta \to 0.$$

By the definition of $\widetilde{p}_\delta$ we have

$$\int_\square \widetilde{p}_\delta^2 \, \mathrm{d}\mu_\delta = \sum_{i=1}^{4} \int_{D_i} \lambda_i^2 \, \mathrm{d}\mu_\delta + \int_{D_0} \widetilde{p}_\delta^2 \, \mathrm{d}\mu_\delta.$$

The second term on the right-hand side vanishes while the first one tends to the integral $\int_\square p^2 \, \mathrm{d}\mu$, and the required convergence follows.

Remark 9. In the above proof we assumed all the weights θ_j to be equal to 1. For arbitrary set of weights θ_j, $\theta_j > 0$, one can adopt the above construction by making the width of the bars D_j equal to $\theta_j\delta$, $j = 1, 2, 3, 4$.

(ii) *Junctions*. Here we use the notation introduced in Section 1.3 (see Fig. 10).

For the sake of brevity we only consider the intersection of the square $Q = (-1, 1)^2$ with one bar $\Pi_\delta = (0, 2) \times (-\delta, \delta)$ related to the segment $I = \{x \mid 0 \leqslant x_1 \leqslant 2,\ x_2 = 0\}$, and assume that Q and I are elements of a periodic junction. A general periodic junction construction can be managed in the same way.

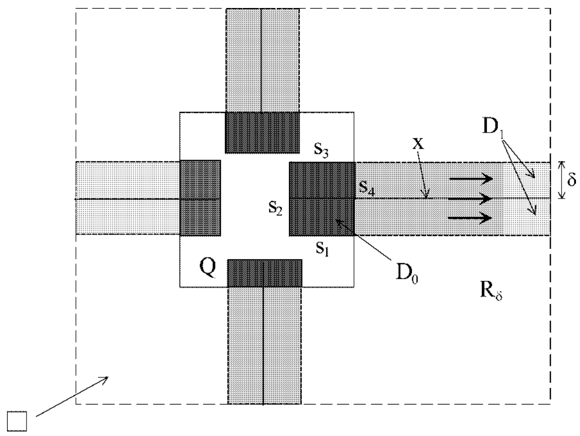

Fig. 10. Cell of periodicity of junction-structure.

Denote

$$
\begin{aligned}
&D_0 = Q \cap \Pi_\delta, && D_1 = \Pi_\delta \setminus \overline{D}_0,\\
&S_4 = \partial D_0 \cap \partial D_1, && S_3 = \{x \mid x_1 = 0,\ -\delta < x_2 < \delta\},\\
&S_1 = \{x \mid 0 < x_1 < 1,\ x_2 = -\delta\}, && S_2 = \{x \mid 0 < x_1 < 1,\ x_2 = \delta\},
\end{aligned}
$$

so that $\partial D_0 = S_1 \cup S_2 \cup S_3 \cup S_4$.

Suppose p is a periodic solenoidal vector-function on the junction involving $Q \cup I$. By Theorem 2 this vector-function admits on $Q \cap I$ the representation $p(x) = p^0(x) + p^1(x)$ with $p^0 \in L_2^{\mathrm{sol}}(\overline{Q} \setminus I)$, $p^1(x) = (p_1^1(x_1), 0)$. Moreover,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x_1} p_1^1(x_1) = p_2^0(x_1, +0) - p_2^0(x_1, -0), \quad 0 < x_1 < 1,\\
&p^1(x_1) = \lambda_1 e_1, \quad 1 < x_1 < 2.
\end{aligned}
\tag{19}
$$

We construct the required family $\widetilde{p}_\delta$ as follows:

- In D_1 we set $\widetilde{p}_\delta = \lambda_1 e_1$.
- In the domain $Q \setminus \overline{D}_0$ we set $\widetilde{p}_\delta = p^0$.
- In the domain D_0 we set $\widetilde{p}_\delta = \nabla\varphi$, where φ is a solution of the Neumann problem

$$
\begin{cases}
\Delta\varphi = 0 \quad \text{in } D_0,\\
\dfrac{\partial\varphi}{\partial n} = \delta p^0 \cdot n \quad \text{on } S_i,\ i = 1, 2, 3,\\
\dfrac{\partial\varphi}{\partial n} = \lambda_1 \quad \text{on } S_4.
\end{cases}
\tag{20}
$$

The compatibility condition for problem (20) reads

$$
\int_{S_1 \cup S_2 \cup S_3} \delta p^0 \cdot n \,\mathrm{d}s + \int_{S_4} \lambda_1 \,\mathrm{d}s = 0.
$$

Thus, we should prove the relation $\lambda_1 = -\int_{S_1 \cup S_2 \cup S_3} p^0 \cdot n \,\mathrm{d}s$.

By the Stokes formula and (19), one has

$$
\int_{S_1 \cup S_2 \cup S_3} \delta p^0 \cdot n \,\mathrm{d}s = \int_0^1 \big(p_2^0(x_1, +0) - p_2^0(x_1, -0)\big) \,\mathrm{d}x_1 = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x_1} p_1^1(x_1) \,\mathrm{d}x_1 = \lambda_1.
$$

This implies the required compatibility condition.

Clearly, the family $\widetilde{p}_\delta$ converges weakly to p, as $\delta \to 0$, in the sense of (4). Let us prove (14), i.e.,

$$
\int_\square \widetilde{p}_\delta^2 \,\mathrm{d}\mu_\delta \to \int_\square p^2 \,\mathrm{d}\mu \quad \text{as } \delta \to 0.
$$

We have

$$
\int_{Q \cup \Pi_\delta} \widetilde{p}_\delta^2 \,\mathrm{d}\mu_\delta = \int_{Q \setminus D_0} \left(p^0\right)^2 \mathrm{d}\mu_\delta + \int_{D_1} (\lambda_1)^2 \,\mathrm{d}\mu_\delta + \int_{D_0} (\nabla\varphi)^2 \,\mathrm{d}\mu_\delta.
$$

Clearly, the first two terms on the right-hand side converge to $\int_Q (p^0)^2\,\mathrm{d}x$ and λ_1^2, respectively. Multiplying Eq. (20) by φ and integrating by parts, one can show that the last term converges to $\int_0^1 (p^1)^2\,\mathrm{d}x_1$. The lemma is proved. □

This completes the proof of Theorem 1.

2. Scalar problems

In this section we compare two different homogenization methods for periodic networks and junctions. The first method involves the direct homogenization procedure based on the analysis on networks and junctions, that was developed in the previous sections. The second method is more classical: we homogenize constructions of small thickness in a usual way and then pass to the limit as the thickness goes to zero. It is shown that these two approaches give the same answer. We deal here with a model problem for one scalar equation, the case of general elliptic operator on networks and junction constructions can be studied in the same manner.

Denote by δ the small parameter which characterizes the fixed thickness of rods, the corresponding structures will be called δ-*structures*. Another small parameter ε will be used to characterize the microscopic length-scale of the whole construction.

Given a □-periodic connected δ-structure $\mathcal{R}_\delta$ and a regular bounded domain $G \subset \mathbb{R}^2$, we define a periodic microstructure $\mathcal{R}_{\delta,\varepsilon}$ by setting $\mathcal{R}_{\delta,\varepsilon} = \varepsilon\mathcal{R}_\delta$, and then consider the homogenization problem in $G \cap \mathcal{R}_{\delta,\varepsilon}$ whose variational formulation reads

$$\inf_{v\in C_0^\infty(G)} \int_{G\cap\mathcal{R}_{\delta,\varepsilon}} \left(\left|\nabla v(x)\right|^2 - 2f(x)v(x)\right) \mathrm{d}\mu_\delta^\varepsilon,$$

where $\mu_\delta^\varepsilon(\mathrm{d}x) = \varepsilon^2\mu_\delta(\varepsilon^{-1}\,\mathrm{d}x)$ and μ_δ is the measure on $\mathcal{R}_\delta$ that has been introduced in the previous section, f is a given function. This is equivalent to say that we consider the homogenization problem in $G \cap \mathcal{R}_{\delta,\varepsilon}$ for a divergence form isotropic operator with the coefficient equal to the density of the measure μ_δ^ε. The Dirichlet boundary condition is stated on the exterior boundary $\partial G \cap \mathcal{R}_{\delta,\varepsilon}$, and the Neumann boundary condition at the boundary of the "microstructure" $\partial(G \cap \mathcal{R}_{\delta,\varepsilon}) \setminus \partial G$. This homogenization problem is well studied, we refer here to [7,8,27,28]. Denote by $A_{\varepsilon,\delta}$ the matrix of coefficients of the original operator and by A_δ^{hom} the constant matrix of coefficients of the homogenized operator.

Remark 10. The asymptotic behaviour of A_δ^{hom}, as $\delta \to 0$, and the properties of the corresponding limit have been investigated in [3] and in [8], where other elliptic problems on reticulated structures have also been considered.

A successful attempt to change the order of passage to the limit in ε and δ in the network homogenization problem, has been made in [11]. This work relies on the extension technique.

Consider also the following "singular" homogenization problem

$$\inf_{v\in C_0^\infty(G)} \int_G \left(\left|\nabla v(x)\right|^2 - 2f(x)v(x)\right) \mathrm{d}\mu^\varepsilon,$$

where $\mu^\varepsilon(\mathrm{d}x) = \varepsilon^2\mu(\varepsilon^{-1}\,\mathrm{d}x)$ and μ is a (singular) □-periodic positive measure, $\mu(□) = 1$. We assume that the measure μ is the weak limit of μ_δ as $\delta \to 0$. The latter problem is not standard. As was

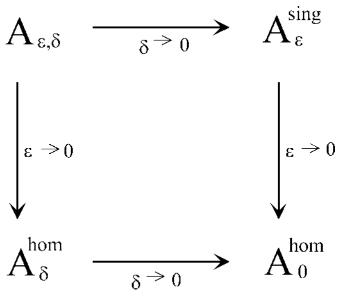

Fig. 11. Homogenization diagram.

shown in [14], this problem can be homogenized and the effective operator is a second order elliptic operator with constant coefficients. Formally, we denote the matrix of coefficients of the singular problem by $A_\varepsilon^{\mathrm{sing}}$; the coefficients of the effective matrix are denoted by A_0^{hom}. The formula that defines the effective matrix A_0^{hom} will be given below.

We want to show that for the singular structures defined above, the corresponding diagram presented at Fig. 11 is commutative.

The operators $A_\varepsilon^{\mathrm{sing}}$ and the related variational problems were studied in [14], where the homogenization result $A_\varepsilon^{\mathrm{sing}} \underset{\varepsilon\to 0}{\longrightarrow} A_0^{\mathrm{hom}}$ was proved.

The parts of the diagram $A_{\varepsilon,\delta} \underset{\delta\to 0}{\longrightarrow} A_\varepsilon^{\mathrm{sing}}$ and $A_\delta^{\mathrm{hom}} \underset{\delta\to 0}{\longrightarrow} A_0^{\mathrm{hom}}$ are studied in this section.

In the above homogenization problems on δ-structures and on the corresponding networks (junction structures) the variational formula for the effective coefficients read respectively:

$$\eta A_\delta^{\mathrm{hom}} \eta = \inf_{u \in C_{\mathrm{per}}^\infty(\mathbb{R}^2)} \int_\square |\eta + \nabla u|^2 \, \mathrm{d}\mu_\delta, \tag{21}$$

$$\eta A_0^{\mathrm{hom}} \eta = \inf_{u \in C_{\mathrm{per}}^\infty(\mathbb{R}^2)} \int_\square |\eta + \nabla u|^2 \, \mathrm{d}\mu, \tag{22}$$

where μ_δ is the periodic measure on a δ-structure that was discussed above, and μ is the limit measure on the corresponding network or junction construction.

Theorem 3. *The homogenized matrices A_δ^{hom} and A_0^{hom} satisfy the following limit relation*

$$A_0^{\mathrm{hom}} = \lim_{\delta\to 0} A_\delta^{\mathrm{hom}}. \tag{23}$$

Proof. Let w make the expression (22) a minimum, i.e.,

$$\eta A_0^{\mathrm{hom}} \eta = \int_\square |\eta + \nabla w|^2 \, \mathrm{d}\mu.$$

Given a sequence of positive α_n, $\alpha_n \to 0$, as $n \to +\infty$, we can find $w_n \in C_{\mathrm{per}}^\infty(\mathbb{R}^2)$ such that

$$\eta A_0^{\mathrm{hom}} \eta + \alpha_n \geqslant \int_\square |\eta + \nabla w_n|^2 \, \mathrm{d}\mu.$$

Since w_n is smooth for each $n > 0$, we have

$$\int_\square |\eta + \nabla w_n|^2 \,\mathrm{d}\mu = \lim_{\delta\to 0} \int_\square |\eta + \nabla w_n|^2 \,\mathrm{d}\mu_\delta.$$

It follow from (21) that $\int_\square |\eta + \nabla w_n|^2 \,\mathrm{d}\mu_\delta \geqslant \eta A_\delta^{\mathrm{hom}}\eta$. Thus, for any $\alpha_n > 0$

$$\eta A_0^{\mathrm{hom}}\eta + \alpha_n \geqslant \int_\square |\eta + \nabla w_n|^2 \,\mathrm{d}\mu = \lim_{\delta\to 0} \int_\square |\eta + \nabla w_n|^2 \,\mathrm{d}\mu_\delta \geqslant \limsup_{\delta\to 0}\ \eta A_\delta^{\mathrm{hom}}\eta \tag{24}$$

for any vectors η. Keeping in mind the arbitrariness of α_n we conclude that

$$\eta A_0^{\mathrm{hom}}\eta \geqslant \limsup_{\delta\to 0}\ \eta A_\delta^{\mathrm{hom}}\eta.$$

On the other hand, the Euler equation for problem (21) reads

$$\int_\square (\eta + \nabla u_\delta)\nabla\varphi \,\mathrm{d}\mu_\delta = 0, \quad \forall\varphi \in C_{\mathrm{per}}^\infty(\mathbb{R}^2).$$

It follows from the variational formulation (21) that the family ∇u_δ is bounded, thus $\nabla u_\delta \rightharpoonup v$ in the sense of (4) and taking into account the lower semicontinuity of a weak limit, we get

$$\lim_{\delta\to 0} \int_\square |\eta + \nabla u_\delta|^2 \,\mathrm{d}\mu_\delta \geqslant \int_\square |\eta + v|^2 \,\mathrm{d}\mu. \tag{25}$$

The fact that the vector-function v is potential follows from Theorem 1 for networks and from Proposition 10 for junctions. This completes the proof of Theorem 3. □

The top arrow of the diagram (see Fig. 11) which represents the Γ-convergence of the respective variational functionals (see [9,16]), can be justified in the same way as above with the evident simplifications.

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