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Homogenization of the Spectral Problem for Periodic Elliptic Operators with Sign-Changing Density Function

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Abstract

The paper deals with the asymptotic behaviour of spectra of second order self-adjoint elliptic operators with periodic rapidly oscillating coefficients in the case when the density function (the factor on the spectral parameter) changes sign. We study the Dirichlet problem in a regular bounded domain and show that the spectrum of this problem is discrete and consists of two series, one of them tending towards $+\infty$ and another towards $-\infty$. The asymptotic behaviour of positive and negative eigenvalues and their corresponding eigenfunctions depends crucially on whether the average of the weight function is positive, negative or equal to zero. We construct the asymptotics of eigenpairs in all three cases.

1. Introduction

The paper focuses on the homogenization of the Dirichlet spectral problem

$$\begin{aligned} -\mathrm{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u\right) &= \rho\left(\frac{x}{\varepsilon}\right)\lambda\, u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

stated in a regular bounded domain $\Omega \subset \mathbb{R}^n$ for a second order symmetric uniformly elliptic operator

$$\mathcal{L}^{\varepsilon} = \mathrm{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla\right)$$

with periodic rapidly oscillating coefficients, ε being a small positive parameter.

Regarding the density function, ρ, we assume that it is periodic and changes sign, that is, both the sets $\{x \in \Omega\,:\,\rho(x/\varepsilon) < 0\}$ and $\{x \in \Omega\,:\,\rho(x/\varepsilon) > 0\}$ are of positive measure. The last assumption makes the problem under consideration nonstandard.

Some of the results of the paper are generalized to the case of formally self-adjoint elliptic systems with periodic coefficients.

It was shown in our preceding work [11] that for each $\varepsilon > 0$ the spectrum of (1) is discrete and consists of the following two infinite sequences

$$0 < \lambda_1^{\varepsilon,+} \leqq \lambda_2^{\varepsilon,+} \leqq \cdots \leqq \lambda_j^{\varepsilon,+} \leqq \cdots, \quad \lim_{j\to\infty} \lambda_j^{\varepsilon,+} = +\infty,$$

and

$$0 > \lambda_1^{\varepsilon,-} \geqq \lambda_2^{\varepsilon,-} \geqq \cdots \geqq \lambda_j^{\varepsilon,-} \geqq \cdots, \quad \lim_{j\to\infty} \lambda_j^{\varepsilon,-} = -\infty.$$

The asymptotic behaviour of the eigenpairs depends crucially on whether the average of ρ is positive, negative or equal to zero. We further construct the asymptotics in all three cases.

I. If the average of ρ is strictly positive, then the positive eigenvalues and the corresponding eigenfunctions show the same regular limit behaviour as in the case of point-wise positive spectral density. Namely, for any $j \geqq 1$ the eigenvalue $\lambda_j^{\varepsilon,+}$ converges, as $\varepsilon \to 0$, to the jth eigenvalue of the limit spectral problem. The corresponding eigenfunctions converge along subsequences.

II. If the mean value of ρ is equal to zero, then the limit spectral problem generates a quadratic operator pencil, and the eigenvalues $\lambda_j^{\varepsilon,+}$, $j = 1, 2, \ldots$, are of order $1/\varepsilon$ so that the normalized quantities $\varepsilon\lambda_j^{\varepsilon,+}$ converge to eigenvalues of the limit spectral problem.

III. For ρ having negative average the positive eigenvalues $\lambda_j^{\varepsilon,+}$ tend to infinity at the rate $1/\varepsilon^2$, and the corresponding eigenfunctions prove to be rapidly oscillating. In this case we use the factorization technique in order to construct the eigenfunctions asymptotics. The eigenvalues $\lambda_j^{\varepsilon,+}$ admit the following representation

$$\lambda_j^{\varepsilon,+} = \frac{1}{\varepsilon^2}\mu_1 + \kappa_j + o(1),$$

where μ_1 is the first positive eigenvalue of the periodic spectral problem

$$\operatorname{div}(a(y)\nabla v) = \mu\, \rho(y)\, v,$$

and κ_j are eigenvalues of the limit Dirichlet spectral problem with constant coefficients, the limit operator being different from the homogenized operator of A^ε.

Under some additional assumptions on the Bloch spectrum of the studied periodic operator, this result can be generalized to divergence form formally self-adjoint elliptic systems possessing the so-called polynomial property. This analysis requires applying Floquet–Bloch theory; we deal with elliptic systems in the last section of the paper, where we provide all the necessary definitions.

The study of the negative part of the spectrum is reduced to that of the positive part, if we replace ρ with $-\rho$.

Previously, a spectral problem with sign-changing density for the Laplace operator has been considered in [15]; in this work the limit behaviour of the spectrum has been studied under the assumption that the density consists of a fixed positive part and asymptotically vanishing negative part. The limit behaviour of the spectrum of the Dirichlet problem for a divergence form self-adjoint elliptic system with periodic coefficients in the case *I, II* has been studied in [11], however, the most irregular case of oscillating eigenfunctions has not yet been considered.

There is a vast literature on homogenization of spectral problems in the case of point-wise positive weight ρ. These problems have been studied in [9,10] and then in many other papers. The homogenization of spectral problems in perforated domains has been studied in [17] followed by many other works on the subject. The limit behaviour of a spectrum of an elasticity system in a perforated domain has been considered in [20]. In [5] the authors have generalized the results obtained in [20] by making weaker the assumptions on the regularity of the inclusions and external forces.

The spectral problems for locally periodic symmetric second order elliptic operators with large potential have been studied in [1]. The work [2] dealt with the asymptotic behaviour of a spectrum for a periodic symmetric elliptic system with large potential.

In the present paper we construct and justify the asymptotics of a spectrum for all possible values of the average of ρ. Some of the obtained results are then generalized for symmetric elliptic systems.

2. Problem setup

Let Ω be a $C^{2,\delta}$ bounded domain in $\mathbb{R}^d$ with a boundary $\partial\Omega$. We consider the following spectral problem:

$$\begin{cases} \mathcal{L}^\varepsilon u^\varepsilon(x) \equiv -\mathrm{div}\,(a^\varepsilon(x)\nabla u^\varepsilon(x)) = \rho^\varepsilon(x)\lambda^\varepsilon u^\varepsilon(x), & x \in \Omega, \\ u^\varepsilon(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

where

$$a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right), \quad \rho^\varepsilon(x) = \rho\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0.$$

Here $a(y)$ is a symmetric $d \times d$ matrix satisfying the uniform ellipticity condition

$$\sum_{i,j=1}^{d} a_{ij}(y)\xi_i\xi_j \geqq \Lambda|\xi|^2, \quad \xi \in \mathbb{R}^d,$$

for some $\Lambda > 0$. We assume that $a_{ij}(y) \in L^\infty(Y)$ are periodic functions (from this time onward Y denotes the periodicity cell); $\rho(y) \in L^\infty(Y)$ is Y-periodic and changes sign, that is, the sets $\{y \in Y : \rho(y) < 0\}$ and $\{y \in Y : \rho(y) > 0\}$ have positive Lebesgue measures. The weak formulation of spectral problem (2) reads: to find $\lambda^\varepsilon \in \mathbb{C}$ (eigenvalues) and $u^\varepsilon \in H_0^1(\Omega)$, $u^\varepsilon \neq 0$, (eigenfunctions) such that

$$a_\varepsilon(u^\varepsilon, v) = \lambda^\varepsilon(\rho^\varepsilon u^\varepsilon, v)_\Omega, \quad v \in H_0^1(\Omega), \tag{3}$$

where $a_\varepsilon(u, v) = (a^\varepsilon \nabla u, \nabla v)_\Omega$ is a bilinear quadratic form; $(\cdot, \cdot)_\Omega$ is a scalar product in $L^2(\Omega)$. Since function ρ^ε changes sign, problem (3) is not a standard spectral problem, and the existing results on the spectrum of semi-bounded self-adjoint operators with compact resolvents do not apply. To overcome this difficulty, we reduce the studied problem (3) to an equivalent spectral problem for a compact self-adjoint operator.

Denote by $\mathcal{H}$ a space $H_0^1(\Omega)$ equipped with the norm

$$\|u\|_\mathcal{H} = \langle u, u\rangle = a_\varepsilon(u, u). \tag{4}$$

The bilinear form $(\rho^\varepsilon u, v)_\Omega$ on $\mathcal{H}$ defines a bounded linear operator $\mathcal{K}^\varepsilon : \mathcal{H} \to \mathcal{H}$ such that

$$(\rho^\varepsilon u, v)_\Omega = \langle \mathcal{K}^\varepsilon u, v\rangle.$$

By definition, the operator $\mathcal{K}^\varepsilon$ is symmetric and its domain $D(\mathcal{K}^\varepsilon)$ coincides with the whole space $\mathcal{H}$, thus it is self-adjoint. Using the representation of $\mathcal{K}^\varepsilon u$ as a solution of the boundary value problem

$$\begin{cases} -\mathrm{div}\,(a^\varepsilon(x)\nabla(\mathcal{K}^\varepsilon u(x))) = \rho^\varepsilon(x)\,u(x), & x \in \Omega, \\ \mathcal{K}^\varepsilon u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{5}$$

and the compactness of the imbedding $H_0^1(\Omega)$ in $L^2(\Omega)$, one can see that $\mathcal{K}^\varepsilon$ is a compact operator.

In terms of the operator $\mathcal{K}^\varepsilon$, problem (3) takes the form

$$\mathcal{K}^\varepsilon u^\varepsilon = \mu^\varepsilon u^\varepsilon, \quad \mu^\varepsilon = 1/\lambda^\varepsilon. \tag{6}$$

Remark 1. Note that in the case $\rho(y) \geqq 0$ the operator $\mathcal{K}^\varepsilon$ is positive and its spectrum $\sigma(\mathcal{K}^\varepsilon)$ belongs to the segment $[0, k^\varepsilon] \subset \mathbb{R}$, $k^\varepsilon = \|\mathcal{K}^\varepsilon\|$. Moreover, $\mu^\varepsilon = 0$ belongs to the essential spectrum $\sigma_e(\mathcal{K}^\varepsilon)$.

Recall that the essential spectrum of a self-adjoint operator A is by definition

$$\sigma_e(A) = \sigma_p^\infty(A) \cup \sigma_c(A),$$

where $\sigma_p^\infty(A)$ is a set of eigenvalues of infinite multiplicity and $\sigma_c(A)$ is the continuous spectrum (see, for example [4]).

The spectrum of the operator $\mathcal{K}^\varepsilon$ is described by the following statement.

Lemma 1. *Let $\rho(y)$ be such that the Lebesgue's measure of the sets where ρ is positive and negative is greater than zero, in other words,*

$$|\{y : \rho(y) \lesseqgtr 0\}| > 0. \tag{7}$$

Then $\sigma(\mathcal{K}^\varepsilon) \subset [-k^\varepsilon, k^\varepsilon]$, $k^\varepsilon = \|\mathcal{K}^\varepsilon\|$; the point $\mu = 0$ is the only element of the essential spectrum $\sigma_e(\mathcal{K}^\varepsilon)$ (see, for example [4]). Moreover, the discrete spectrum of the operator $\mathcal{K}^\varepsilon$ consists of two infinite monotone sequences

$$\mu_1^{\varepsilon,+} \geqq \mu_2^{\varepsilon,+} \geqq \cdots \geqq \mu_j^{\varepsilon,+} \geqq \cdots \to +0,$$
$$\mu_1^{\varepsilon,-} \leqq \mu_2^{\varepsilon,-} \leqq \cdots \leqq \mu_j^{\varepsilon,-} \leqq \cdots \to -0.$$

Proof. Since the operator $\mathcal{K}^\varepsilon$ is compact and self-adjoint, its spectrum $\sigma(\mathcal{K}^\varepsilon)$ is a countable set of points in $\mathbb{R}$ which does not have any accumulation points except, maybe, for $\mu = 0$. Every nonzero eigenvalue has finite multiplicity.

Let us show that the families $\{\mu_j^{\varepsilon,\pm}\}$ are infinite, and thus converge to 0. We make use of the minimum principle (see [4]) which implies that the eigenvalues $\mu_j^{\varepsilon,\pm}$ can be found from the formula

$$\pm\mu_j^{\varepsilon,\pm} = \max_{\substack{u\in\mathcal{H},\\ \langle u,u\rangle=1}} \pm\langle\mathcal{K}^\varepsilon u, u\rangle,$$

where the minimum is taken over vectors $u \in \mathcal{H}$ which are orthogonal to the linear span of $\{u_k^{\varepsilon,\pm}\}_{k=1}^{j-1}$, and $u_j^{\varepsilon,\pm}$ is a point on the unit sphere in $\mathcal{H}$ at which the minimum is attained. In particular, for the first negative eigenvalue $\mu_1^{\varepsilon,-}$ we have

$$\mu_1^{\varepsilon,-} = \min_{\substack{u\in\mathcal{H},\\ \langle u,u\rangle=1}} \langle\mathcal{K}^\varepsilon u, u\rangle = \min_{\langle u,u\rangle=1} (\rho^\varepsilon u, u)_\Omega.$$

Due to our assumption, the measure of the set $M_1^\varepsilon \equiv \{x \in \Omega : \rho^\varepsilon(x) < 0\}$ is positive. Denote by $\chi(M_1^\varepsilon)$ the characteristic function of M_1^ε, and let φ be a $C_0^\infty(\mathbb{R}^n)$ function such that $\varphi \geqq 0$, $\varphi(x) = 0$ if $|x| \geqq 1$; $\int_\Omega \varphi(x)\,\mathrm{d}x = 1$. We set $\varphi_\delta(x) = \delta^{-n}\varphi(x/\delta)$. Then $\chi_\delta(M_1^\varepsilon) \equiv \chi(M_1^\varepsilon) * \varphi_\delta$ converges to $\chi(M_1^\varepsilon)$ in $L^2(\Omega)$ as $\delta \to 0$. As usual, the sign " $*$ " indicates the convolution of two functions, that is

$$\chi_\delta\left(M_1^\varepsilon\right) = \int_{\mathbb{R}^d} \chi(M_1^\varepsilon)(x-z)\varphi_\delta(z)\,\mathrm{d}z.$$

If we choose a constant c_0 such that

$$c_0^2\left\langle\chi_\delta\left(M_1^\varepsilon\right), \chi_\delta\left(M_1^\varepsilon\right)\right\rangle = 1,$$

then the function $c_0\chi_\delta(M_1^\varepsilon)$ can be used as a test function in the expression for $\mu_1^{\varepsilon,-}$, and thus

$$\mu_1^{\varepsilon,-} \leqq c_0^2\left(\rho^\varepsilon\chi_\delta\left(M_1^\varepsilon\right), \chi_\delta\left(M_1^\varepsilon\right)\right)_\Omega \underset{\delta\to0}{\longrightarrow} c_0^2\left(\rho^\varepsilon\chi\left(M_1^\varepsilon\right), \chi\left(M_1^\varepsilon\right)\right)_\Omega.$$

Hence, in view of definition of the set M_1^ε, the last expression implies $\mu_1^{\varepsilon,-} < 0$.

The second eigenvalue can be found from the formula

$$\mu_2^{\varepsilon,-} = \min_{\substack{\langle u,u\rangle=1\\ \langle u,u_1^{\varepsilon,-}\rangle=0}} \langle\mathcal{K}^\varepsilon u, u\rangle = \min_{\substack{\langle u,u\rangle=1\\ \langle u,u_1^{\varepsilon,-}\rangle=0}} (\rho^\varepsilon u, u)_\Omega.$$

It is not difficult to see that the set

$$\Theta = \{w \in H_0^1(\Omega) : \langle w, u_1^{\varepsilon,-}\rangle = 0, (\rho^\varepsilon w, w)_\Omega < 0\}$$

is not empty. Indeed, it suffices to consider two functions in $H_0^1(\Omega)$, say $\psi_1^\varepsilon(x)$ and $\psi_2^\varepsilon(x)$, with disjoint supports such that $(\rho^\varepsilon\psi_1^\varepsilon, \psi_1^\varepsilon)_\Omega < 0$ and $(\rho^\varepsilon\psi_2^\varepsilon, \psi_2^\varepsilon)_\Omega < 0$. Choosing a suitable linear combination $\gamma_1\psi_1^\varepsilon + \gamma_2\psi_2^\varepsilon$, one readily gets an element of Θ. This yields $\mu_2^{\varepsilon,-} < 0$.

In the same way, one proves that for any $m \geqq 1$ the set

$$\left\{ w \in H_0^1(\Omega) : (\rho^\varepsilon w, w)_\Omega < 0, \left\langle w, u_k^{\varepsilon,-} \right\rangle = 0, k = 1, \dots, m \right\}$$

is not empty. Thus, we have shown that $\mu_j^{\varepsilon,-} < 0$ for any j.

Due to the compactness of $\mathcal{K}^\varepsilon$, for any $\varepsilon > 0$ we have

$$\lim_{j\to\infty} \mu_j^{\varepsilon,-} = 0.$$

Similar arguments for positive eigenvalues $\mu_j^{\varepsilon,+}$ give the desired statement. Lemma 1 is proved. □

Taking into account the relation $\mu_j^{\varepsilon,\pm} = 1/\lambda_j^{\varepsilon,\pm}$, we obtain the following theorem.

Theorem 1. *Under assumption (7), the operator $\mathcal{L}^\varepsilon$ has a discreet spectrum which consists of two sequences*

$$\begin{aligned} 0 < \lambda_1^{\varepsilon,+} \leqq \lambda_2^{\varepsilon,+} \leqq \cdots \leqq \lambda_j^{\varepsilon,+} \leqq \cdots \to +\infty, \\ 0 > \lambda_1^{\varepsilon,-} \geqq \lambda_2^{\varepsilon,-} \geqq \cdots \geqq \lambda_j^{\varepsilon,-} \geqq \cdots \to -\infty. \end{aligned} \tag{8}$$

The corresponding eigenfunctions $u_j^{\varepsilon,\pm}$ satisfy the orthogonality and normalization condition

$$\langle u_i^{\varepsilon,\pm}, u_j^{\varepsilon,\pm} \rangle = \delta_{i,j}. \tag{9}$$

3. The case $\langle\rho\rangle > 0$: formal asymptotic expansion

We represent a solution $(\lambda^\varepsilon, u^\varepsilon)$ of problem (3) in the form

$$\begin{aligned} u^\varepsilon(x) &= u^0(x) + \varepsilon N(y)^{\mathrm{T}} \nabla u^0(x) + \varepsilon^2 w(x, y) + \cdots, \quad y = \frac{x}{\varepsilon}, \\ \lambda^\varepsilon &= \lambda^0 + \cdots; \end{aligned} \tag{10}$$

here $N(y)$ and $w(x, y)$ are Y-periodic in y. Let us substitute ansätze (10) into (2) and collect terms which are power-like with respect to ε. This yields the following problems on the periodicity cell Y:

$$\begin{cases} -\mathrm{div}_y(a(y)\nabla_y N_k(y)) = \mathrm{div}_y a_{\cdot k}(y), \quad k = 1, \dots, d, \ \ y \in Y, \\ N_k \in H_\#^1(Y), \end{cases} \tag{11}$$

where $a_{\cdot k}$ is a kth column of the matrix $a(y)$,

$$\begin{cases} -\mathrm{div}_y(a(y)\nabla_y w(x, y)) = \mathrm{div}_x(a(y)\nabla_x u^0(x)) + \lambda^0 \rho(y) u^0(x) \\ +\mathrm{div}_x[a(y)\nabla_y(N(y)^{\mathrm{T}}\nabla_x u^0(x))] + \mathrm{div}_y[a(y)\nabla_x(N(y)^{\mathrm{T}}\nabla_x u^0(x))], \quad y \in Y, \\ w(x, y) \text{ is periodic in } y. \end{cases} \tag{12}$$

If the mean-value of the right-hand side in (12) is equal to zero, then a solution of periodic problem (12) exists and is unique up to an additive constant (see, for example, Section 1.1 in [21]). The compatibility condition in problem (12) reads

$$\int_Y \Big\{\mathrm{div}_x(a(y)\nabla_x u^0(x)) + \lambda^0\rho(y)u^0(x) + \mathrm{div}_x\left[a(y)\nabla_y(N(y)^{\mathrm{T}}\,\nabla_x u^0(x))\right]$$
$$+\mathrm{div}_y\left[a(y)\nabla_x(N(y)^{\mathrm{T}}\nabla_x u^0(x))\right]\Big\}\;\mathrm{d}y = 0.$$

From the last equality one derives the following equation for $u^0(x)$:

$$\begin{cases} \mathcal{L}^{\mathrm{hom}}u^0(x) \equiv -\mathrm{div}(a^{\mathrm{hom}}\nabla u^0(x)) = \lambda^0\langle\rho\rangle u^0(x), & x\in\Omega,\\ u^0(x)=0, & x\in\partial\Omega, \end{cases} \tag{13}$$

where the constant matrix a^{hom} has the form

$$a^{\mathrm{hom}} = \int_Y \left[a(y) + a(y)\nabla_y N(y)^{\mathrm{T}}\right]\,\mathrm{d}y, \tag{14}$$

in other words,

$$a_{ij}^{\mathrm{hom}} = \int_Y \left[a_{ij}(y) + a_{ik}(y)\partial_{y_k}N_j(y)\right]\,\mathrm{d}y,\quad i,j=1,\dots,d.$$

This matrix a^{hom} is symmetric and positive definite (see, for instance [21]).

In view of Remark 1, Dirichlet problem (13) has the discreet spectrum

$$0<\lambda_1^{0,+}<\lambda_2^{0,+}\leqq\cdots\leqq\lambda_j^{0,+}\leqq\cdots\to+\infty. \tag{15}$$

Note that the first eigenvalue $\lambda_1^{0,+}$ is simple (see, for example, [8]). The corresponding eigenfunctions can be chosen to satisfy the orthogonality and normalization condition

$$\left(a^{\mathrm{hom}}\nabla u_i^{0,+},\nabla u_j^{0,+}\right)_\Omega = \langle\rho\rangle\left|\lambda_i^{0,+}\right|\left(u_i^{0,+},u_j^{0,+}\right)_\Omega = \delta_{i,j},\quad i,j=1,\dots,d. \tag{16}$$

Remark 2. Since Ω is a $C^{2,\delta}$ domain, and the homogenized equation in (13) has constant coefficients, then $u_j^{0,+}$ are $C^{2,\delta}(\overline{\Omega})$ functions (see [8]). Moreover, in the interior of the domain Ω the eigenfunctions $u_j^{0,+}$ are C^∞ functions (see [8]).

The next statement characterizes the asymptotic behaviour of the positive part of the spectrum $\sigma(\mathcal{L}^\varepsilon)$ as $\varepsilon\to 0$.

Theorem 2. *Assume that $a_{ij},\rho\in L^\infty(Y)$ are periodic functions, and $\langle\rho\rangle>0$. Then the following statements hold true:*

1. *Let $\lambda_j^{0,+}$ be an eigenvalue of the limit spectral problem (13), and assume that the multiplicity of λ_j^0 is equal to $\varkappa_j^+$, $\varkappa_j^+ \geqq 1$, so that $\lambda_{j-1}^{0,+} < \lambda_j^{0,+} = \lambda_{j+1}^{0,+} = \cdots = \lambda_{j+\varkappa_j^+-1}^{0,+} < \lambda_{j+\varkappa_j^+}^{0,+}$. Then there exist $\varepsilon_j > 0$ and a constant c_j such that for $\varkappa_j^+$ eigenvalues $\lambda_j^{\varepsilon,+}, \ldots, \lambda_{j+\varkappa_j^+-1}^{\varepsilon,+}$ of problem (2) and only for them the inequality holds*

$$\left|\lambda_q^{\varepsilon,+} - \lambda_j^{0,+}\right| \leqq c_j \varepsilon^{1/2}, \quad \varepsilon \in (0, \varepsilon_j).$$

Moreover, for $q \notin \{j, j+1, \ldots, j+\varkappa_j^+ - 1\}$ the inequality holds

$$|\lambda_q^{\varepsilon,+} - \lambda_j^{0,+}| \geqq \tilde{c}_j, \quad \varepsilon \in (0, \varepsilon_j),$$

with some $\tilde{c}_j > 0$.

2. *There exists a unitary $\varkappa_j^+ \times \varkappa_j^+$ matrix β^ε such that*

$$\left\| u_p^{\varepsilon,+} - \sum_{k=j}^{j+\varkappa_j^+-1} \beta_{kp}^\varepsilon \widetilde{U}_k^{\varepsilon,+} \right\|_{H^1(\Omega)} \leqq C_j\, \varepsilon^{1/2}, \quad p = j, \ldots, j+\varkappa_j^+ - 1, \tag{17}$$

where

$$\widetilde{U}_k^{\varepsilon,\pm}(x) = u_k^{0,+}(x) + \varepsilon\, N^\varepsilon(x)^{\mathrm{T}}\, \nabla u_k^{0,+}(x). \tag{18}$$

Here $N^\varepsilon(x) = N(x/\varepsilon)$, the vector-function $N(y)$ is a solution of problem (11); the eigenfunctions $u_k^{0,+}$ of limit spectral problem (13) satisfy normalization condition (16).

"Almost eigenfunctions" $\{\widetilde{U}_k^{\varepsilon,+}\}$ are "almost" orthogonal and normalized in the sense of the following inequality:

$$\left|\langle \widetilde{U}_k^{\varepsilon,+}, \widetilde{U}_l^{\varepsilon,+}\rangle - \delta_{k,l}\right| \leqq C\, \varepsilon^{1/2}. \tag{19}$$

Remark 3. Since $\lambda_1^{0,+}$ is simple, then by Theorem 2, for sufficiently small ε the eigenvalue $\lambda_1^{\varepsilon,+}$ is simple, as well.

Theorem 2 can be also reformulated as a convergence result.

Corollary 1. *For the positive eigenvalues (8) and (15) the following convergence result holds:*

$$\lambda_j^{\varepsilon,+} \to \lambda_j^{0,+}, \quad \varepsilon \to 0.$$

If $\lambda_j^{0,+}$ is simple, then $\lambda_j^{\varepsilon,+}$ is also simple and the corresponding eigenfunctions satisfy the relations:

- $$u_j^{\varepsilon,+} \xrightarrow[\varepsilon\to 0]{} u_j^{0,+} \text{ strongly in } L^2(\Omega);$$
- $$u_j^{\varepsilon,+} - \varepsilon(N^\varepsilon)^{\mathrm{T}}\nabla u_j^{0,+} \xrightarrow[\varepsilon\to 0]{} u_j^{0,+} \text{ strongly in } H^1(\Omega);$$
- $a^\varepsilon \nabla u_j^{\varepsilon,+} \xrightarrow[\varepsilon\to 0]{} a^{\mathrm{hom}}\nabla u_j^{0,+}$ *weakly in* $L^2(\Omega)$, *where* $\langle\cdot\rangle$ *denotes the mean value over* Y.

The proofs of Theorem 2, as well as Corollary 1, are similar to the proofs of the corresponding statements in the case $\langle\rho\rangle = 0$ but a little bit less technical, that is why we omit them here.

Remark 4. Theorem 2 also applies to the negative part of the spectrum in the case $\langle\rho\rangle < 0$. Indeed, it suffices to replace ρ with $-\rho$ in (2).

4. The asymptotics of the spectrum in the case $\langle\rho\rangle = 0$

4.1. Preliminary notes

We begin by estimating $|\lambda_1^{\varepsilon,\pm}|$ from below. Multiplying the equality $\mathcal{L}^\varepsilon u_1^{\varepsilon,+} = \lambda_1^{\varepsilon,+}\rho^\varepsilon u_1^{\varepsilon,+}$ by $u_1^{\varepsilon,+}$ and integrating the result over Ω, we obtain

$$\int_\Omega \left(\nabla u_1^{\varepsilon,+}\right)^{\mathrm{T}} a^\varepsilon \left(\nabla u_1^{\varepsilon,+}\right)\,\mathrm{d}x = \lambda_1^{\varepsilon,+}\int_\Omega \rho^\varepsilon \left(u_1^{\varepsilon,+}\right)^2\,\mathrm{d}x. \tag{20}$$

Since $\langle\rho\rangle = 0$, there is a periodic vector-function $J \in (L^\infty_\#(Y))^n \cap (H^1_\#(Y))^n$ such that $\rho(y) = \operatorname{div} J(y)$ and $\langle J\rangle = 0$. This yields

$$\begin{aligned}\lambda_1^{\varepsilon,+}\int_\Omega \rho^\varepsilon(x)\left(u_1^{\varepsilon,+}(x)\right)^2\,\mathrm{d}x &= 2\varepsilon\lambda_1^{\varepsilon,+}\int_\Omega u_1^{\varepsilon,+}(x) J\left(\frac{x}{\varepsilon}\right)^{\mathrm{T}}\nabla u_1^{\varepsilon,+}(x)\,\mathrm{d}x\\ &\leqq C\varepsilon\lambda_1^{\varepsilon,+}\int_\Omega \left|u_1^{\varepsilon,+}(x)\right|\left|\nabla u_1^{\varepsilon,+}(x)\right|\,\mathrm{d}x \leqq C\varepsilon\lambda_1^{\varepsilon,+}\left\|\nabla u_1^{\varepsilon,+}\right\|^2_{L^2(\Omega)}.\end{aligned}$$

Combining this inequality with (20), we conclude that

$$\lambda_1^{\varepsilon,+} \geqq c\varepsilon^{-1}, \quad c > 0.$$

Similarly, $|\lambda_1^{\varepsilon,-}| \geqq c\varepsilon^{-1}$.

Let us show that if $\langle\rho\rangle = 0$, then

$$|\lambda_1^{\varepsilon,\pm}| \leqq C\varepsilon^{-1}.$$

To this end we use the variational principle (see, for example [4]):

$$\lambda_1^{\varepsilon,\pm} = \pm \min_{\substack{v\in\mathcal{H}\\ (\rho^\varepsilon v,v)_\Omega=\pm 1}} \int_\Omega \nabla^{\mathrm{T}} v(x) a^\varepsilon(x)\nabla v(x)\,\mathrm{d}x. \tag{21}$$

Denote

$$v^\varepsilon(x) = c^\varepsilon\varphi(x)[1+\varepsilon\rho^\varepsilon(x)],$$

with $\varphi \in C_0^\infty(\Omega)$, $\varphi \not\equiv 0$. We choose the constant c^ε such that

$$1 = (c^\varepsilon)^2 \int_\Omega \rho^\varepsilon(x)(\varphi(x))^2[1 + 2\varepsilon\rho^\varepsilon(x) + \varepsilon^2(\rho^\varepsilon(x))^2]\,\mathrm{d}x.$$

Using, again, the representation $\rho(y) = \operatorname{div} J(y)$, neglecting the terms of order ε^2, and integrating by parts, we obtain

$$\begin{aligned} 1 &= (c^\varepsilon)^2\varepsilon \int_\Omega \left\{\operatorname{div} J\left(\frac{x}{\varepsilon}\right)(\varphi(x))^2 + 2(\rho^\varepsilon(x))^2(\varphi(x))^2 + O(\varepsilon)\right\}\,\mathrm{d}x \\ &= -(c^\varepsilon)^2\left\{\varepsilon\int_\Omega \left(J\left(\frac{x}{\varepsilon}\right)\right)^{\mathrm{T}}\nabla(\varphi(x))^2\,\mathrm{d}x + 2\varepsilon\int_\Omega (\rho^\varepsilon(x))^2(\varphi(x))^2\,\mathrm{d}x + O(\varepsilon^2)\right\}. \end{aligned} \tag{22}$$

Since $\langle J\rangle = 0$, each component of this vector-function admits the representation $J_k = \operatorname{div}\widetilde{J}_{\cdot k}$, where $\widetilde{J}_{\cdot k}$ are periodic vector functions. This allows us to integrate by parts in (22) once more and to derive

$$1 = (c^\varepsilon)^2\left(2\varepsilon\langle\rho^2\rangle\|\varphi\|^2_{L^2(\Omega)} + O(\varepsilon^2)\right).$$

Therefore,

$$(c^\varepsilon)^2 = \left(2\,\varepsilon\,\langle\rho^2\rangle\,\|\varphi\|^2_{L^2(\Omega)}\right)^{-1} + O(\varepsilon).$$

Taking v^ε as a test function in (21), one sees that

$$\lambda_1^{\varepsilon,+} \leqq \int_\Omega \nabla^{\mathrm{T}} v^\varepsilon(x) a^\varepsilon(x)\nabla v^\varepsilon(x)\,\mathrm{d}x \leqq C\varepsilon^{-1}.$$

The negative eigenvalue $\lambda_1^{\varepsilon,-}$ can be estimated in the same way.

4.2. Formal asymptotic expansion

Bearing in mind the estimates from the previous subsection, we are looking for a solution of problem (2) in the form

$$\begin{aligned} u^\varepsilon(x) &= u^0(x) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \cdots, \quad y = \frac{x}{\varepsilon}, \\ \lambda^\varepsilon &= \varepsilon^{-1}\nu + \cdots, \end{aligned} \tag{23}$$

where ν, $u^0(x)$, $u_1(x,y)$ and $u_2(x,y)$ are to be determined. We suppose that $u_1(x,y)$ and $u_2(x,y)$ are Y-periodic in y. Substituting asymptotic ansätze (23) into (2) and collecting terms of order ε^{-1}, we obtain the following equation for the unknown function $u_1(x,y)$:

$$\begin{cases} -\operatorname{div}_y\left(a(y)\nabla_y u_1(x,y)\right) = \operatorname{div}_y\left(a(y)\nabla_x u^0(x)\right) + \nu\rho(y)u^0(x), & y \in Y, \\ u_1(x,\cdot) \in H^1_\#(Y). \end{cases}$$

Note that, since $a(y)$ is periodic and $\langle\rho\rangle = 0$, the compatibility condition is satisfied and the last problem has a unique solution of zero average. The specific form of the right-hand side of the equation suggests the following representation for the solution:

$$u_1(x, y) = N(y)^{\mathrm{T}} \nabla_x u^0(x) + \nu N^0(y) u^0(x).$$

Thus, the unknown vector-function N and the function N^0 solve the problems

$$\begin{cases} -\mathrm{div}_y (a(y) \nabla_y N_k(y)) = \mathrm{div}_y a_{\cdot k}(y), \quad k = 1, \dots, d, \ \ y \in Y, \\ N_k \in H^1_{\#}(Y), \end{cases} \tag{24}$$

where $a_{\cdot k}$ is a kth column in the matrix $a(y)$,

$$\begin{cases} -\mathrm{div}_y (a(y) \nabla_y N^0(y)) = \rho(y), \quad y \in Y, \\ N^0 \in H^1_{\#}(Y). \end{cases} \tag{25}$$

Notice that the compatibility conditions in problems (24) and (25) are satisfied.

Then, collecting the terms of order ε^0, we get an equation for the function $u_2(x, y)$ on the periodicity cell Y, namely

$$\begin{cases} -\mathrm{div}_y \left(a(y) \nabla_y u_2(x, y)\right) = \mathrm{div}_y \left(a(y) \nabla_x (N(y)^{\mathrm{T}} \nabla_x u^0(x))\right) \\ +\nu \mathrm{div}_y \left(a(y) \nabla_x u^0(x) N^0(y)\right) + \mathrm{div}_x \left(a(y) \nabla_y N(y) \nabla_x u^0(x)\right) \\ +\nu \mathrm{div}_x \left(a(y) \nabla_y N^0(y) u^0(x)\right) + \mathrm{div}_x \left(a(y) \nabla_x u^0(x)\right) \\ +\nu \rho(y) N(y)^{\mathrm{T}} \nabla_x u^0(x) + \nu^2 \rho(y) N^0(y) u^0(x), \quad y \in Y, \\ u_2(x, \cdot) \in H^1_{\#}(Y). \end{cases} \tag{26}$$

Owing to the periodicity of the matrix $a(y)$, the vector function $N(y)$ and the function $N^0(y)$, we have

$$\int_Y \mathrm{div}_y \left(a(y) \nabla_x (N(y)^{\mathrm{T}} \nabla_x u^0(x))\right) + \nu \, \mathrm{div}_y \left(a(y) \nabla_x u^0(x) N^0(y)\right) \, \mathrm{d}y = 0.$$

Thus, the compatibility condition in problem (26) reads

$$\begin{aligned} &\mathrm{div}_x \left\{ \int_Y \left(a(y) + a(y) \, \nabla_y N(y)\right) \, \mathrm{d}y \, \nabla_x u^0(x) \right\} \\ &+\nu \, \mathrm{div}_x \left\{ \int_Y a(y) \nabla_y N^0(y) \, \mathrm{d}y \, u^0(x) \right\} \\ &+\nu \int_Y \rho(y) N(y)^{\mathrm{T}} \, \mathrm{d}y \, \nabla_x u^0(x) + \nu^2 \int_Y \rho(y) N^0(y) \, \mathrm{d}y \, u^0(x) = 0. \end{aligned} \tag{27}$$

Let us rearrange (27) using equations for N and N^0:

$$\begin{aligned} \int_Y \rho(y) \, N(y)^{\mathrm{T}} \, \mathrm{d}y &= -\int_Y \mathrm{div}_y \left(a(y) \nabla_y N^0(y)\right) \, N(y)^{\mathrm{T}} \, \mathrm{d}y \\ &= \int_Y {\nabla_y}^{\mathrm{T}} N^0(y) \, a(y) \, \nabla_y N(y)^{\mathrm{T}} \, \mathrm{d}y, \end{aligned} \tag{28}$$

$$\operatorname{div}_x \left\{ \int_Y a(y)\, \nabla_y N^0(y)\, \mathrm{d}y\, u^0(x) \right\} = -\operatorname{div}_x \left\{ \int_Y \operatorname{div}_y a(y)\, N^0(y)\, \mathrm{d}y\, u^0(x) \right\}$$

$$= \operatorname{div}_x \left\{ \int_Y \operatorname{div}_y \left(a(y) \nabla_y N(y)^{\mathrm{T}} \right) N^0(y)\, \mathrm{d}y\, u^0(x) \right\}$$

$$= - \int_Y \nabla_y^{\mathrm{T}} N^0(y)\, a(y)\, \nabla_y N(y)^{\mathrm{T}}\, \mathrm{d}y, \tag{29}$$

$$\nu^2 \int_Y \rho(y)\, N^0(y)\, \mathrm{d}y\, u^0(x) = -\nu^2 \int_Y \operatorname{div}_y \left(a(y) \nabla_y N^0(y) \right) N^0(y)\, \mathrm{d}y\, u^0(x)$$

$$= \nu^2 \int_Y \nabla_y^{\mathrm{T}} N^0(y)\, a(y)\, \nabla_y N^0(y)\, \mathrm{d}y\, u^0(x) \equiv \nu^2 \kappa^2 u^0(x), \tag{30}$$

where we have set

$$\kappa^2 = \int_Y \nabla_y^{\mathrm{T}} N^0(y)\, a(y)\, \nabla_y N^0(y)\, \mathrm{d}y. \tag{31}$$

Notice that the sum of the right-hand sides in (28) and (29) is equal to zero. Consequently, (27) (supplemented with an appropriate boundary condition) takes the form

$$\begin{cases} \mathcal{L}^{\mathrm{hom}} u^0(x) \equiv -\operatorname{div}_x \left(a^{\mathrm{hom}} \nabla_x u^0(x) \right) = \nu^2 \kappa^2 u^0(x), & x \in \Omega, \\ u^0(x) = 0, & x \in \partial\Omega, \end{cases} \tag{32}$$

with a positive definite symmetric homogenized matrix a^{hom} defined by the formula

$$a^{\mathrm{hom}} \equiv \int_Y \left(a(y) + a(y)\, \nabla_y N(y) \right) \mathrm{d}y. \tag{33}$$

Although (32) is a spectral problem for a quadratic operator pencil with respect to ν, it is not difficult to characterize its spectrum introducing the new spectral parameter $\tau = \nu^2$. Indeed, the spectrum of (32) consists of two sequences

$$\begin{aligned} 0 < \quad & \nu_1^{0,+} < \nu_2^{0,+} \leqq \cdots \leqq \nu_j^{0,+} \leqq \cdots \to +\infty, \\ 0 > \quad & \nu_1^{0,-} > \nu_2^{0,-} \geqq \cdots \geqq \nu_j^{0,-} \geqq \cdots \to -\infty. \end{aligned} \tag{34}$$

with $\nu_j^{0,-} = -\nu_j^{0,+}$, $j = 1, 2, \ldots$, and with the corresponding eigenfunctions $u_j^{0,+} = u_j^{0,-}$. In what follows, omitting the indices $\pm$, we will denote them u_j^0. The notation $\varkappa_j$ will be used for the multiplicity of $\nu_j^{0,\pm}$.

For the eigenfunctions u_j^0 we choose the following orthogonality and normalization condition:

$$\left(a^{\mathrm{hom}} \nabla u_i^0, \nabla u_j^0 \right)_\Omega + \nu_i^0 \nu_i^0 \kappa^2 \left(u_i^0, u_j^0 \right)_\Omega = \delta_{i,j}. \tag{35}$$

Although, at first sight, such a choice seems to be odd, it ensures the convergence of energies. It should be noted that the first positive and negative eigenvalues $\nu_1^{0,\pm}$ are simple.

Remark 5. Since Ω is a $C^{2,\delta}$-domain, then, as in the case $\langle\rho\rangle > 0$, u_j^0 are $C^{2,\delta}(\overline{\Omega})$-functions (see [8]). Moreover, in the interior of the domain Ω the eigenfunctions u_j^0 are C^∞-functions (see [8]).

4.3. Justification procedure in the case $\langle\rho\rangle = 0$

Let $\nu_j^{0,\pm}$ be eigenvalues of the operator pencil (32) of multiplicity $\varkappa_j$ that is $\pm\nu_{j-1}^{0,\pm} < \pm\nu_j^{0,\pm} = \pm\nu_{j+1}^{0,\pm} = \cdots = \pm\nu_{j+\varkappa_j-1}^{0,\pm} < \pm\nu_{j+\varkappa_j}^{0,\pm}$, and $\{u_p^0\}$, $p = j, \dots, j+\varkappa_j - 1$, be the eigenfunctions of the limit spectral problem (32) corresponding to $\nu_j^{0,\pm}$. We denote $\Omega_\gamma = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geqq \gamma\}$. Let χ_ε be a cut-off function which is equal to 0 in $\Omega \setminus \Omega_{h\varepsilon}$, $h > \sqrt{d}$, equal to 1 in $\Omega_{2h\varepsilon}$, and is such that

$$0 \leqq \chi_\varepsilon(x) \leqq 1, \quad |\nabla\chi_\varepsilon(x)| \leqq C\varepsilon^{-1}. \tag{36}$$

The justification procedure will rely on the lemma about "almost eigenvalues and eigenfunctions" (see, for example [12,18,20]).

Lemma 2. *Given a self-adjoint operator $\mathcal{K}^\varepsilon : \mathcal{H} \to \mathcal{H}$, let $\nu \in \mathbb{R}$ and $v \in \mathcal{H}$ be such that*

$$\|v\|_{\mathcal{H}} = 1, \quad \delta \equiv \|\mathcal{K}^\varepsilon\, v - \nu\, v\|_{\mathcal{H}} < |\nu|.$$

Then there exists an eigenvalue μ_l^ε of the operator $\mathcal{K}^\varepsilon$ such that

$$\left|\mu_l^\varepsilon - \nu\right| \leqq \delta.$$

Moreover, for any $\delta_1 \in (\delta, |\nu|)$ there exist coefficients $\{b_j^\varepsilon\} \in \mathbb{R}$ satisfying

$$\left\| v - \sum b_j^\varepsilon u_j^\varepsilon \right\|_{\mathcal{H}} \leqq 2\frac{\delta}{\delta_1},$$

where the sum is taken over all the eigenvalues of the operator $\mathcal{K}^\varepsilon$ in the segment $[\nu - \delta_1, \nu + \delta_1]$, and $\{u_j^\varepsilon\}$ are the corresponding eigenfunctions orthonormalized in $\mathcal{H}$. The coefficients b_j^ε are normalized by $\sum |b_j^\varepsilon|^2 = 1$.

For an arbitrary $p \in \{j, \dots, j+\varkappa_j - 1\}$, denote

$$U_p^{\varepsilon,\pm}(x) \equiv u_p^0(x) + \varepsilon\, \chi_\varepsilon(x)\, N\left(\frac{x}{\varepsilon}\right)^{\mathrm{T}} \nabla u_p^0(x) + \varepsilon\, \nu_j^{0,\pm}\, N^0\left(\frac{x}{\varepsilon}\right) u_p^0(x), \tag{37}$$

where $N(y)$ and $N^0(y)$ solve problems (24) and (25), respectively. The normalized functions $\mathcal{U}_p^{\varepsilon,+} \equiv \|U_p^{\varepsilon,+}\|_{\mathcal{H}}^{-1}\, U_p^{\varepsilon,+}$ and the numbers $\varepsilon(\nu_j^{0,+})^{-1}$ will play the role of $v \in \mathcal{H}$ and $\nu \in \mathbb{R}$ in Lemma 2. Notice that ∇u_p^0 need not be equal to zero on the boundary $\partial\Omega$; the cut-off function has been introduced in order to make the approximate solution (37) an element of the space $\mathcal{H}$.

We are going to estimate

$$\begin{aligned}
&\left\|\mathcal{K}^\varepsilon\,\mathcal{U}_p^{\varepsilon,\pm}-\varepsilon(\nu_j^{0,\pm})^{-1}\,\mathcal{U}_p^{\varepsilon,\pm}\right\|_{\mathcal{H}}=\sup_{\substack{v\in\mathcal{H}\\ \langle v,v\rangle=1}}\left|\langle\mathcal{K}^\varepsilon\,\mathcal{U}_p^{\varepsilon,\pm}-\varepsilon(\nu_j^{0,\pm})^{-1}\,\mathcal{U}_p^{\varepsilon,\pm},v\rangle\right|\\
&=\varepsilon(\nu_j^{0,\pm})^{-1}\,\|U_p^{\varepsilon,\pm}\|_{\mathcal{H}}^{-1}\sup_{\substack{v\in\mathcal{H}\\ \langle v,v\rangle=1}}\left|\langle\varepsilon^{-1}\,\nu_j^{0,\pm}\;\mathcal{K}^\varepsilon\,U_p^{\varepsilon,\pm}-U_p^{\varepsilon,\pm},v\rangle\right|\\
&=\varepsilon(\nu_j^{0,\pm})^{-1}\,\|U_p^{\varepsilon,\pm}\|_{\mathcal{H}}^{-1}\sup_{\substack{v\in\mathcal{H}\\ \langle v,v\rangle=1}}\left|a_\varepsilon(U_p^{\varepsilon,\pm},v)-\varepsilon^{-1}\,\nu_j^{0,\pm}\left(\rho^\varepsilon\,U_p^{\varepsilon,\pm},v\right)_\Omega\right|\\
&=\varepsilon(\nu_j^{0,\pm})^{-1}\|U_p^{\varepsilon,\pm}\|_{\mathcal{H}}^{-1}\sup_{\substack{v\in\mathcal{H}\\ \langle v,v\rangle=1}}\left|\left(\mathcal{L}^\varepsilon\,U_p^{\varepsilon,\pm}-\varepsilon^{-1}\,\nu_j^{0,\pm}\,\rho^\varepsilon\,U_p^{\varepsilon,\pm},v\right)_\Omega\right|\\
&=\varepsilon(\nu_j^{0,\pm})^{-1}\,\|U_p^{\varepsilon,\pm}\|_{\mathcal{H}}^{-1}\sup_{\substack{v\in\mathcal{H}\\ \langle v,v\rangle=1}}\left|\varepsilon^{-1}(I_1^\varepsilon,v)_\Omega+\varepsilon^0(I_2^\varepsilon,v)_\Omega+\varepsilon^1(I_3^\varepsilon,v)_\Omega\right|,
\end{aligned}$$

where

$$\begin{aligned}
I_1^\varepsilon(x)&=\Big\{-\mathrm{div}_y\Big[a(y)\nabla_y N(y)\,\nabla_x u_p^0(x))\Big]\,\chi_\varepsilon(x)-\mathrm{div}_y\Big[a(y)\nabla_x u_p^0(x)\Big]\\
&\qquad-\nu_j^{0,\pm}\,\mathrm{div}_y\Big[a(y)\nabla_y N^0(y)\,u_p^0(x)\Big]-\nu_j^{0,\pm}\,\rho(y)\,u_p^0(x)\Big\}\Big|_{y=x/\varepsilon}\\
&=-\,\mathrm{div}_y a_{\cdot i}(y)\,\partial_{x_i}u_p^0(x)\;(1-\chi_\varepsilon(x))\Big|_{y=x/\varepsilon};\\
I_2^\varepsilon(x)&=\Big\{-\mathrm{div}_x\Big[a(y)\nabla_y N(y)\,\nabla u_p^0(x)\Big]\,\chi_\varepsilon(x)-{\nabla_x}^{\mathrm{T}}\chi_\varepsilon(x)\,a(y)\nabla_y N(y)\,\nabla u_p^0(x)\\
&\qquad-\nu_j^{0,\pm}\,\mathrm{div}_x\Big[a(y)\nabla_y N^0(y)\,u_p^0(x)\Big]-\mathrm{div}_x\Big[a(y)\nabla u_p^0(x)\Big]\\
&\qquad-\nu_j^{0,\pm}\rho(y)\,\chi_\varepsilon(x)\,N(y)^{\mathrm{T}}\,\nabla_x u_p^0(x)-(\nu_j^{0,\pm})^2\,\rho(y)\,N^0(y)\,u_p^0(x)\Big\}\Big|_{y=x/\varepsilon};\\
I_3^\varepsilon(x)&=\Big\{-\Big[\mathrm{div}_x+\varepsilon^{-1}\mathrm{div}_y\Big]\Big(a(y)\nabla_x\Big(\chi_\varepsilon(x)N(y)^{\mathrm{T}}\,\nabla u_p^0(x)\Big)\Big)\\
&\qquad-\nu_j^{0,\pm}\,\Big[\mathrm{div}_x+\varepsilon^{-1}\mathrm{div}_y\Big]\Big(a(y)\,N^0(y)\,\nabla_x u_p^0(x)\Big)\Big\}\Big|_{y=x/\varepsilon}.
\end{aligned}$$

Integrating by parts, and considering the regularity of u_p^0, we obtain

$$\begin{aligned}
\varepsilon^{-1}\,\left|\left(I_1^\varepsilon,v\right)_\Omega\right|&=\varepsilon^{-1}\left|\int_\Omega\varepsilon\,\nabla_y^{\mathrm{T}}\left((1-\chi_\varepsilon(x))\,v(x)\right)\,a_{\cdot i}\left(\frac{x}{\varepsilon}\right)\,\partial_{x_i}u_p^0(x)\,\mathrm{d}x\right|\\
&\leqq C_1\int_{\Omega\setminus\Omega_{h\varepsilon}}\{|v(x)|+|\nabla_x v(x)|\}\,(1-\chi_\varepsilon(x))\,\mathrm{d}x\\
&\quad+C_2\int_{\Omega\setminus\Omega_{h\varepsilon}}|v(x)|\,|\nabla_x\chi_\varepsilon(x)|\,\mathrm{d}x.
\end{aligned}$$

By (36), Lemma 3 formulated below, and the Cauchy–Bunyakovsky inequality, we get

$$\varepsilon^{-1}\,\left|\left(I_1^\varepsilon,v\right)_\Omega\right|\leqq C\sqrt{\varepsilon};\tag{38}$$

here we have also used the fact that the measure of $\Omega \setminus \Omega_{h\varepsilon}$ is of order ε. The proofs of the following auxiliary inequalities of Hardy's type can be found, for example, in [12].

Lemma 3. *Let $v \in H_0^1(\Omega)$. Then*

$$\|v\|_{L^2(\partial\Omega_\gamma)} \leqq C\,\sqrt{\gamma}\,\|\nabla v\|_{L^2(\Omega)};$$
$$\|v\|_{L^2(\Omega\setminus\Omega_\gamma)} \leqq C\,\gamma\,\|\nabla v\|_{L^2(\Omega)}.$$

Denote by $W_\#^{1,\infty}(Y)$ a space of periodic functions v with the norm

$$\|v\|_{W_\#^{1,\infty}(Y)} = \|v\|_{L^\infty(Y)} + \|\nabla v\|_{L^\infty(Y)}.$$

Lemma 4. *Let $\langle g\rangle$ be the mean value of g over the periodicity cell Y, $f \in H^1(\Omega)$ and $g \in L^2(Y)$ (or, alternatively, $f \in W_\#^{1,\infty}(Y)$ and $g \in L^1(Y)$). Then the following inequality is valid:*

$$\left|\int_\Omega \chi_\varepsilon(x) f(x) g\left(\frac{x}{\varepsilon}\right)\,\mathrm{d}x - \langle g\rangle \int_\Omega f(x)\,\mathrm{d}x\right| \leqq C\,\varepsilon\|f\|\,\|g\|$$

with the corresponding norms of the functions f and g, the constant C does not depend on ε, f and g.

The proof of Lemma 4 can be found, for example, in [6,11].

In order to estimate $(I_2^\varepsilon, v)_\Omega$ we rearrange I_2^ε as follows:

$$\begin{aligned} I_2^\varepsilon(x) = &-\Big\{\mathrm{div}_x\left[a(y)\nabla_y N(y)\,\nabla_x u_p^0(x)\right] + \mathrm{div}_x\left[a(y)\nabla_x u_p^0(x)\right] \\ &+\nu_j^{0,\pm}\,\mathrm{div}_x\left[a(y)\nabla_y N^0(y)\,u_p^0(x)\right] + \nu_j^{0,\pm}\rho(y)\,N(y)^{\mathrm{T}}\,\nabla_x u_p^0(x) \\ &+\,(\nu_j^{0,\pm})^2\,\rho(y)\,N^0(y)\,u_p^0(x)\Big\}\Big|_{y=x/\varepsilon}\;\chi_\varepsilon(x) \\ &-\Big[\nu_j^{0,\pm}\,\mathrm{div}_x\left[a(y)\nabla_y N^0(y)\,u_p^0(x)\right] + \mathrm{div}_x\left[a(y)\nabla_x u_p^0(x)\right] \\ &+(\nu_j^{0,\pm})^2\,\rho(y)\,N^0(y)\,u_p^0(x)\Big]\Big|_{y=x/\varepsilon}\;(1-\chi_\varepsilon(x)) \\ &-\,\nabla_x^{\mathrm{T}}\chi_\varepsilon(x)\,a(y)\nabla_y N(y)\,\nabla_x u_p^0(x)\Big|_{y=x/\varepsilon} \\ \equiv\;& f_1^\varepsilon(x)\,\chi_\varepsilon(x) + f_2^\varepsilon(x)\,(1-\chi_\varepsilon(x)) + f_3^\varepsilon(x); \end{aligned}$$

Since the expression in the braces has zero mean [see (27)], we, by Lemma 4, have

$$\left|\left(f_1^\varepsilon\,\chi_\varepsilon, v\right)_\Omega\right| \leqq C\,\varepsilon. \tag{39}$$

Taking into account the boundedness of the coefficients, Remark 5, formula (36) and the properties of N^0 as a solution of (25), one can check that

$$\begin{aligned} \left|\left(f_2^\varepsilon\,(1-\chi_\varepsilon), v\right)_\Omega\right| &\leqq C\,\sqrt{\varepsilon}\,\|\nabla_y N^0\|_{L^2(\Omega\setminus\Omega_{h\varepsilon})}\,\|v\|_{L^2(\Omega\setminus\Omega_{h\varepsilon})} \\ &+C|\Omega\setminus\Omega_{h\varepsilon}|^{1/2}\,\|v\|_{L^2(\Omega\setminus\Omega_{h\varepsilon})} \leqq C\,\varepsilon^{3/2}\|v\|_{H^1(\Omega)}. \end{aligned}$$

Here $|\Omega \setminus \Omega_{h\varepsilon}|$ is the Lebesgue measure of the set $\Omega \setminus \Omega_{h\varepsilon}$. By similar arguments, we derive the estimate

$$\begin{aligned}\left|\left(f_3^\varepsilon \nabla \chi_\varepsilon, v\right)_\Omega\right| &\leqq C\,\varepsilon^{-1} \left(\int_{\Omega\setminus\Omega_{h\varepsilon}} |\nabla_y N(y)^{\mathrm{T}}|^2\Big|_{y=x/\varepsilon}\, \mathrm{d}x\right)^{1/2} \|v\|_{L^2(\Omega\setminus\Omega_{h\varepsilon})}\\ &\leqq C\,\sqrt{\varepsilon}\,\|v\|_{H^1(\Omega)}.\end{aligned}$$

Consequently,

$$\left|\left(I_2^\varepsilon, v\right)_\Omega\right| \leqq C\,\sqrt{\varepsilon}. \tag{40}$$

In view of (36) integrating by parts yields

$$\begin{aligned}\left|\varepsilon\,(I_3^\varepsilon, v)_\Omega\right| \leqq\ & \varepsilon \left|\int_\Omega \nabla^{\mathrm{T}} v(x)\, a^\varepsilon(x)\, \nabla\chi_\varepsilon(x)\, N^{\mathrm{T}}\left(\frac{x}{\varepsilon}\right)\, \nabla u_p^0(x)\,\mathrm{d}x\right|\\ &+\varepsilon\left|\int_\Omega \nabla^{\mathrm{T}} v(x)\, a^\varepsilon(x)\, \nabla_x\, \left(N(y)^{\mathrm{T}}\, \nabla u_p^0(x)\right)\Big|_{y=x/\varepsilon}\, \chi_\varepsilon(x)\,\mathrm{d}x\right|\\ &+\varepsilon\left|\int_\Omega \nabla^{\mathrm{T}} v(x)\, a^\varepsilon(x)\, \nabla u_p^0(x)\, N^0\left(\frac{x}{\varepsilon}\right)\,\mathrm{d}x\right|\\ \leqq\ & C\,|\Omega\setminus\Omega_{h\varepsilon}|^{1/2}\|\nabla v\|_{L^2(\Omega)} + C\varepsilon\|\nabla v\|_{L^2(\Omega)}.\end{aligned}$$

Finally,

$$\left|\varepsilon\,\left(I_3^\varepsilon, v\right)_\Omega\right| \leqq C\,\sqrt{\varepsilon}. \tag{41}$$

Lemma 5. *"Almost eigenfunctions" $\mathcal{U}_p^{\varepsilon,\pm} = \|U_p^{\varepsilon,\pm}\|_{\mathcal{H}}^{-1}\, U_p^{\varepsilon,\pm}$, $p = j, \ldots, j + \varkappa_j - 1$, where $U_p^{\varepsilon,\pm}$ is defined by (37), are "almost orthonormal". Namely, the following inequality holds true:*

$$\left|\left\langle \mathcal{U}_p^{\varepsilon,\pm}, \mathcal{U}_q^{\varepsilon,\pm}\right\rangle - \delta_{p,q}\right| \leqq C\,\varepsilon, \quad p, q = j, \ldots, j + \varkappa_j - 1. \tag{42}$$

Proof. 1. First, we calculate the gradient of the function $U_p^{\varepsilon,\pm}$:

$$\begin{aligned}\nabla U_p^{\varepsilon,\pm}(x) =& \left\{\nabla_x u_p^0(x) + \nabla_y N(y)\, \nabla_x u_p^0(x) + \nu_j^{0,\pm} \nabla_y N^0(y)\, u_p^0(x)\right\}\Big|_{y=x/\varepsilon} \chi_\varepsilon(x)\\ &+\varepsilon\,\left\{\nabla_x (N(y)^{\mathrm{T}} \nabla_x u_p^0(x)) + \nu_j^{0,\pm} \nabla_y N^0(y)\, u_p^0(x)\right\}\Big|_{y=x/\varepsilon} \chi_\varepsilon(x)\\ &+ \Big[\left\{\nabla_x u_p^0(x) + \nu_j^{0,\pm} \nabla_y N^0(y)\, u_p^0(x)\right\}(1 - \chi_\varepsilon(x))\\ &+\varepsilon\, N(y)^{\mathrm{T}} \nabla_x u_p^0(x)\, \nabla_x \chi_\varepsilon(x)\Big]\Big|_{y=x/\varepsilon}\\ \equiv\ & \chi_\varepsilon(x)\, J_{1p}^\varepsilon(x) + \varepsilon\chi_\varepsilon(x) J_{2p}^\varepsilon(x) + J_{3p}^\varepsilon(x).\end{aligned}$$

Then $\langle U_p^{\varepsilon,\pm}, U_q^{\varepsilon,\pm}\rangle$ takes the form

$$\begin{aligned}
&\left(a^\varepsilon \nabla U_p^{\varepsilon,\pm}, \nabla U_q^{\varepsilon,\pm}\right)_\Omega = \left(a^\varepsilon \chi_\varepsilon J_{1p}^\varepsilon, \chi_\varepsilon J_{1q}^\varepsilon\right)_\Omega \\
&\quad +\varepsilon \left(a^\varepsilon \chi_\varepsilon J_{1p}^\varepsilon, \chi_\varepsilon J_{2q}^\varepsilon\right)_\Omega + \left(a^\varepsilon \chi_\varepsilon J_{1p}^\varepsilon, J_{3q}^\varepsilon\right)_\Omega \\
&\quad +\varepsilon \left(a^\varepsilon \chi_\varepsilon J_{2p}^\varepsilon, \chi_\varepsilon J_{1q}^\varepsilon\right)_\Omega + \varepsilon^2 \left(a^\varepsilon \chi_\varepsilon J_{2p}^\varepsilon, \chi_\varepsilon J_{2q}^\varepsilon\right)_\Omega + \varepsilon \left(a^\varepsilon \chi_\varepsilon J_{2p}^\varepsilon, J_{3q}^\varepsilon\right)_\Omega \\
&\quad + \left(a^\varepsilon J_{3p}^\varepsilon, \chi_\varepsilon J_{1q}^\varepsilon\right)_\Omega + \varepsilon \left(a^\varepsilon J_{3p}^\varepsilon, \chi_\varepsilon J_{2q}^\varepsilon\right)_\Omega + \left(a^\varepsilon J_{3p}^\varepsilon, J_{3q}^\varepsilon\right)_\Omega .
\end{aligned}$$

2. Let us proceed with proving that

$$\left|\left(a^\varepsilon \chi_\varepsilon J_{1p}^\varepsilon, \chi_\varepsilon J_{1q}^\varepsilon\right)_\Omega - \delta_{p,q}\right| \leqq C\,\varepsilon. \tag{43}$$

We have

$$\begin{aligned}
&\left(a^\varepsilon \chi_\varepsilon J_{1p}^\varepsilon, \chi_\varepsilon J_{1q}^\varepsilon\right)_\Omega = \int_\Omega \chi_\varepsilon^2(x)\,\nabla_x^{\mathrm T} u_p^0(x) \\
&\quad \times \left[a(y) + (\nabla_y N(y)^{\mathrm T})^{\mathrm T} a(y)\right]\Big|_{y=x/\varepsilon} \nabla_x u_p^0(x)\,\mathrm d x \\
&\quad +\int_\Omega \chi_\varepsilon^2(x)\,\nu_j^{0,\pm} \left\{\nabla_y^{\mathrm T} N^0(y)\,a(y) + \nabla_y^{\mathrm T} N^0(y)\,a(y)\,\nabla_y N(y)^{\mathrm T}\right\}\Big|_{y=x/\varepsilon} u_q^0(x)\,\nabla_x u_p^0(x)\,\mathrm d x \\
&\quad +\int_\Omega \chi_\varepsilon^2(x)\,\nabla_x^{\mathrm T} u_q^0(x) \left\{a(y)\,\nabla_y N(y)^{\mathrm T} + (\nabla_y N(y)^{\mathrm T})^{\mathrm T} a(y)\,\nabla_y N(y)^{\mathrm T}\right\}\Big|_{y=x/\varepsilon} \nabla_x u_p^0(x)\,\mathrm d x \\
&\quad +\int_\Omega \chi_\varepsilon^2(x)\,\nu_j^{0,\pm}\,\nabla_x^{\mathrm T} u_q^0(x) \left\{a(y)\nabla_y^{\mathrm T} N^0(y) + \nabla_y^{\mathrm T} N^0(y)\,a(y)\,\nabla_y N(y)^{\mathrm T}\right\}\Big|_{y=x/\varepsilon} u_p^0(x)\,\mathrm d x \\
&\quad + \int_\Omega \chi_\varepsilon^2(x)\,(\nu_j^{0,\pm})^2\,\nabla_y^{\mathrm T} N^0(y)\,a(y)\,\nabla_y N^0(y)\Big|_{y=x/\varepsilon} u_p^0(x)\,u_q^0(x)\,\mathrm d x.
\end{aligned}$$

Notice that the mean value of the expression in square brackets coincides with the homogenized matrix [see (33)]. Integrating by parts, one can show that all expressions in braces have zero mean, and, by definition,

$$\int_Y \nabla_y^{\mathrm T} N^0(y)\,a(y)\,\nabla_y N^0(y)\,\mathrm d y \equiv \kappa^2.$$

Thus, by Lemma 4,

$$\left|\left(a^\varepsilon\,\chi_\varepsilon J_{1p}^\varepsilon, \chi_\varepsilon J_{1q}^\varepsilon\right)_\Omega - \left(a^{\mathrm{hom}}\,\nabla u_p^0, \nabla u_q^0\right)_\Omega - \left(\nu_j^{0,\pm}\right)^2 \kappa^2 \left(u_p^0, u_q^0\right)_\Omega\right| \leqq C\,\varepsilon.$$

Taking into account the orthogonality and normalization condition (35), we obtain

$$\left|\left(a^\varepsilon\,\chi_\varepsilon J_{1p}^\varepsilon, \chi_\varepsilon J_{1q}^\varepsilon\right)_\Omega - \delta_{p,q}\right| \leqq C\,\varepsilon.$$

In particular, the $L^2(\Omega)$-norm of $\chi_\varepsilon J_{1p}^\varepsilon$ is bounded for a small $\varepsilon > 0$:

$$\left\|\chi_\varepsilon J_{1p}^\varepsilon\right\|_{L^2(\Omega)}^2 \leqq 1 + C\,\varepsilon \leqq \tilde C. \tag{44}$$

3. At this step we show that

$$\left|(a^\varepsilon\,\nabla U_p^{\varepsilon,\pm},\,\nabla U_q^{\varepsilon,\pm})_\Omega-(a^\varepsilon\,\chi_\varepsilon\,J_{1p}^\varepsilon,\,\chi_\varepsilon\,J_{1q}^\varepsilon)_\Omega\right|\leqq C\,\varepsilon. \tag{45}$$

Combining (44) with the evident estimate

$$\left|\varepsilon^2\int_\Omega\chi_\varepsilon^2(x)\,\left(J_{2p}^\varepsilon(x)\right)^2\,\mathrm{d}x\right|\leqq C\varepsilon^2, \tag{46}$$

we obtain

$$2\varepsilon\,\left|\left(a^\varepsilon\,\chi_\varepsilon\,J_{1p}^\varepsilon,\,\chi_\varepsilon\,J_{2q}^\varepsilon\right)_\Omega\right|\leqq C\,\varepsilon. \tag{47}$$

From (36), Remark 5 and the bound $|\Omega\setminus\Omega_{h\varepsilon}|\leqq c\varepsilon$, it follows that

$$\begin{aligned}\left\|J_{3p}^\varepsilon\right\|_{L^2(\Omega)}^2&\leqq\int_\Omega(1-\chi_\varepsilon(x))^2\left(|\nabla_x u_p^0(x)|^2+\nu_j^{0,\pm}|\nabla_y N^0(y)|^2\Big|_{y=x/\varepsilon}\left(u_p^0(x)\right)^2\right)\mathrm{d}x\\&\quad+\varepsilon^2\int_\Omega|\nabla_x\chi_\varepsilon(x)|^2\left|N(x/\varepsilon)^{\mathrm{T}}\,\nabla_x u_p^0(x)\right|^2\mathrm{d}x\\&\leqq C(|\Omega\setminus\Omega_{h\varepsilon}|+\varepsilon+\varepsilon^2\,\varepsilon^{-2}|\Omega\setminus\Omega_{h\varepsilon}|)\leqq C\,\varepsilon.\end{aligned}$$

The last estimate, together with (46), gives

$$\varepsilon\left|\left(a^\varepsilon\,\chi_\varepsilon\,J_{2p}^\varepsilon,\,J_{3q}^\varepsilon\right)_\Omega\right|\leqq C\,\varepsilon^{3/2}. \tag{48}$$

As regards to the term $(a^\varepsilon\chi_\varepsilon J_{1p}^\varepsilon,\,J_{3q}^\varepsilon)_\Omega$, it is not difficult to show that

$$\left|\left(a^\varepsilon\chi_\varepsilon J_{1p}^\varepsilon,\,J_{3q}^\varepsilon\right)_\Omega\right|\leqq C\varepsilon. \tag{49}$$

Now (46)–(49) imply (45) which, in turn, together with (43) leads to the inequality

$$\left|\left(a^\varepsilon\nabla U_p^{\varepsilon,\pm},\,\nabla U_q^{\varepsilon,\pm}\right)_\Omega-\delta_{p,q}\right|\leqq C\,\varepsilon. \tag{50}$$

In particular, the last estimate yields

$$\left\|U_p^{\varepsilon,\pm}\right\|_{\mathcal{H}}^2\geqq\frac{1}{2},\quad \varepsilon\in(0,\varepsilon_0). \tag{51}$$

Since $\mathcal{U}_p^{\varepsilon,\pm}=\|U_p^{\varepsilon,\pm}\|_{\mathcal{H}}^{-1}\,U_p^{\varepsilon,\pm}$, we have

$$\begin{aligned}\left|\left\langle\mathcal{U}_p^{\varepsilon,\pm},\mathcal{U}_q^{\varepsilon,\pm}\right\rangle-\delta_{p,q}\right|&=\left\|U_p^{\varepsilon,\pm}\right\|_{\mathcal{H}}^{-2}\left|\left\langle U_p^{\varepsilon,\pm},\,U_q^{\varepsilon,\pm}\right\rangle-\left\|U_p^{\varepsilon,\pm}\right\|_{\mathcal{H}}^2\delta_{p,q}\right|\\&\leqq\left\|U_p^{\varepsilon,\pm}\right\|_{\mathcal{H}}^{-2}\left|\left\langle U_p^{\varepsilon,\pm},\,U_q^{\varepsilon,\pm}\right\rangle-\delta_{p,q}\right|\\&\quad+\left\|U_p^{\varepsilon,\pm}\right\|_{\mathcal{H}}^{-2}\delta_{p,q}\left|1-\left\|U_p^{\varepsilon,\pm}\right\|_{\mathcal{H}}^{-2}\right|.\end{aligned}$$

The last inequality, (50) and (51) result in (42). The lemma is proved. □

Taking into account (38), (40), (41) and (51), we obtain the estimate

$$\left\| \mathcal{K}^\varepsilon \,\mathcal{U}_p^{\varepsilon,\pm} - \varepsilon(\nu_j^{0,\pm})^{-1}\,\mathcal{U}_p^{\varepsilon,\pm} \right\|_{\mathcal{H}} \leqq C\,\varepsilon^{3/2}. \tag{52}$$

By Lemma 2, there exists an eigenvalue $\mu_{q_j}^{\varepsilon,\pm}$ of the operator $\mathcal{K}^\varepsilon$, where q_j might depend on ε, such that

$$\left| \mu_{q_j}^{\varepsilon,\pm} - \varepsilon \left(\nu_j^{0,\pm} \right)^{-1} \right| \leqq C\,\varepsilon^{3/2}. \tag{53}$$

Since $\lambda_{q_j}^{\varepsilon,\pm} = (\mu_{q_j}^{\varepsilon,\pm})^{-1}$, there exist $\varepsilon_j > 0$ and a constant c_j such that

$$\left| \lambda_{q_j}^{\varepsilon,\pm} - \varepsilon^{-1}\nu_j^{0,\pm} \right| \leqq c_j\,\varepsilon^{-1/2}, \quad \varepsilon \in (0, \varepsilon_j).$$

Moreover, letting δ_1 in Lemma 2 be equal to $\Theta_j\,\varepsilon^{3/2}$ (the constant Θ_j will be chosen below), we conclude that there exists a $K_j(\varepsilon) \times \varkappa_j$ constant matrix α^ε such that

$$\left\| \mathcal{U}_p^{\varepsilon,\pm} - \sum_{k=J_j}^{J_j+K_J(\varepsilon)-1} \alpha_{kp}^\varepsilon\, u_k^{\varepsilon,\pm} \right\|_{\mathcal{H}} \leqq 2\frac{C\varepsilon^{3/2}}{\delta_1} \leqq C_j\Theta_j^{-1} \quad p = j, \ldots, j + \varkappa_j - 1,$$

here $\mu_{J_j(\varepsilon)}^{\varepsilon,\pm}, \ldots, \mu_{J_j(\varepsilon)+K_j(\varepsilon)-1}^{\varepsilon,\pm}$ are all the eigenvalues of the operator $\mathcal{K}^\varepsilon$ which satisfy the estimate

$$\left| \mu_k^{\varepsilon,\pm} - \varepsilon \left(\nu_j^{0,\pm} \right)^{-1} \right| \leqq \Theta_j\,\varepsilon^{3/2}. \tag{54}$$

Since the eigenvalues $\nu_j^{0,\pm}$ do not depend on ε, one can choose the constants $\varepsilon_j > 0$ such that the intervals $(\varepsilon(\nu_j^{0,\pm})^{-1} - \Theta_j\varepsilon^{3/2}, \varepsilon(\nu_j^{0,\pm})^{-1} + \Theta_j\varepsilon^{3/2})$ and $(\varepsilon(\nu_k^{0,\pm})^{-1} - \Theta_k\varepsilon^{3/2}, \varepsilon(\nu_k^{0,\pm})^{-1} + \Theta_k\varepsilon^{3/2})$ do not intersect under the condition $\nu_j^{0,\pm} \neq \nu_k^{0,\pm}$ and $\varepsilon < \min\{\varepsilon_j, \varepsilon_k\}$. Then the eigenvalue sets $\{\mu_k^{\varepsilon,\pm}\}$ related to different $\nu_j^{0,\pm}$ in (54) do not intersect for a small ε.

Thus, we conclude that, for any $\nu_j^{0,\pm}$ of multiplicity $\varkappa_j$, there exist $K_J(\varepsilon)$ eigenvalues $\lambda_k^{\varepsilon,\pm}$ of problem (2) such that

$$\left| \varepsilon\,\lambda_k^{\varepsilon,\pm} - \nu_j^{0,\pm} \right| \leqq \Theta_j\,\varepsilon^{1/2}, \quad \varepsilon \in (0, \varepsilon_j), \tag{55}$$

and the functions $\mathcal{U}_p^{\varepsilon\pm}$ admit the approximation

$$\left\| \mathcal{U}_p^{\varepsilon,\pm} - \sum_{k=J_j(\varepsilon)}^{J_j(\varepsilon)+K_J(\varepsilon)-1} \alpha_{kp}^\varepsilon\, u_k^{\varepsilon,\pm} \right\|_{\mathcal{H}} \leqq C_j\,\Theta_j^{-1}, \quad p = j, \ldots, j + \varkappa_j - 1, \tag{56}$$

Denote $J(j) = \min\{i \in \mathbb{Z}^+ : \nu_i^{0,\pm} = \nu_j^{0,\pm}\}$. The main result of this section is given in the following theorem.

Theorem 3. *Assume that $a_{ij}, \rho \in L^\infty(Y)$ are periodic functions, and the function ρ has zero mean. Let $\nu_j^{0,\pm}$ be an eigenvalue of the Dirichlet problem (32) of multiplicity $\varkappa_j$. Then the following statements hold true:*

1. *For each $j = 1, 2, \dots,$ there exist $\varepsilon_j > 0$ and a constant c_j such that only the eigenvalues $\lambda_{J(j)}^{\varepsilon,\pm}, \dots, \lambda_{J(j)+\varkappa_j-1}^{\varepsilon,\pm}$ of problem (2) satisfy the inequality*

$$\left|\varepsilon\,\lambda_q^{\varepsilon,\pm} - \nu_j^{0,\pm}\right| \leqq c_j\,\varepsilon^{1/2}, \quad \varepsilon \in (0, \varepsilon_j).$$

2. *There exists a unitary $\varkappa_j \times \varkappa_j$ matrix β^ε such that*

$$\left\| u_p^{\varepsilon,\pm} - \sum_{k=J(j)}^{J(j)+\varkappa_j-1} \beta_{kp}^\varepsilon\, \widetilde{U}_k^{\varepsilon,\pm} \right\|_{H^1(\Omega)} \leqq C_j\,\varepsilon^{1/2}, \quad p = J(j), \dots, J(j)+\varkappa_j-1, \tag{57}$$

where

$$\widetilde{U}_k^{\varepsilon,\pm}(x) = u_k^0(x) + \varepsilon\, N\left(\frac{x}{\varepsilon}\right)^{\mathrm{T}} \nabla u_k^0(x) + \varepsilon\,\nu_j^{0,\pm}\, N^0\left(\frac{x}{\varepsilon}\right) u_k^0(x). \tag{58}$$

Here the functions N, N^0 solve problems (24) and (25), respectively; eigenfunctions u_k^0 of the limit problem (32) satisfy the orthogonality and normalization condition (35).

"Almost eigenfunctions" $\{\widetilde{U}_k^{\varepsilon,\pm}\}$ are "almost" orthogonal and normalized in the following sense:

$$\left|\left\langle \widetilde{U}_k^{\varepsilon,\pm}, \widetilde{U}_l^{\varepsilon,\pm} \right\rangle - \delta_{k,l}\right| \leqq C\,\varepsilon^{1/2}. \tag{59}$$

Remark 6. Since both $\nu_1^{0,+}$ and $\nu_1^{0,-}$ are simple, for $\varepsilon \in (0, \varepsilon_1)$, eigenvalues $\lambda_1^{\varepsilon,\pm}$ are simple owing to Theorem 3.

Proof of Theorem 3. The proof consists of the several steps. First, we show that columns of the matrix α^ε are "almost" orthonormal, and from this deduce that $K_J(\varepsilon) \geqq \varkappa_j$. Then we prove that $J_j(\varepsilon) = J(j)$ and $K_J(\varepsilon) = \varkappa_j$. Finally, using properties of the matrix α^ε, we derive (57).

1. A simple transformation gives

$$\begin{aligned}\left\langle \mathcal{U}_p^{\varepsilon,\pm}, \mathcal{U}_q^{\varepsilon,\pm}\right\rangle &= \left\langle \mathcal{U}_p^{\varepsilon,\pm} - \sum_{k=J_j}^{J_j+K_J(\varepsilon)-1} \alpha_{kp}^\varepsilon u_k^{\varepsilon,\pm}, \mathcal{U}_q^{\varepsilon,\pm}\right\rangle \\ &\quad + \left\langle \sum_{k=J_j}^{J_j+K_J(\varepsilon)-1} \alpha_{kp}^\varepsilon u_k^{\varepsilon,\pm}, \mathcal{U}_q^{\varepsilon,\pm} - \sum_{k=J_j}^{J_j+K_J(\varepsilon)-1} \alpha_{kq}^\varepsilon u_k^{\varepsilon,\pm}\right\rangle \\ &\quad + \sum_{k=J_j}^{J_j+K_J(\varepsilon)-1} \alpha_{kp}^\varepsilon\,\alpha_{kq}^\varepsilon.\end{aligned}$$

Taking estimates (42) and (56) into account, we obtain

$$\left|\sum_{k=J_j}^{J_j+K_J(\varepsilon)-1} \alpha^\varepsilon_{kp}\,\alpha^\varepsilon_{kq} - \delta_{p,q}\right| \leqq C\,\Theta_j^{-1}, \quad p,q = J(j),\dots,J(j)+\varkappa_j-1,$$

and, in other words,

$$\left|\left(\alpha^\varepsilon_{\cdot p}\right)^{\mathrm{T}}\alpha^\varepsilon_{\cdot q} - \delta_{p,q}\right| \leqq C\,\Theta_j^{-1}, \quad p,q = J(j),\dots,J(j)+\varkappa_j-1, \tag{60}$$

where $\alpha^\varepsilon_{\cdot p}$ denotes a pth column in the matrix α^ε. The last inequality means that the vectors $\{\alpha^\varepsilon_{\cdot p}\}_{p=J(j)}^{J(j)+\varkappa_j-1}$ are asymptotically orthonormal. This property implies the linear independence of the vectors $\{\alpha^\varepsilon_{\cdot p}\}$. Indeed, assume that $\{\alpha^\varepsilon_{\cdot p}\}_{p=J(j)}^{J(j)+\varkappa_j-1}$ are not linearly independent. Then there exist constants $c_{J(j)},\dots,c_{J(j)+\varkappa_j-1}$ such that

$$\sum_{k=J(j)}^{J(j)+\varkappa_j-1} c_k\,\alpha^\varepsilon_{\cdot k} = 0.$$

Without loss of generality, we assume that $c_{J(j)} = 1 \geqq \max_k |c_k|$. Then

$$\alpha^\varepsilon_{\cdot,J(j)} + \sum_{k>J(j)} c_k\,\alpha^\varepsilon_{\cdot k} = 0.$$

Multiplying the last equality by $\alpha^\varepsilon_{\cdot,J(j)}$ and using (60), we obtain the inequality

$$\left|(\alpha^\varepsilon_{\cdot,J(j)})^{\mathrm{T}}\,\alpha^\varepsilon_{\cdot,J(j)}\right| \leqq C_j\,\Theta_j^{-1},$$

which contradicts (60) if Θ_j is small. Thus, the vectors $\{\alpha^\varepsilon_{\cdot p}\}_{p=J(j)}^{J(j)+\varkappa_j-1}$ of length $K_J(\varepsilon)$ are linearly independent. Obviously, it is possible only in the case $K_J(\varepsilon) \geqq \varkappa_j$.

2. Our next goal is to prove that any accumulating point of the sequence $\varepsilon\lambda_j^{\varepsilon,\pm}$, as $\varepsilon \to 0$, is an eigenvalue of problem (32).

Lemma 6. *Assume that, for an infinitesimal positive sequence* $\{\varepsilon_k\}$*, there exists a sequence* $\{j(k)\}$ *such that*

$$\varepsilon_k\lambda_{j(k)}^{\varepsilon_k,+} \underset{k\to\infty}{\longrightarrow} \beta \quad \text{or} \quad \varepsilon_k\lambda_{j(k)}^{\varepsilon_k,-} \underset{k\to\infty}{\longrightarrow} \beta.$$

Then β *is an eigenvalue of the limit problem* (32)*. For any* j*, perhaps along a subsequence,*

$$\varepsilon\lambda_j^{\varepsilon,\pm} \underset{\varepsilon\to 0}{\longrightarrow} \beta_j,$$

where β_j *is an eigenvalue of the Dirichlet problem* (32)*.*

Proof of Lemma 6. Note, first, that due to the inequality $K_j(\varepsilon) \geqq \varkappa_j$ and (55), the sequence $\{\varepsilon\lambda_j^{\varepsilon,\pm}\}$ is bounded for any j. Therefore, the second statement of Lemma follows from the first one.

Since the eigenpair $\{\lambda_{j(k)}^{\varepsilon_k,\pm}, u_{j(k)}^{\varepsilon_k,\pm}\}$ solves problem (3), integrating by parts yields

$$\left(u_{j(k)}^{\varepsilon_k,\pm}, \mathcal{L}^{\varepsilon_k} V - \lambda_{j(k)}^{\varepsilon_k,\pm} \rho^{\varepsilon_k} V\right)_{\Omega} = 0, \quad V \in H_0^1(\Omega). \tag{61}$$

In view of the normalization condition (9), up to a subsequence, $u_{j(k)}^{\varepsilon_k,\pm}$ converges weakly in $H_0^1(\Omega)$ to some function $\bar{u}^{\pm}$:

$$u_{j(k)}^{\varepsilon_k,\pm} \to \bar{u}^{\pm} \text{ weakly in } H_0^1(\Omega), \;\; \varepsilon_k \to 0. \tag{62}$$

In order to show that β is an eigenvalue of problem (32), for any $v \in C_0^\infty(\Omega)$ we substitute into (61) a test function in the form

$$V^\varepsilon(x) \equiv v(x) + \varepsilon\, N\left(\frac{x}{\varepsilon}\right)^{\mathrm{T}} \nabla_x v(x) + \varepsilon^2\, \lambda_j^{\varepsilon,\pm}\, N^0\left(\frac{x}{\varepsilon}\right) v(x)$$

where N and N^0 solve problems (24) and (25), respectively. Let us calculate the expression $\mathcal{L}^\varepsilon V^\varepsilon - \lambda_j^{\varepsilon,\pm}\rho^\varepsilon V^\varepsilon$:

$$\begin{aligned}
\mathcal{L}^\varepsilon V^\varepsilon(x) - \lambda_j^{\varepsilon,\pm}\rho^\varepsilon(x) V^\varepsilon(x) = \Big\{ & -\mathrm{div}_x\, (a(y)\nabla_x v(x)) - \mathrm{div}_x\, \big(a(y)\nabla_y N(y)\nabla_x v(x)\big) \\
& -\varepsilon\, \lambda_j^{\varepsilon,\pm}\, \mathrm{div}_x \left(a(y)\nabla_y N^0(y)\, v(x)\right) - \varepsilon\, \lambda_j^{\varepsilon,\pm}\, \rho(y)\, N(y)^{\mathrm{T}}\, \nabla_x v(x) \\
& -(\varepsilon\, \lambda_j^{\varepsilon,\pm})^2 \rho(y)\, N^0(y)\, v(x) \Big\} \Big|_{y=x/\varepsilon} \\
& -\varepsilon\, [\mathrm{div}_x + \varepsilon^{-1}\mathrm{div}_y] \left(a(y)\nabla_x (N(y)^{\mathrm{T}}\, \nabla_x v(x))\right) \Big|_{y=x/\varepsilon} \\
& -\varepsilon^2 \lambda_j^{\varepsilon,\pm}\;\; [\mathrm{div}_x + \varepsilon^{-1}\mathrm{div}_y] \left(a(y)\, N^0(y)\, \nabla_x v(x))\right) \Big|_{y=x/\varepsilon} \\
\equiv\; & I_1^\varepsilon(x,y)\big|_{y=x/\varepsilon} + I_2^\varepsilon(x) + I_3^\varepsilon(x).
\end{aligned}$$

Recalling the definition of a^{hom} and κ, one sees that the mean value of the expression in braces takes the form

$$\int_Y I_1^\varepsilon(x,y)\, \mathrm{d}y = -\mathrm{div}(a^{\mathrm{hom}} \nabla v) - \left(\varepsilon\, \lambda_j^{\varepsilon,\pm}\right)^2 \kappa^2\, v(x).$$

In view of (62), we have

$$\left(u_{j(k)}^{\varepsilon_k,\pm}, \int_Y I_1^\varepsilon(\cdot, y)\, \mathrm{d}y\right)_{\Omega} \to \left(\bar{u}^{\pm}, -\mathrm{div}(a^{\mathrm{hom}} \nabla v) - \beta^2\kappa^2\, v(x)\right)_{\Omega}, \quad \varepsilon_k \to 0.$$

Considering (62) and the smoothness of v, Lemma 4 provides that

$$\left(u_{j(k)}^{\varepsilon_k,\pm}, \mathcal{I}_1^\varepsilon - \int_Y I_1^\varepsilon(\cdot, y)\, \mathrm{d}y\right)_{\Omega} \to 0, \quad \varepsilon \to 0, \quad \mathcal{I}_1^\varepsilon(x) \equiv I_1^\varepsilon\left(x, \frac{x}{\varepsilon}\right).$$

Then, integrating by parts and using the boundedness of $a(y)$ and regularity properties of N and N^0, we estimate $(u^{\varepsilon_k,\pm}_{j(k)}, I^{\varepsilon}_2)_\Omega$ and $(u^{\varepsilon_k,\pm}_{j(k)}, I^{\varepsilon}_3)_\Omega$ as follows

$$\begin{aligned}\left|\left(u^{\varepsilon_k,\pm}_{j(k)}, I^{\varepsilon}_2\right)_\Omega\right| &= \left|\varepsilon_k \int_\Omega \nabla^{\mathrm{T}} u^{\varepsilon_k,\pm}_{j(k)}\, a^{\varepsilon_k}(x)\, \nabla_x (N(y)^{\mathrm{T}} \nabla_x v(x))\big|_{y=x/\varepsilon_k} \,\mathrm{d}x\right| \\ &\leqq C\,\varepsilon_k \|\nabla u^{\varepsilon_k,\pm}_{j(k)}\|_{L^2(\Omega)} \leqq C\varepsilon_k; \\ \left|\left(u^{\varepsilon_k,\pm}_{j(k)}, I^{\varepsilon}_3\right)_\Omega\right| &= \left|\varepsilon_k^2\, \lambda^{\varepsilon_k,\pm}_{j(k)} \int_\Omega \nabla^{\mathrm{T}} u^{\varepsilon_k,\pm}_{j(k)}\, a^{\varepsilon_k}(x)\, N^0\left(\frac{x}{\varepsilon_k}\right) \nabla_x v(x)\,\mathrm{d}x\right| \\ &\leqq C\,\varepsilon_k \|\nabla u^{\varepsilon_k,\pm}_{j(k)}\|_{L^2(\Omega)} \leqq C\varepsilon_k.\end{aligned}$$

In such a way passing to the limit in the integral identity (61) leads to the equality

$$\left(\bar{u}^{\pm}, \mathcal{L}^{\mathrm{hom}} v - (\beta)^2 \kappa^2 v\right)_\Omega = 0, \quad v \in C_0^\infty(\Omega).$$

Integrating by parts gives

$$\left(\mathcal{L}^{\mathrm{hom}} \bar{u}^{\pm} - (\beta)^2 \kappa^2 \bar{u}^{\pm}, v\right)_\Omega = 0, \quad v \in C_0^\infty(\Omega).$$

Since the space $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, the last equality holds for any $v \in H_0^1(\Omega)$, that means $\{\beta, \bar{u}^{\pm}\}$ to be an eigenpair of problem (32) if $\bar{u}^{\pm} \neq 0$. Let us assume that $u^{\varepsilon_k,\pm}_{j(k)}$ converges weakly in $H^1(\Omega)$ to $\bar{u}^{\pm} \equiv 0$, as $\varepsilon_k \to 0$. By the definition of the eigenpair $\{\lambda^{\varepsilon_k,\pm}_{j(k)}, u^{\varepsilon_k,\pm}_{j(k)}\}$ and normalization condition (9) we have

$$1 = a^{\varepsilon_k}\left(u^{\varepsilon_k,\pm}_{j(k)}, u^{\varepsilon_k,\pm}_{j(k)}\right) = \lambda^{\varepsilon_k,\pm}_{j(k)}\left(\rho^{\varepsilon_k} u^{\varepsilon_k,\pm}_{j(k)}, u^{\varepsilon_k,\pm}_{j(k)}\right)_\Omega.$$

In the same way as in the proof of Lemma 4 one shows that

$$\left(\rho^{\varepsilon_k} u^{\varepsilon_k,\pm}_{j(k)}, u^{\varepsilon_k,\pm}_{j(k)}\right)_\Omega \leqq C\varepsilon_k \left\|u^{\varepsilon_k,\pm}_{j(k)}\right\|_{L^2(\Omega)} \left\|\nabla u^{\varepsilon_k,\pm}_{j(k)}\right\|_{L^2(\Omega)}$$

Combining the last two relations and taking into account the estimate $|\varepsilon_k \lambda^{\pm}_{j(k)}| \leqq C$, we conclude that

$$\left\|u^{\varepsilon_k,\pm}_{j(k)}\right\|_{L^2(\Omega)} \geqq C > 0.$$

Thus, $\|\bar{u}^{\pm}\| \geqq C > 0$. This contradicts our assumption that $\bar{u}^{\pm} = 0$. Lemma 6 is proved. □

Assume that $K_J(\varepsilon_k) > \varkappa_j$ for some sequence $\varepsilon_k \to 0$. It means that there exist $c_j > 0$ and at least $\varkappa_j + 1$ eigenvalues $\lambda^{\varepsilon_k,\pm}_l$ of problem (2) such that

$$\left|\varepsilon_k \lambda^{\varepsilon_k,\pm}_l - \nu^{0,\pm}_j\right| \leqq c_j \varepsilon_k^{1/2}, \quad l = J_j(\varepsilon_k), \ldots, J_j(\varepsilon_k) + \varkappa_j.$$

Then by Lemma 6 the corresponding eigenfunctions $u^{\varepsilon_k,\pm}_l$ converge to eigenfunctions $\{\bar{u}^{\pm}_r\}_{r=1}^{\varkappa_j+1}$ of the Dirichlet problem (32):

$$\mathcal{L}^{\mathrm{hom}} \bar{u}^{\pm}_r = \left(\nu^{0,\pm}_j\right)^2 \kappa^2 \bar{u}^{\pm}_r, \quad r = 1, 2, \ldots, \varkappa_j + 1.$$

It is straightforward to check that the functions $\{\bar{u}_r^{\pm}\}_{r=1}^{\varkappa_j+1}$ are linearly independent. Therefore, the multiplicity of $\nu_j^{0,\pm}$ is greater than or equal to $\varkappa_j + 1$, which contradicts our assumption. Thus, $K_J(\varepsilon) = \varkappa_j$. Combining this relation with the fact that for each $j \in \mathbb{Z}^+$, any accumulating point of the sequence $\varepsilon\lambda_j^{\varepsilon,\pm}$, as $\varepsilon \to 0$, is an eigenvalue of the homogenized problem, we conclude that

$$\lim_{\varepsilon\to 0} \varepsilon\lambda_j^{\varepsilon,\pm} = \nu_j^{0,\pm}.$$

This completes the proof of the first statement of Theorem 3.

3. In order to prove the second statement in the theorem, we come back to bound (52) and apply the estimate in Lemma 2 with $\delta_1 = c_j\varepsilon$, c_j being a sufficiently small constant. In our case this estimate reads

$$\left\| \mathcal{U}_p^{\varepsilon,\pm} - \sum_{k\in S(j,\varepsilon)} \alpha_{kp}^{\varepsilon} u_k^{\varepsilon,\pm} \right\|_{\mathcal{H}} \leqq 2\frac{C\varepsilon^{3/2}}{\delta_1} \leqq C_j\varepsilon^{1/2}, \quad p = j, \dots, j + \varkappa_j - 1,$$

here $S(j,\varepsilon)$ is the set of eigenvalues $\mu_k^{\varepsilon,\pm}$ of the operator $\mathcal{K}^{\varepsilon}$ which satisfy the estimate

$$\left| \mu_k^{\varepsilon,\pm} - \varepsilon \left(\nu_j^{0,\pm}\right)^{-1} \right| \leqq c_j\, \varepsilon; \tag{63}$$

the constant matrix α^{ε} is such that

$$\left| \left(\alpha_{\cdot p}^{\varepsilon}\right)^{\mathrm{T}} \alpha_{\cdot q}^{\varepsilon} - \delta_{p,q} \right| \leqq C_j\, \varepsilon^{1/2}, \quad p, q = J(j), \dots, J(j) + \varkappa_j - 1, \tag{64}$$

From the first statement of the theorem we deduce that for sufficiently small $\varepsilon > 0$ the set $S(j,\varepsilon)$ coincides with the set $\{J(j), \dots, J(j) + \varkappa_j - 1\}$. Hence, we have

$$\left\| \mathcal{U}_p^{\varepsilon,\pm} - \sum_{k=J(j)}^{J(j)+\varkappa_j-1} \alpha_{kp}^{\varepsilon} u_k^{\varepsilon,\pm} \right\|_{\mathcal{H}} \leqq 2\frac{C\varepsilon^{3/2}}{\delta_1} \leqq C_j\varepsilon^{1/2}, \quad p = j, \dots, j+\varkappa_j-1, \tag{65}$$

with $\varkappa_j \times \varkappa_j$ matrix α^{ε} which meets (64). It remains to use the following simple statement.

Lemma 7. *For any $n \times n$ matrix A satisfying an equality*

$$\|A^{\mathrm{T}}A - \mathbb{I}; \mathbb{R}^n \to \mathbb{R}^n\| = \gamma \in (0, 1),$$

there exists a unitary matrix B such that

$$\|AB - \mathbb{I}; \mathbb{R}^n \to \mathbb{R}^n\| \leqq \gamma;$$

where $\mathbb{I}$ is a unit matrix and

$$\|D; \mathbb{R}^n \to \mathbb{R}^n\| = \sup_{\substack{\xi\in\mathbb{R}^n \\ \|\xi;\mathbb{R}^n\|=1}} \|D\xi; \mathbb{R}^n\|.$$

We omit the proof of this lemma which can be found in[12].

According to (64) and Lemma 7, there exists a unitary $\varkappa_j \times \varkappa_j$ matrix β^ε such that

$$\|\alpha^\varepsilon \beta^\varepsilon - \mathbb{I}; \mathbb{R}^{\varkappa_j} \to \mathbb{R}^{\varkappa_j}\| \leqq C\varepsilon^{1/2}. \tag{66}$$

If we denote by $\mathcal{U}_{\bar{J}}^{\varepsilon,\pm}$, $U_{\bar{J}}^{\varepsilon,\pm}$ and $u_{\bar{J}}^{\varepsilon,\pm}$ the vectors $(\mathcal{U}_{J(j)}^{\varepsilon,\pm}, \dots, \mathcal{U}_{J(j)+\varkappa_j-1}^{\varepsilon,\pm})^{\mathrm{T}}$, $(U_{J(j)}^{\varepsilon,\pm}, \dots, U_{J(j)+\varkappa_j-1}^{\varepsilon,\pm})^{\mathrm{T}}$ and $(u_{J(j)}^{\varepsilon,\pm}, \dots, u_{J(j)+\varkappa_j-1}^{\varepsilon,\pm})^{\mathrm{T}}$, respectively, then

$$\left\|u_{\bar{J}}^{\varepsilon,\pm} - \beta^\varepsilon U_{\bar{J}}^{\varepsilon,\pm}\right\|_{\mathcal{H}^{\varkappa_j}} \leqq 2\left\|\alpha^\varepsilon u_{\bar{J}}^{\varepsilon,\pm} - \alpha^\varepsilon\beta^\varepsilon U_{\bar{J}}^{\varepsilon,\pm}\right\|_{\mathcal{H}^{\varkappa_j}} \leqq \left\|\alpha^\varepsilon u_{\bar{J}}^{\varepsilon,\pm} - \mathcal{U}_{\bar{J}}^{\varepsilon,\pm}\right\|_{\mathcal{H}^{\varkappa_j}} + \left\|U_{\bar{J}}^{\varepsilon,\pm} - \mathcal{U}_{\bar{J}}^{\varepsilon,\pm}\right\|_{\mathcal{H}^{\varkappa_j}} + \left\|U_{\bar{J}}^{\varepsilon,\pm} - \alpha^\varepsilon\beta^\varepsilon U_{\bar{J}}^{\varepsilon,\pm}\right\|_{\mathcal{H}^{\varkappa_j}} \leqq C_j\varepsilon^{1/2};$$

here we have also used (50), (66) and (65). The last inequality implies that

$$\left\|u_p^{\varepsilon,\pm} - \sum_{m=j}^{j+\varkappa_j-1} \beta_{mp}^\varepsilon U_m^{\varepsilon,\pm}\right\|_{\mathcal{H}} \leqq C_j\varepsilon^{1/2}.$$

In order to replace here $U_m^{\varepsilon,\pm}$ given by formula (37) with $\widetilde{U}_m^{\varepsilon,\pm}$ defined by (58), we estimate the $H^1(\Omega)$ norm of the difference

$$\|U_m^{\varepsilon,\pm} - \widetilde{U}_m^{\varepsilon,\pm}\|_{H^1(\Omega)} = \varepsilon^2 \left\|(1-\chi_\varepsilon)\,(N^\varepsilon)^{\mathrm{T}}\,\nabla u_m^0\right\|_{H^1(\Omega)}$$

with $N^\varepsilon(x) = N^\varepsilon(x/\varepsilon)$. Considering the properties of χ_ε and $N(y)$, it is straightforward to check that

$$\|U_m^{\varepsilon,\pm} - \widetilde{U}_m^{\varepsilon,\pm}\|_{H^1(\Omega)} \leqq C\varepsilon^{1/2}, \tag{67}$$

which, in turn, results in (57).

Lemma 5 states that the functions $\{\mathcal{U}_p^{\varepsilon,\pm}\}_{p=j}^{j+\varkappa_j-1}$ corresponding to the same eigenvalue $\nu_j^{0,\pm}$ are almost orthonormal. Let u_q^0 be an eigenfunction of the limit problem (32) which corresponds to $\nu_m^{0,\pm}$. Using formula (23) we construct $\mathcal{U}_q^{\varepsilon,\pm} = \|U_q^{\varepsilon,\pm}\|_{\mathcal{H}}^{-1}\ U_q^{\varepsilon,\pm}$. By (52), we have

$$\mathcal{K}^\varepsilon\mathcal{U}_p^{\varepsilon,\pm} = \varepsilon(\nu_j^{\varepsilon,\pm})^{-1}\mathcal{U}_p^{\varepsilon,\pm} + \theta_p^\varepsilon(x), \quad \|\theta_p^\varepsilon\|_{\mathcal{H}} \leqq C\varepsilon^{3/2};$$
$$\mathcal{K}^\varepsilon\mathcal{U}_q^{\varepsilon,\pm} = \varepsilon(\nu_m^{\varepsilon,\pm})^{-1}\mathcal{U}_q^{\varepsilon,\pm} + \theta_q^\varepsilon(x), \quad \|\theta_q^\varepsilon\|_{\mathcal{H}} \leqq C\varepsilon^{3/2}.$$

Multiplying the last two relations by $\mathcal{U}_q^{\varepsilon,\pm}$ and $\mathcal{U}_p^{\varepsilon,\pm}$ in $\mathcal{H}$, respectively, and subtracting them from each other, we obtain

$$\left\langle\mathcal{U}_p^{\varepsilon,\pm}, \mathcal{U}_q^{\varepsilon,\pm}\right\rangle = \varepsilon^{-1}\,\frac{\nu_j^{0,\pm}\,\nu_m^{0,\pm}}{\nu_j^{0,\pm} - \nu_m^{0,\pm}}\left[\left\langle\theta_q^\varepsilon, \mathcal{U}_p^{\varepsilon,\pm}\right\rangle - \left\langle\theta_p^\varepsilon, \mathcal{U}_q^{\varepsilon,\pm}\right\rangle\right] \leqq C\varepsilon^{1/2}. \tag{68}$$

This completes the proof of Theorem 3. □

From Theorem 3 we obtain the following convergence result.

Corollary 2. *For the sequences of eigenvalues* (8) *and* (34) *the following convergence result holds:*

$$\varepsilon\lambda_j^{\varepsilon,\pm} \to \nu_j^{0,\pm}, \quad \varepsilon \to 0.$$

Moreover, if $\nu_j^{0,\pm}$ *is a simple eigenvalue, then* $\lambda_j^{\varepsilon,\pm}$ *is also simple, for a small* ε, *and the corresponding eigenfunctions satisfy the relations:*

- $u_j^{\varepsilon,\pm} \underset{\varepsilon\to 0}{\longrightarrow} u_j^0$ *strongly in* $L^2(\Omega)$;
- $u_j^{\varepsilon,\pm} - \varepsilon N\left(\frac{x}{\varepsilon}\right)^{\mathrm{T}} \nabla u_j^0 - \varepsilon\, \nu_j^{0,\pm} N^0\left(\frac{x}{\varepsilon}\right) u_j^0 \underset{\varepsilon\to 0}{\longrightarrow} u_j^0$ *strongly in* $H^1(\Omega)$;
- $a^\varepsilon\ \nabla u_j^{\varepsilon,\pm} \underset{\varepsilon\to 0}{\rightharpoonup} a^{\mathrm{hom}}\,\nabla u_j^0 + \nu_j^{0,\pm}\,\langle a\,\nabla N^0\rangle\, u_j^0$ *weakly in* $L^2(\Omega)$,

where $\langle\cdot\rangle$ *denotes the mean value over* Y.

Proof. All the statements, except for the last one, are immediate consequences of Theorem 3. In order to prove the convergence of fluxes, we estimate, for an arbitrary function $v \in C_0^\infty(\Omega)$, the following expression

$$\begin{aligned}
&\left|\left(a^\varepsilon \nabla u_j^{\varepsilon,\pm} - a^{\mathrm{hom}}\,\nabla u_j^0 - \nu_j^{0,\pm}\,\langle a\,\nabla N^0\rangle\, u_j^0\ ,\ v\right)_\Omega\right| \\
&\leqq C\ \left\|\nabla(u_j^{\varepsilon,\pm} - \widetilde{U}_j^{\varepsilon,\pm})\right\|_{L^2(\Omega)}\ \|v\|_{L^2(\Omega)} \\
&\quad + \left|\left(a^\varepsilon\ \nabla\widetilde{U}_j^{\varepsilon,\pm} - a^{\mathrm{hom}}\,\nabla u_j^0 - \nu_j^{0,\pm}\,\langle a\,\nabla N^0\rangle\, u_j^0, v\right)_\Omega\right|.
\end{aligned}$$

By Theorem 3,

$$\left\|\nabla(u_j^{\varepsilon,\pm} - \widetilde{U}_j^{\varepsilon,\pm})\right\|_{L^2(\Omega)} \leqq C_j \varepsilon^{1/2}.$$

A straightforward calculation gives

$$\begin{aligned}
&a(y)\,\nabla\widetilde{U}_j^{\varepsilon,\pm}(x) - a^{\mathrm{hom}}\,\nabla u_j^0(x) - \nu_j^{0,\pm}\,\langle a\,\nabla N^0\rangle\, u_j^0(x) \\
&= \left\{(a(y) + a(y)\nabla_y N(y)) - a^{\mathrm{hom}}\right\}\Big|_{y=x/\varepsilon}\ \nabla_x u_j^0(x) \\
&\quad + \nu_j^{0,\pm}\ \left\{a(y)\nabla_y N^0(y) - \langle a\,\nabla N^0\rangle\right\}\Big|_{y=x/\varepsilon}\ u_j^0(x) \\
&\quad + \varepsilon\, a(y)\,\nabla_x(N(y)^{\mathrm{T}}\,\nabla_x u_j^0(x))\Big|_{y=x/\varepsilon} + \varepsilon\,\nu_j^{0,\pm}\,a(y)\,N^0(y)\,\nabla_x u_j^0(x)\Big|_{y=x/\varepsilon}.
\end{aligned}$$

The first two items on the right-hand side have zero mean, thus, by Lemma 4

$$\begin{aligned}
&\left|\int_\Omega \left\{(a(y) + a(y)\nabla_y N(y)) - a^{\mathrm{hom}}\right\}\Big|_{y=x/\varepsilon}\ v(x)\,\nabla u_j^0(x)\ \mathrm{d}x\right| \leqq C\varepsilon; \\
&\left|\int_\Omega \left\{a(y)\nabla_y N^0(y) - \langle a\,\nabla N^0\rangle\right\}\Big|_{y=x/\varepsilon}\ u_j^0(x)\ v(x)\,\mathrm{d}x\right| \leqq C\varepsilon;
\end{aligned}$$

Finally, using the boundedness of $a_{ij}(y)$, properties of $N^0(y)$ and the smoothness of $u_j^0(x)$ we have

$$\varepsilon \left| \int_\Omega a(y)\, \nabla_x (N(y)^{\mathrm{T}}\, \nabla_x u_j^0(x)) \Big|_{y=x/\varepsilon} v(x)\,\mathrm{d}x \right| \leqq C\varepsilon;$$

$$\varepsilon\, \nu_j^{0,\pm} \left| \int_\Omega a(y)\, N^0(y) \Big|_{y=x/\varepsilon} \nabla_x u_j^0(x)\; v(x)\,\mathrm{d}x \right| \leqq C\varepsilon.$$

Summing up the obtained estimates, we arrive at the last statement in the corollary. □

5. Negative part of the spectrum in the case $\langle\rho\rangle > 0$

In this section we assume that $\langle\rho\rangle > 0$ and deal with the negative part of the spectrum of spectral problem (2) or the much more general spectral problem

$$\begin{cases} \mathcal{L}\left(\dfrac{x}{\varepsilon}, \nabla_x\right) u(x) = \lambda\rho\left(\dfrac{x}{\varepsilon}\right) u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{cases} \tag{69}$$

for a formally self-adjoint differential (non-necessary scalar) operator of the form

$$\mathcal{L}(y, \nabla_y) = \overline{\mathcal{D}(-\nabla_y)}^{\mathrm{T}} \mathcal{A}(y)\mathcal{D}(\nabla_y)$$

with $[0,1]^d$-periodic complex coefficients, the overline symbol stands for the complex conjugation. As above we denote $Y = [0,1]^d$.

In the case of the scalar elliptic operator with real coefficients we make the same assumptions on the coefficients as those in Section 2.

In the more general case of problem (69), we assume that

1. $\mathcal{D}(\nabla_y)$ is an $N \times K$ matrix of homogeneous first-order differential operators with constant coefficients;
2. $\mathcal{A}(y)$ is a positive definite Hermitian $N \times N$ matrix; the elements of $\mathcal{A}$ and their partial derivatives are Hölder continuous $[0,1]^d$-periodic functions. The function ρ is Hölder continuous.
3. There exists a positive $n_0 \in \mathbb{Z}$ such that for any row $(P_1(\xi), \ldots, P_K(\xi))$ of homogeneous polynomials of degree $n > n_0$ with $\xi = (\xi_1, \ldots, \xi_d)$, there is a row $(Q_1(\xi), \ldots, Q_N(\xi))$ of homogeneous polynomials such that $P(\xi) = Q(\xi)\mathcal{D}(\xi)$. This property is called the algebraic completeness of $\mathcal{D}(\xi)$ (see [16]).

Under these assumptions, the operator $\mathcal{L}$ is formally positive and elliptic. Moreover, it possesses the so-called polynomial property (see [13,14]):

There is a finite dimensional subspace $\mathcal{P}$ of vector-valued polynomials such that, for any bounded domain $\Omega \subset \mathbb{R}^d$, the relation

$$(\mathcal{A}\mathcal{D}(\nabla)u, \mathcal{D}(\nabla)u)_{(L^2(\Omega))^K} = 0, \quad u \in C^1(\bar{\Omega}),$$

holds if and only if $u \in \mathcal{P}|_\Omega$.

Lemma 8. *Under assumptions* 1. *and* 3. *for any* $v \in H_0^1(\Omega)^K$ *the inequality holds true*

$$\|v\|_{H_0^1(\Omega)^K} \leqq C\|\mathcal{D}(\nabla_x)v\|_{L^2(\Omega)^N} \tag{70}$$

with $C = C(\mathcal{D}, \Omega)$.

Proof. We set $v = 0$ for $x \in \mathbb{R}^d \setminus \Omega$ and keep for the extended function the same notation v, then $v \in H^1(\mathbb{R}^d)$. Applying Fourier transform to $\mathcal{D}(\nabla_x)v$ we obtain

$$\|\mathcal{D}(\nabla_x)v\|^2_{L^2(\mathbb{R}^d)^N} = \|\mathcal{D}(\xi)\widehat{v}(\xi)\|^2_{L^2(\mathbb{R}^d)^N}. \tag{71}$$

Let us show that for some constant $C = C(\mathcal{D}) > 0$

$$|\mathcal{D}(\xi)\eta|^2 \geqq C|\xi|^2\,|\eta^2|, \quad \xi \in \mathbb{R}^d,\ \eta \in \mathbb{R}^K. \tag{72}$$

If we assume that (72) fails to hold, then there exist $\check{\xi} \neq 0$ and $\check{\eta} \neq 0$ such that $\mathcal{D}(\check{\xi})\check{\eta} = 0$. Letting

$$P(\xi) = \check{\eta}^t|\xi|^{2n_0}$$

and using property 3, we conclude that there is a vector of homogeneous polynomials $Q(\xi) = (Q_1(\xi), \dots, Q_N(\xi))$ such that $\check{\eta}^t|\xi|^{2n_0} = Q(\xi)\mathcal{D}(\xi)$. Taking the inner product with η and substituting $\eta = \check{\eta}$, we obtain the equality

$$|\check{\eta}|^2|\check{\xi}|^{2n_0} = Q(\check{\xi})\mathcal{D}(\check{\xi})\check{\eta} = 0,$$

which contradicts our assumption that $\check{\xi} \neq 0$ and $\check{\eta} \neq 0$.

Substituting (72) in (71) we derive (70). □

It follows from Lemma 8 and properties of $\mathcal{A}$ that $(\mathcal{A}\mathcal{D}(\nabla)u, \mathcal{D}(\nabla)v)_{(L^2(\Omega))^K}$ forms a scalar product in $H_0^1(\Omega)^K$. Following the line of the proof of Theorem 1, one can easily justify the following assertion (see also [11]).

Lemma 9. *Under Conditions 1–3, the spectrum of problem* (69) *is discrete and consists of the positive and negative sequences:*

$$0 < \lambda_1^{\varepsilon,+} \leqq \lambda_2^{\varepsilon,+} \leqq \cdots \leqq \lambda_j^{\varepsilon,+} \leqq \cdots \to +\infty,$$
$$0 > \lambda_1^{\varepsilon,-} \geqq \lambda_2^{\varepsilon,-} \geqq \cdots \geqq \lambda_j^{\varepsilon,-} \geqq \cdots \to -\infty.$$

In the case under consideration, the negative part of the spectra of problems (1) and (69) shows rather singular behaviour, as $\varepsilon \to 0$. The analysis of eigenpairs with negative eigenvalues will rely on the Floquet-Bloch decomposition. For the reader's convenience we recall briefly the results on the Bloch spectrum.

Given an operator of the form

$$\mathcal{L}_\mu(y, \nabla_y) = \mathcal{L}(y, \nabla_y) + \mu\,\rho(y) = \overline{\mathcal{D}(-\nabla_y)}^{\mathrm{T}}\mathcal{A}(y)\mathcal{D}(\nabla_y) + \mu\,\rho(y)$$

with a periodic $\rho \in L^{\infty}(Y)$, $\mu \in \mathbb{R}$, we introduce the operator

$$\mathcal{L}^{\eta}_{\mu}(y, \nabla_y) = \mathrm{e}^{-i2\pi y\cdot\eta}\, \mathcal{L}_{\mu}(y, \nabla_y)\, \mathrm{e}^{i2\pi y\cdot\eta}$$
$$= \overline{\mathcal{D}(-(\nabla_y + i2\pi\eta))}^{\mathrm{T}}\, \mathcal{A}(y)\, \mathcal{D}(\nabla_y + i2\pi\eta) + \mu\, \rho(y),$$

and consider a family of spectral problems parameterized by $\eta \in \mathbb{R}^d$: find $\nu(\eta, \mu) \in \mathbb{R}$ and $\theta(y, \eta; \mu)$ (not identically zero) such that

$$\begin{cases} \mathcal{L}^{\eta}_{\mu}(y, \nabla_y)\, \theta(y, \eta; \mu) = \nu(\eta, \mu)\, \theta(y, \eta; \mu), \quad y \in Y, \\ \theta(y, \eta; \mu) \text{ is } Y-\text{periodic}. \end{cases}$$

It is well-known (see, for example [3]) that under the above conditions 1–3, for each $\eta \in \mathbb{R}^d$, the operator $\mathcal{L}^{\eta}_{\mu}(y, \nabla_y)$ has a discrete spectrum

$$\nu_1(\eta, \mu) \leqq \nu_2(\eta, \mu) \leqq \cdots \leqq \nu_j(\eta, \mu) \underset{j\to\infty}{\longrightarrow} +\infty.$$

in $L^2_{\#}(Y)^K$ (or, equivalently, in $H^1_{\#}(Y)^K$). The corresponding eigenfunctions $\theta_j(\cdot, \eta; \mu)$ form an orthonormal basis in $L^2_{\#}(Y)^K$. Moreover, $\nu_j(\eta, \mu)$ are Y- periodic and continuous for all j, and the corresponding normalized eigenfunctions $\theta_j(y, \eta; \mu)$, $\|\theta_j(\cdot, \eta; \mu)\|_{L^2(Y)} = 1$, are measurable in η. In [19] it was shown that $\nu_j(\eta, \mu)$ and $\theta_j(\cdot, \eta; \mu)$ are analytic functions η everywhere except for the subset of measure zero, where $\nu_j(\eta; \mu)$ changes its multiplicity. The family $\{\nu_j(\xi), \theta_j(y, \xi)\}$ is called the Bloch spectrum of operator $\mathcal{L}$.

Any function $g \in L^2(\mathbb{R}^d)^K$ admits the representation (see [3,7])

$$g(y) = \int_{Y^*} \sum_{j=1}^{\infty} \hat{g}_j(\eta)\, \theta_j(x, \eta; \mu)\, \mathrm{e}^{i2\pi y\cdot\eta}\, \mathrm{d}\eta$$

with

$$\hat{g}_j(\eta) = \int_{\mathbb{R}^d} g(y)^{\mathrm{T}}\, \theta_j(y, \eta; \mu)\, \mathrm{e}^{-i2\pi y\cdot\eta}\, \mathrm{d}y$$

and $Y^* = [0, 1]^d$. Furthermore,

$$\int_{\mathbb{R}^d} f(x)^{\mathrm{T}}\, \overline{g(y)}\, \mathrm{d}y = \int_{Y^*} \sum_{j=1}^{\infty} \hat{f}_j(\eta)\, \overline{\hat{g}_j(\eta)}\, \mathrm{d}\eta, \quad f,\ g \in L^2(\mathbb{R}^d)^K. \tag{73}$$

We will exploit a rescaled version of the above formulae. Given $g \in L^2(\mathbb{R}^d)^K$, we define jth Bloch coefficient of g by

$$\hat{g}^{\varepsilon}_j(\xi) = \varepsilon^{-d/2} \int_{\mathbb{R}^d} g(x)^{\mathrm{T}}\, \theta_j\left(\frac{x}{\varepsilon}, \varepsilon\xi; \mu\right)\, \mathrm{e}^{-i2\pi x\cdot\xi}\, \mathrm{d}x.$$

Then

$$g(x) = \varepsilon^{d/2} \int_{\varepsilon^{-1}Y^*} \sum_{j=1}^{\infty} \hat{g}^{\varepsilon}_j(\xi)\, \theta_j\left(\frac{x}{\varepsilon}, \varepsilon\xi; \mu\right)\, \mathrm{e}^{i2\pi x\cdot\xi}\, \mathrm{d}\xi, \tag{74}$$

where (x, ξ) and (y, η) are related by

$$y = \frac{x}{\varepsilon}, \quad \eta = \varepsilon\,\xi.$$

In the rescaled coordinates, formula (73) takes the form

$$\varepsilon^{-d} \int_{\mathbb{R}^d} f(x)^{\mathrm{T}}\, \overline{g(x)}\, \mathrm{d}x = \int_{\varepsilon^{-1}Y^*} \sum_{j=1}^{\infty} \hat{f}_j^{\varepsilon}(\xi)\, \overline{\hat{g}_j^{\varepsilon}(\xi)}\, \mathrm{d}\xi.$$

In the sequel it is convenient to extend the elements of the functional spaces $L^2(\Omega)^K$ and $H_0^1(\Omega)^K$ by letting them equal zero in $\mathbb{R}^d \setminus \Omega$. We keep the notation for the extended functions.

One can see that, for any $f,\ g \in H_0^1(\Omega)^K$, the following equality holds true:

$$\left(\mathcal{L}_\mu\left(\frac{x}{\varepsilon}, \nabla_x\right) f, g\right)_{L^2(\Omega)^K} = \varepsilon^d \int_{\varepsilon^{-1}Y^*} \sum_{j=1}^{\infty} \frac{\nu_j(\varepsilon\xi, \mu)}{\varepsilon^2}\, \hat{f}_j^{\varepsilon}(\xi)\, \overline{\hat{g}_j^{\varepsilon}(\xi)}\, \mathrm{d}\xi, \quad (75)$$

where

$$\begin{aligned}\mathcal{L}_\mu\left(\frac{x}{\varepsilon}, \nabla_x\right) f(x) &= \varepsilon^{-2}\, \mathcal{L}_\mu(y, \nabla_y)\Big|_{y=x/\varepsilon} f(x)\\ &= \overline{\mathcal{D}(-\nabla_x)}^{\mathrm{T}} \mathcal{A}\left(\frac{x}{\varepsilon}\right) \mathcal{D}(\nabla_x)\, f(x) + \frac{\mu}{\varepsilon^2}\, \rho\left(\frac{x}{\varepsilon}\right) f(x).\end{aligned}$$

Indeed, making use of (74) and taking into account the linearity of $\mathcal{L}_\mu$ and definition of $\theta_j(\cdot, \eta; \mu)$, we have

$$\begin{aligned}\mathcal{L}_\mu\left(\frac{x}{\varepsilon}, \nabla_x\right) f(x) &= \varepsilon^{d/2} \int_{\varepsilon^{-1}Y^*} \sum_{j=1}^{\infty} \hat{f}_j^{\varepsilon}(\xi)\, \mathcal{L}_\mu\left(\frac{x}{\varepsilon}, \nabla_x\right) \left[\theta_j\left(\frac{x}{\varepsilon}, \varepsilon\xi; \mu\right) \mathrm{e}^{i2\pi x\cdot\xi}\right] \mathrm{d}\xi\\ &= \varepsilon^{d/2} \int_{\varepsilon^{-1}Y^*} \sum_{j=1}^{\infty} \hat{f}_j^{\varepsilon}(\xi)\, \mathrm{e}^{i2\pi x\cdot\xi}\, \frac{\nu_j(\varepsilon\xi, \mu)}{\varepsilon^2}\, \theta_j\left(\frac{x}{\varepsilon}, \varepsilon\xi; \mu\right) \mathrm{d}\xi.\end{aligned}$$

Here we have used the relation

$$\begin{aligned}&\mathcal{L}_\mu^{\varepsilon\xi}\left(\frac{x}{\varepsilon}, \nabla_x\right) \theta_j\left(\frac{x}{\varepsilon}, \varepsilon\xi; \mu\right)\\ &\equiv \mathrm{e}^{-i2\pi x\cdot\xi}\, \overline{\mathcal{D}(-\nabla_x)}^{\mathrm{T}} \mathcal{A}\left(\frac{x}{\varepsilon}\right) \mathcal{D}(\nabla_x)\, \mathrm{e}^{i2\pi x\cdot\xi}\, \theta_j\left(\frac{x}{\varepsilon}, \varepsilon\xi; \mu\right)\\ &\quad + \frac{\mu}{\varepsilon^2}\, \rho\left(\frac{x}{\varepsilon}\right) \theta_j\left(\frac{x}{\varepsilon}, \varepsilon\xi; \mu\right)\\ &= \frac{\nu_j(\varepsilon\xi, \mu)}{\varepsilon^2}\, \theta_j\left(\frac{x}{\varepsilon}, \varepsilon\xi; \mu\right).\end{aligned} \quad (76)$$

Finally, recalling the normalization condition for $\theta_j(\cdot, \eta; \mu)$, we obtain (75).

Consider now the auxiliary spectral problem

$$\mathcal{L}(y, \nabla_y) v(y) \equiv \overline{\mathcal{D}(-\nabla_y)}^{\mathrm{T}} \mathcal{A}(y) \mathcal{D}(\nabla_y)\, v(y) = \mu\, \rho(y)\, v(y), \quad (77)$$

with $v \in H_\#^1(Y)^K$ (or, equivalently, $v \in L_\#^2(Y)^K$).

Theorem 4. *Let $\langle\rho\rangle \neq 0$. Then the spectrum of* (77) *consists of zero and two infinite sequences*

$$0 < \mu_1^+ \leqq \mu_2^+ \leqq \cdots \leqq \mu_j^+ \leqq \ldots \underset{j\to\infty}{\longrightarrow} +\infty,$$

$$0 > \mu_1^- \geqq \mu_2^- \leqq \cdots \geqq \mu_j^- \geqq \cdots \underset{j\to\infty}{\longrightarrow} -\infty.$$

Proof. Without loss of generality, we assume that $\langle\rho\rangle = 1$. Consider an operator

$$\mathcal{L}_\alpha(y, \nabla_y) = \mathcal{L}(y, \nabla_y) + \alpha\,\rho(y) = \overline{\mathcal{D}(-\nabla_y)}^{\mathrm{T}}\mathcal{A}(y)\mathcal{D}(\nabla_y) + \alpha\,\rho(y)$$

Lemma 10. *There exists $\alpha_0 > 0$ such that for any $\alpha \in (0, \alpha_0)$ and for all $u \in H^1_\#(Y)^K$ the inequality holds*

$$(\mathcal{L}_\alpha(y, \nabla_y)u, u) \geqq \gamma(\alpha)\|u\|^2_{L^2(Y)^K}, \quad \gamma(\alpha) > 0. \tag{78}$$

Proof. Denote $\langle u\rangle = \int_Y u(y)\,\mathrm{d}y$. Then $u = \langle u\rangle + \tilde{u}$ with $\tilde{u} = u - \langle u\rangle$. By [16] we have

$$\|\nabla u\|_{L^2(Y)^{Kd}} \leqq c(\|\mathcal{D}(\nabla_y)u\|_{L^2(Y)^N} + \|u\|_{L^2(Y)^K}).$$

This implies

$$\|\nabla u\|^2_{L^2(Y)^{Kd}} \leqq c\left((\mathcal{L}(y, \nabla_y)u, u)_{L^2(Y)^K} + \|u\|^2_{L^2(Y)^K}\right). \tag{79}$$

Let us show that for any $\tilde{u} \in H^1_\#(Y)^K$ such that $\langle\tilde{u}\rangle = 0$ the bound holds

$$\|\tilde{u}\|^2_{H^1(Y)^K} \leqq c_1\|\mathcal{D}(\nabla_y)\tilde{u}\|^2_{L^2(Y)^K}. \tag{80}$$

To this end, it suffices to develop $\tilde{u}$ in Fourier series $\tilde{u} = \sum\limits_{k\in\mathbb{Z}\setminus\{0\}} \eta_k \exp(2\pi k y)$. Then the bound (80) can be obtained in exactly the same way as estimate (70). We leave the details to the reader.

The estimate (80) implies the inequality

$$\|\tilde{u}\|^2_{H^1(Y)^K} \leqq c_1\left(\mathcal{L}(y, \nabla_y)\tilde{u}, \tilde{u}\right)_{L^2(Y)^K}. \tag{81}$$

Considering the fact that $\rho \in L^\infty_\#(Y)$, we deduce

$$\begin{aligned}\alpha(\rho u, u)_{L^2(Y)^K} &= \alpha|\langle u\rangle|^2 + 2\alpha(\langle u\rangle, \rho\tilde{u})_{L^2(Y)^K} + \alpha(\rho\tilde{u}, \tilde{u})_{L^2(Y)^K}\\ &\geqq \alpha|\langle u\rangle|^2 - \frac{\alpha}{2}|\langle u\rangle|^2 - 2\alpha\|\rho\|^2_{L^\infty(Y)}\|\tilde{u}\|^2_{L^2(Y)^K} - \alpha\|\rho\|_{L^\infty(Y)}\|\tilde{u}\|^2_{L^2(Y)^K}.\end{aligned}$$

Summing up (81) and the last inequality, we get

$$\left(\mathcal{L}_\alpha(y, \nabla_y)u, u\right)_{L^2(Y)^K} \geqq ((c_1)^{-1} - 2\alpha\|\rho\|^2_{L^\infty(Y)} - \alpha\|\rho\|_{L^\infty(Y)})\|\tilde{u}\|^2_{L^2(Y)^K} + \frac{\alpha}{2}|\langle u\rangle|^2.$$

Choosing $\alpha_0 > 0$ to satisfy the inequality

$$(c_1)^{-1} - 2\alpha_0\|\rho\|^2_{L^\infty(Y)} - \alpha_0\|\rho\|_{L^\infty(Y)} \geqq \frac{\alpha_0}{2},$$

we obtain (78) for all $\alpha \in (0, \alpha_0)$ with $\gamma(\alpha) = \alpha/2$. This completes the proof of lemma. □

Applying exactly the same arguments as those used in the proof of Lemma 1, we conclude that the spectrum of problem

$$(\mathcal{L}_{\alpha_0}(y,\nabla_y)u(y) = \tilde{\mu}\rho(y)u(y)$$

is discrete and consists of two series

$$0 < \tilde{\mu}_1^+ \leqq \tilde{\mu}_2^+ \leqq \cdots \leqq \tilde{\mu}_j^+ \leqq \cdots \underset{j\to\infty}{\longrightarrow} +\infty,$$
$$0 > \tilde{\mu}_1^- \geqq \tilde{\mu}_2^- \leqq \cdots \geqq \tilde{\mu}_j^- \geqq \cdots \underset{j\to\infty}{\longrightarrow} -\infty.$$

Then the spectrum of (77) is also discrete and consists of $(\tilde{\mu}_j^\pm - \alpha_0)$. Clearly, 0 is an eigenvalue of (77), and the desired statement follows. □

Using variational arguments, one can also show that

$$-\mu_1^- = \inf\{(\mathcal{A}\mathcal{D}(\nabla_y)v, \mathcal{D}(\nabla_y)v)_{L^2(Y)^K} \;:\; v \in H^1_\#(Y)^K,\; (\rho v, v)_{L^2(Y)^K} = -1\}. \tag{82}$$

In particular, $-\mu_1^-$ can be determined as the maximum of all positive numbers $\mu > 0$ such that the quadratic form

$$\mathcal{E}_\mu(v,v) = (\mathcal{A}\mathcal{D}(\nabla_y)v, \mathcal{D}(\nabla_y)v)_{L^2(Y)^K} + \mu(\rho v, v)_{L^2(Y)^K} \tag{83}$$

is non-negative:

$$-\mu_1^- = \sup\{\mu > 0 \;:\; \mathcal{E}_\mu(v,v) \geqq 0 \text{ for all } v \in H^1_\#(Y)^K\}. \tag{84}$$

The statement below is an immediate consequence of the polynomial property of operator $\mathcal{D}$ and the fact that $\nu_j(\eta,\mu)$ depends continuously on μ and η.

Proposition 1. *There is $\mu_0 > 0$ such that $\nu_1(\eta,\mu) > 0$ for all $\mu \in (0,\mu_0)$ and $\eta \in Y^*$.*

Proof. Assume that in any neighbourhood of zero there exists a point μ such that $\nu_1(\eta,\mu) \leqq 0$ for some η. In other words, assume that there exist a sequence $\mu_k \to 0$, as $k \to \infty$, and $\eta_k \in Y^*$ such that $\nu_1(\eta_k,\mu_k) \leqq 0$. Or, equivalently,

$$\left(\mathcal{L}_{\mu_k}^{\eta_k}\theta_1(\cdot,\eta_k;\mu_k), \theta_1(\cdot,\eta_k;\mu_k)\right)_{L^2(Y)^K} \leqq 0.$$

Without loss of generality, we assume that $\eta_k \to \eta_\infty$, as $k \to \infty$. Making use of the last inequality, due to the normalization condition, we have

$$\|\theta_1(\cdot,\eta_k;\mu_k)\|_{H^1(Y)^K} \leqq C,$$

where the constant C does not depend on k. Thus, up to a subsequence, $\theta_1(\cdot,\eta_k;\mu_k)$ converges weakly in $H^1(Y)$ to a function θ_∞, as $k \to \infty$, such that

$$\left(\mathcal{L}_0^{\eta_\infty}\theta_\infty, \theta_\infty\right)_{L^2(Y)^K} \leqq 0.$$

Taking into account the polynomial property and the periodicity of the vector θ_∞, one can see that θ_∞ is a constant vector such that $|\theta_\infty|^2 = 1$, and $\eta_\infty = 0$. Thus,

$|\theta_1(\cdot, \eta_k; \mu_k)| = 1 + \gamma_k$, where $\|\gamma_k\|_{L^2(Y)} \to 0$ as $k \to \infty$. Using this representation one readily gets

$$\left(\mathcal{L}_{\mu_k}^{\eta_k} \theta_1(\cdot, \eta_k; \mu_k), \theta_1(\cdot, \eta_k; \mu_k)\right)_{L^2(Y)} > 0.$$

We have arrived at contradiction. The proposition is proved. □

The last statement allows us to define the following quantity:

$$\bar{\mu} = \sup\{\mu > 0 \;:\; \nu_1(\eta, \mu) > 0 \text{ for all } \eta \in Y^*\}. \tag{85}$$

Proposition 2. *The constant $\bar{\mu}$ defined by (85) possesses the following properties:*

1. $\bar{\mu} < \infty$;
2. *There exists a point $\eta_0 \in Y^*$ such that $\nu_1(\eta_0, \bar{\mu}) = 0$. Moreover, η_0 is a minimal point (strict or not) of the function $\eta \mapsto \nu_1(\eta, \bar{\mu})$.*
3. $\nu_1(\eta_0, \bar{\mu} + \delta) < 0$ *for any* $\delta > 0$.

Proof. 1. It is sufficient to show that there exists $\mu_0 < \infty$ such that $\nu_1(\eta, \mu) < 0$ for all $\eta \in Y^*$ and $\mu > \mu_0$. By the minimum principle, we have

$$\nu_1(\eta, \mu) = \inf_{\substack{v \in H^1_\#(Y)^K \\ \|v\|_{L^2(Y)^K} = 1}} \left(\mathcal{L}_\mu^\eta v, v\right)_{L^2(Y)^K}.$$

Consider the eigenfunction $v_1^-(y)$ corresponding to the eigenvalue μ_1^- of the operator $\mathcal{L}(y, \nabla_y)$. Then $\int_Y \rho(y)\,|v_1^-(y)|^2\,\mathrm{d}y < 0$ and

$$\nu_1(\eta, \mu) \leqq \int_Y \left(D(\nabla_y + i2\pi\eta)v_1^-(y)\right)^{\mathrm{T}} \mathcal{A}(y)\, \overline{\left(D(\nabla_y + i2\pi\eta)v_1^-(y)\right)}\,\mathrm{d}y$$
$$+\mu \int_Y \rho(y)\,|v_1^-(y)|^2\,\mathrm{d}y.$$

Notice that the first term in the sum on the right-hand side is a uniformly continuous function η. Thus, one can find $\mu_0 > 0$ such that $\nu_1(\eta, \mu) < 0$, $\eta \in Y^*$, $\mu > \mu_0$.

2. Assume that $\nu_1(\eta, \bar{\mu}) > \delta_0 > 0$ for any $\eta \in Y^*$. Since ν_1 is a uniformly continuous function on $Y^* \times [\bar{\mu} - 1, \bar{\mu} + 1]$, for any $\delta > 0$, there exists $\gamma > 0$ such that

$$|\nu_1(\mu, \eta) - \nu_1(\bar{\mu}, \eta)| < \delta$$

as soon as $|\mu - \bar{\mu}| < \gamma$. For $\delta = \delta_0/2$ we obtain an inequality

$$\nu_1(\mu, \eta) \geqq \frac{\delta_0}{2}, \quad \mu \in (\bar{\mu}, \bar{\mu} + \gamma),$$

which contradicts the definition of $\bar{\mu}$. Thus, there exists $\eta_0 \in Y^*$ such that $\nu_1(\eta_0, \bar{\mu}) = 0$.

Assume that η_0 is not a minimum point of $\nu_1(\cdot, \bar{\mu})$. In other words, assume that $\nu_1(\eta_1, \bar{\mu}) < 0$ for some $\eta_1 \in Y^*$. By the continuity arguments, there exists $\delta_0 > 0$ such that $\nu_1(\eta_1, \bar{\mu} - \delta) < 0$, $\delta \in (0, \delta_0)$, that again contradicts the definition of $\bar{\mu}$.

3. For any $\eta \in Y^*$, by the minimum principle, we have

$$\begin{aligned}\nu_1(\eta, \bar{\mu}+\delta) &= \inf_{\substack{v \in H^1_\#(Y)^K \\ \|v\|_{L^2(Y)^K}=1}} \left(\mathcal{L}^{\eta}_{\bar{\mu}+\delta}\, v, v\right)_{L^2(Y)^K} \\ &= \inf_{\substack{v \in H^1_\#(Y)^K \\ \|v\|_{L^2(Y)^K}=1}} \left\{\left(\mathcal{L}^{\eta}_{\bar{\mu}}\, v, v\right)_{L^2(Y)^K} + \delta\ (\rho v, v)_{L^2(Y)^K}\right\}.\end{aligned}$$

Choosing $\eta = \eta_0$ (the minimum point of $\nu_1(\cdot, \bar{\mu})$), we use $\theta_1(y, \eta_0; \bar{\mu})$ as a test function in the last formula and obtain

$$\nu_1(\eta_0, \bar{\mu}+\delta) \leqq \nu_1(\eta_0, \bar{\mu}) + \delta \int_Y \rho(y)|\theta_1(y, \eta_0; \bar{\mu})|^2\,\mathrm{d}y.$$

As was proved above, $\nu_1(\eta_0, \bar{\mu}) = 0$. Thus,

$$\begin{aligned}&\int_Y \rho(y)|\theta_1(y, \eta_0; \bar{\mu})|^2\,\mathrm{d}y \\ &\quad= \left(\overline{\mathcal{D}(-(\nabla_y + i2\pi\eta))}^{\mathrm{T}}\, \mathcal{A}(\cdot)\, \mathcal{D}(\nabla_y + i2\pi\eta)\, \theta_1(\cdot, \eta_0; \bar{\mu}), \theta_1(\cdot, \eta_0; \bar{\mu})\right) < 0,\end{aligned}$$

and, consequently, $\nu_1(\eta_0, \bar{\mu}+\delta) < 0$ for any $\delta > 0$. □

Due to the minimum principle, $\mathcal{E}^{\mu}(v, v) \geqq 0$, $v \in H^1_\#(Y)^K$, if and only if $\nu_1(0, \mu) \geqq 0$. Therefore, in view of (84), we have

$$-\mu_1^- = \sup\{\mu > 0 \,:\, \nu_1(0, \mu) \geqq 0\}.$$

Considering Proposition 2, one can see that

$$-\mu_1^- = \sup\{\mu > 0 \,:\, \nu_1(0, \mu) > 0\}.$$

Obviously,

$$-\mu_1^- \geqq \sup\{\mu > 0 \,:\, \nu_1(\eta, \mu) > 0 \text{ for all } \eta \in Y^*\} = \bar{\mu}.$$

The next lemma concerns the case of a scalar elliptic operator.

Lemma 11. *For the scalar symmetric elliptic operator $\mathcal{L}_\mu$ defined by*

$$\mathcal{L}_\mu v(y) = -\mathrm{div}(a(y)\nabla_y v(y)) + \mu\rho(y)v(y)$$

with the real coefficients a_{ij} and ρ, the minimum of the corresponding Bloch eigenvalue $\nu_1(\eta, \mu)$ is attained at zero, that is

$$\nu_1(0, \mu) = \min_{\eta \in Y^*} \nu_1(\eta, \mu), \quad \nu_1(\eta, \mu) > \nu_1(0, \mu) \text{ if } \eta \in Y^* \setminus \{0\}.$$

Furthermore, the eigenvalue $\nu_1(0, \mu)$ is simple.

Proof. Denote

$$\mathcal{L}_\mu^\eta v = \mathrm{e}^{-i2\pi y\cdot\eta}\,\mathcal{L}_\mu\,\mathrm{e}^{i2\pi y\cdot\eta}\,v.$$

Without loss of generality, we assume that the first eigenvalue of the operator $\mathcal{L}_\mu$ is equal to 0, that is $\nu_1(0,\mu)=0$. The corresponding eigenfunction of $\mathcal{L}_\mu$ is denoted by θ_1. Notice that, $\nu_1(0,\mu)$ is simple and, under proper normalization, $\theta_1(y)>0$, $y\in Y$. We need to show that $\nu_1(\eta,\mu)>0$ for $\eta\in Y^*\setminus\{0\}$ or, equivalently, that $(\mathcal{L}_\mu^\eta v, v)_{L^2(Y)}>0$ for $\eta\in Y^*\setminus\{0\}$, $v\in H^1_\#(Y)$.

Let us represent $v\in H^1_\#(Y)$ as the product $v(y)=\theta_1(y)\,w(y)$. Then

$$\begin{aligned}\left(\mathcal{L}_\mu^\eta v, v\right)_{L^2(Y)} &= (\mathrm{e}^{-i2\pi y\cdot\eta}\,\mathcal{L}_\mu\,\mathrm{e}^{i2\pi y\cdot\eta}\,\theta_1\,w, \theta_1\,w)_{L^2(Y)}\\ &= (\mathrm{e}^{-i2\pi y\cdot\eta}\,\theta_1\,\mathcal{L}_\mu\,\theta_1\,\mathrm{e}^{i2\pi y\cdot\eta}\,w, w)_{L^2(Y)}.\end{aligned}$$

Straightforward computations yield

$$\theta_1\,\mathcal{L}_\mu\,\theta_1 w = -\mathrm{div}\left(a\,\theta_1^2\nabla w\right);$$

here we have used the fact that $\nu_1(0,\mu)=0$. Then

$$\begin{aligned}&\left(\mathcal{L}_\mu^\eta v, v\right)_{L^2(Y)}\\ &\quad= \int_Y (\theta_1(y))^2\,(\nabla_y w(y)+i2\pi\eta\,w(y))^{\mathrm{T}}\,a(y)\,\overline{(\nabla_y w(y)+i2\pi\eta\,w(y))}\,\mathrm{d}y.\end{aligned}$$

Recalling that w is a periodic function, one shows that the last integral is equal to zero only if $\eta=0$ and w is a constant function. Lemma 11 is proved. □

Remark 7. In view of Lemma 11, in the case of a scalar elliptic operator with real coefficients $\bar\mu=-\mu_1^-$ and $\nu_1(0,-\mu_1^-)=0$.

We proceed with the first result on the negative part of the spectrum of the original problem.

Theorem 5. *Under the assumption $\langle\rho\rangle>0$, the principal negative eigenvalue in (69) satisfies the relation*

$$\lambda_1^{\varepsilon,-}\leqq-\frac{1}{\varepsilon^2}\bar\mu.$$

with $\bar\mu$ defined in (85).

Proof. The proof relies on the representation (75) and on the variational representation of $\lambda_1^{\varepsilon,-}$. We recall that

$$-\lambda_1^{\varepsilon,-} = \sup\left\{\lambda>0\,:\,\mathcal{E}_\lambda^\varepsilon(u,u)\geqq 0 \text{ for any } u\in H_0^1(\Omega)^K\right\} \tag{86}$$

with

$$\mathcal{E}_\lambda^\varepsilon(u,u)=\int_\Omega\left((\mathcal{D}(\nabla_x)u)^{\mathrm{T}}\mathcal{A}\left(\frac{x}{\varepsilon}\right)\overline{\mathcal{D}(\nabla_x)u}+\lambda\rho\left(\frac{x}{\varepsilon}\right)|u|^2\right)\mathrm{d}x$$

By the definition of $\bar{\mu}$, for any $\hat{\mu} \in (0, \bar{\mu})$ the inequality holds true

$$\nu_j(\eta, \hat{\mu}) > 0, \quad j = 1, 2, \dots, \ \eta \in Y^*. \tag{87}$$

Therefore, according to (75), for any $u \in H_0^1(\Omega)^K$,

$$\begin{aligned} \mathcal{E}^\varepsilon_{\hat{\mu}/\varepsilon^2}(u, u) &= \int_\Omega \left((\mathcal{D}(\nabla_x)u)^{\mathrm{T}} \mathcal{A}\left(\frac{x}{\varepsilon}\right) \overline{\mathcal{D}(\nabla_x)u} + \frac{\hat{\mu}}{\varepsilon^2} \rho\left(\frac{x}{\varepsilon}\right) |u|^2 \right) \mathrm{d}x \\ &= \varepsilon^d \int_{\varepsilon^{-1}Y^*} \sum_{j=1}^\infty \frac{\nu_j(\varepsilon\xi, \hat{\mu})}{\varepsilon^2} \left| \hat{u}_j^\varepsilon(\xi) \right|^2 \mathrm{d}\xi \end{aligned} \tag{88}$$

with

$$\hat{u}_j^\varepsilon(\xi) = \varepsilon^{-d/2} \int_{\mathbb{R}^d} u(x) \theta_j\left(\frac{x}{\varepsilon}, \varepsilon\xi; \hat{\mu}\right) \mathrm{e}^{i2\pi x \cdot \xi} \, \mathrm{d}x.$$

Since, owing to (87), we have $\nu_j(\varepsilon\xi, \hat{\mu}) > 0$ for any ξ, then $\mathcal{E}^\varepsilon_{\hat{\mu}/\varepsilon^2}(u, u) \geqq 0$ for any $u \in H_0^1(\Omega)^K$. Due to (86) this yields $-\lambda_1^{\varepsilon,-} \geqq \hat{\mu}/\varepsilon^2$ for any $\hat{\mu} < \bar{\mu}$. Thus,

$$-\lambda_1^{\varepsilon,-} \geqq \bar{\mu}/\varepsilon^2,$$

which completes the proof. □

5.1. Negative part of the spectrum in the case $\langle\rho\rangle > 0$. *Scalar operator with real coefficients.*

In this section we consider scalar operators with real coefficients, so that the studied eigenproblem takes the form

$$\begin{cases} -\mathrm{div}\left(a\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon(x)\right) = \lambda^\varepsilon \rho\left(\frac{x}{\varepsilon}\right) u^\varepsilon(x), & x \in \Omega, \\ u^\varepsilon(x) = 0, & x \in \partial\Omega. \end{cases} \tag{89}$$

As was already noted in Remark 7, in the case under consideration we have $\bar{\mu} = -\mu_1^-$ so that, by Theorem 5,

$$\lambda_1^{\varepsilon,-} \leqq \varepsilon^{-2} \mu_1^-.$$

Furthermore, by the standard variational arguments (see [8]), or Krein–Rutman theorem, $\nu_1(0, -\mu_1^-) = 0$ is a simple eigenvalue, and, under proper sign choice, $\theta_1(y, 0; -\mu_1^-) > 0$. For brevity, we denote

$$\vartheta(y) = \theta_1(y, 0; -\mu_1^-) > 0.$$

In the next step we factorize spectral problem (89) and introduce a new spectral parameter as follows

$$u^\varepsilon(x) = \vartheta\left(\frac{x}{\varepsilon}\right) v^\varepsilon(x), \quad \lambda^\varepsilon = \frac{1}{\varepsilon^2}\mu_1^- + \kappa^\varepsilon, \tag{90}$$

therefore, $u_j^\varepsilon(x) = \vartheta(x/\varepsilon)v_j^\varepsilon(x)$. Multiplying the resulting equation by $\vartheta(x/\varepsilon)$, after straightforward rearrangements, we derive the spectral problem for v^ε and κ^ε:

$$\begin{cases} -\mathrm{div}\left(\vartheta\left(\frac{x}{\varepsilon}\right)^2 a\left(\frac{x}{\varepsilon}\right) \nabla_x v^\varepsilon(x)\right) = \kappa^\varepsilon \vartheta\left(\frac{x}{\varepsilon}\right)^2 \rho\left(\frac{x}{\varepsilon}\right) v^\varepsilon(x), & x \in \Omega, \\ v^\varepsilon(x) = 0, & x \in \partial\Omega. \end{cases} \tag{91}$$

Notice that since $0 < \vartheta_- \leqq \vartheta(y) \leqq \vartheta^+ < +\infty$, the new spectral problem (91) is equivalent to the original one (89).

Considering (82), we conclude that $\int_Y \vartheta(y)^2 \rho(y)\, \mathrm{d}y < 0$. Under proper normalization of $\vartheta(y)$ one can assume without loss of generality that

$$\int_Y \vartheta(y)^2 \rho(y)\, \mathrm{d}y = -1. \tag{92}$$

Denote, for the sake of brevity, $a_\vartheta(y) = \vartheta(y)^2 a(y)$ and $\rho_\vartheta(y) = \vartheta(y)^2 \rho(y)$. Since, due to the Hölder continuity property of $\vartheta(y)$, $\vartheta \in L^\infty(\mathbb{R}^d)$, then $a_\vartheta(y) \in L^\infty(\mathbb{R}^d)$ and $\rho_\vartheta(y) \in L^\infty(\mathbb{R}^d)$. In view of (92), and due to the uniform ellipticity of the operator on the left-hand side of (91), Theorem 2 applies to the negative part of the spectrum of (91).

For (91) the effective spectral problem reads (see Section 3)

$$\begin{cases} -\mathrm{div}\left(a_\vartheta^{\mathrm{hom}} \nabla v^0(x)\right) = -\kappa^0 v^0(x), & x \in \Omega, \\ v^0(x) = 0, & x \in \partial\Omega, \end{cases} \tag{93}$$

where

$$a_\vartheta^{\mathrm{hom}} = \int_Y \left[a_\vartheta(y) + a_\vartheta(y)\nabla_y N_\vartheta(y)^{\mathrm{T}}\right] \mathrm{d}y$$

and N_ϑ is a unique solution of zero mean of the following problem with periodic boundary conditions:

$$\begin{cases} -\mathrm{div}_y(a_\vartheta(y)\nabla_y N_\vartheta(y)) = \mathrm{div}_y a_\vartheta(y), & y \in Y, \\ N_\vartheta \in H^1_\#(Y)^d. \end{cases}$$

Denote by κ_j^0 and $v_j^0(x)$ the eigenvalues and corresponding eigenfunctions of spectral problem (93). Notice that

$$0 > \kappa_1^0 > \kappa_2^0 \geqq \kappa_3^0 \geqq \cdots \geqq \kappa_j^0 \geqq \cdots \to -\infty \quad \text{as } j \to \infty.$$

Proposition 3. *For the eigenvalues κ_j^ε and κ_j^0 of problems (91) and (93), respectively, the following convergence result occurs:*

$$\kappa_j^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \kappa_j^0, \quad j = 1, 2, \ldots$$

If $\kappa_j^0 = \kappa_{j+1}^0 = \cdots = \kappa_{j+\varkappa_j-1}^0$ is an eigenvalue of multiplicity $\varkappa_j$ of problem (93), then there is a unitary $\varkappa_j \times \varkappa_j$ matrix $\beta^\varepsilon = \beta^\varepsilon(j)$ such that

$$\left\| \left(v_j^\varepsilon, v_{j+1}^\varepsilon, \ldots, v_{j+\varkappa_j-1}^\varepsilon\right)^{\mathrm{T}} - \beta^\varepsilon \left(V_j^\varepsilon, V_{j+1}^\varepsilon, \ldots, V_{j+\varkappa_j-1}^\varepsilon\right)^{\mathrm{T}} \right\|_{(H^1(\Omega))^{\varkappa_j}} \leqq C_j \varepsilon^{1/2}$$

with

$$V_k^\varepsilon(x) = v_k^0(x) + \varepsilon N_\vartheta\left(\frac{x}{\varepsilon}\right)\nabla v_k^0(x).$$

Recalling (90) we obtain the following result.

Theorem 6. *If* $\langle\rho\rangle > 0$ *then, under our standing assumptions,*

1. *for any* $j \in \mathbb{N}$

$$\lambda_j^{\varepsilon,-} - \frac{\mu_1^-}{\varepsilon^2} \underset{\varepsilon\to 0}{\longrightarrow} \kappa_j^0$$

Moreover,

$$\left|\lambda_j^{\varepsilon,-} - \frac{\mu_1^-}{\varepsilon^2} - \kappa_j^0\right| \leqq c_j\varepsilon^{1/2}.$$

2. *Let* $\kappa_j^0 = \kappa_{j+1}^0 = \cdots = \kappa_{j+\varkappa_j-1}^0$ *be an eigenvalue of multiplicity* $\varkappa_j$ *of problem* (93)*, then there is a unitary* $\varkappa_j \times \varkappa_j$ *matrix* $\beta^\varepsilon = \beta^\varepsilon(j)$ *such that*

$$\left\|\left(u_j^{\varepsilon,-}, u_{j+1}^{\varepsilon,-}, \ldots, u_{j+\varkappa_j-1}^{\varepsilon,-}\right)^{\mathrm{T}} - p\left(\frac{x}{\varepsilon}\right)\beta^\varepsilon\left(v_j^0, v_{j+1}^0, \ldots, v_{j+\varkappa_j-1}^0\right)^{\mathrm{T}}\right\|_{(L^2(\Omega))^{\varkappa_j}} \leqq C_j\varepsilon^{1/2}.$$

Remark 8. The last theorem states, in particular, that the eigenfunctions $u_j^{\varepsilon,-}(x)$, $\|u_j^{\varepsilon,-}\|_{L^2(\Omega)} = 1$, admit, perhaps along a subsequence, the asymptotic representation

$$u_j^{\varepsilon,-}(x) = \theta_1\left(\frac{x}{\varepsilon}, 0; -\mu_1^-\right) v^0(x) + \gamma^\varepsilon(x),$$

where $v^0(x)$ is an eigenfunction of the limit problem

$$\begin{cases} -\mathrm{div}\left(a_\vartheta^{\mathrm{hom}}\nabla v^0(x)\right) = -\kappa_j^0\, v^0(x), & x \in \Omega, \\ v^0(x) = 0, & x \in \partial\Omega, \end{cases}$$

normalized by $\|v^0\|_{L^2(\Omega)} = 1$, and the $L^2(\Omega)$ norm of γ^ε vanishes, as $\varepsilon \to 0$.

Remark 9. When studying the negative part of the spectrum, it is convenient to use the normalization $\|u_j^{\varepsilon,-}\|_{L^2(\Omega)} = 1$ for the eigenfunctions $u_j^{\varepsilon,-}$ instead of (9), because, due to the presence of the fast oscillation, under the later normalization condition the $L^\infty(\Omega)$ norm of $u_j^{\varepsilon,-}$ vanishes, as $\varepsilon \to 0$.

5.2. Negative part of the spectrum. General case.

In this section we consider general spectral problem (69) and assume that conditions 1–3 in Section 5 are fulfilled. Our goal is to obtain a lower bound for $\lambda_1^{\varepsilon,-}$.

Theorem 7. *If $\langle\rho\rangle > 0$, then the first negative eigenvalue $\lambda_1^{\varepsilon,-}$ of problem (69) satisfies the estimate*

$$-\frac{\bar{\mu}}{\varepsilon^2} - C \leqq \lambda_1^{\varepsilon,-} \leqq -\frac{\bar{\mu}}{\varepsilon^2} \tag{94}$$

with a constant C which does not depend on ε.

Proof. Let $\eta_0 \in Y^*$ be a minimum point of $\nu_1(\eta, \bar{\mu})$ on Y^*; if it is not unique, we choose one of them. Denote

$$\vartheta_{\eta_0}(y) = \theta_1(y, \eta_0; \bar{\mu}).$$

By virtue of Proposition 2, we have

$$\nu_1(\eta_0, \bar{\mu}) = 0, \quad \nu_1(\eta, \bar{\mu}) \geqq 0 \;\; \eta \in Y^*.$$

Clearly, the sesquilinear form

$$\begin{aligned}\mathcal{E}_{\bar{\mu},\eta}(\vartheta_{\eta_0}, \vartheta_{\eta_0}) = &\int_Y \overline{\big(\mathcal{D}(\nabla_y + 2\pi i\eta)\vartheta_{\eta_0}(y)\big)}^{\mathrm{T}} \mathcal{A}(y)\, \mathcal{D}(\nabla_y + 2\pi i\eta)\vartheta_{\eta_0}(y)\, \mathrm{d}y \\ &+\bar{\mu} \int_Y \rho(y) |\, \vartheta_{\eta_0}(y)|^2\, \mathrm{d}y\end{aligned}$$

considered as a function of $\eta \in Y^*$, is differentiable in η and has a minimum point at η_0. Differentiating this function in η_j, $j = 1, 2, \dots, d$, yields

$$\begin{aligned}&\int_Y \overline{\mathcal{D}(e_j)\vartheta_{\eta_0}(y)}^{\mathrm{T}} \mathcal{A}(y)\mathcal{D}(\nabla_y + 2\pi i\eta)\vartheta_{\eta_0}(y)\, \mathrm{d}y \\ &\quad+ \int_Y \overline{\mathcal{D}(\nabla_y + 2\pi i\eta)\vartheta_{\eta_0}(y)}^{\mathrm{T}} \mathcal{A}(y)\mathcal{D}(e_j)\vartheta_{\eta_0}(y)\, \mathrm{d}y = 0\end{aligned} \tag{95}$$

for any basis vector $e_j \in \mathbb{R}^d$. Notice also that $(\rho\vartheta_{\eta_0}, \vartheta_{\eta_0})_{L^2(\mathbb{Y})^K} < 0$.

We will repeatedly use the following simple estimate: if g is a periodic $L^2(Y)$ function with zero mean value, and φ is a $C^2(\Omega)$ function with a compact support in Ω, then

$$\left|\int_\Omega g\left(\frac{x}{\varepsilon}\right) \varphi(x)\, \mathrm{d}x\right| \leqq C\varepsilon^2. \tag{96}$$

In order to justify this estimate, one can introduce G as a periodic solution of the equation $\Delta G = g$ in the periodicity cell Y, and set $J(y) = \nabla_y G(y)$ so that $g(y) = \mathrm{div}_y J(y)$. Then

$$\int_\Omega g\left(\frac{x}{\varepsilon}\right) \varphi(x)\, \mathrm{d}x = \varepsilon \int_\Omega J\left(\frac{x}{\varepsilon}\right) \cdot \nabla\varphi(x)\, \mathrm{d}x.$$

Repeating this trick once again, we obtain the desired estimate(96).

It is convenient to introduce the notation $\mathcal{A}^\varepsilon(x) = \mathcal{A}(x/\varepsilon)$ and $\rho^\varepsilon(x) = \rho(x/\varepsilon)$, then the variational representation of $\lambda_1^{\varepsilon,-}$ reads

$$-\lambda_1^{\varepsilon,-} = \inf\{(\mathcal{A}^\varepsilon \mathcal{D}(\nabla_x)u, \mathcal{D}(\nabla_x)u)_{L^2(\Omega)^K} \,:\, u \in H_0^1(\Omega)^K,\ (\rho^\varepsilon u, u)_{L^2(\Omega)^K} = -1\}. \tag{97}$$

Let $\psi \in C_0^\infty(\Omega)$ be a scalar function such that

$$\psi(x) \geqq 0,\ \ x \in \Omega; \quad \int_\Omega (\psi(x))^2\,\mathrm{d}x = 1.$$

Assuming the normalization $\int_Y \rho(y)|\,\vartheta_{\eta_0}(y)|^2\,\mathrm{d}y = -1$, we evaluate the quantities

$$(\mathcal{A}^\varepsilon \mathcal{D}(\nabla_x)u, \mathcal{D}(\nabla_x)u)_{L^2(\Omega)^K} \quad \text{and} \quad (\rho^\varepsilon u, u)_{L^2(\Omega)^K}$$

in (97) at the test function

$$U_\varepsilon(x) = \psi(x)\vartheta_{\eta_0}\left(\frac{x}{\varepsilon}\right)\exp\left(\frac{2\pi i \eta_0 \cdot x}{\varepsilon}\right).$$

By (96), we easily derive

$$|(\rho^\varepsilon U_\varepsilon, U_\varepsilon)_{L^2(\Omega)} + 1| \leqq C\varepsilon^2 \tag{98}$$

We proceed by substituting U_ε in the sesquilinear form

$$\begin{aligned}
&(\mathcal{A}^\varepsilon \mathcal{D}(\nabla_x)U_\varepsilon, \mathcal{D}(\nabla_x)U_\varepsilon)_{L^2(\Omega)^K}\\
&= \frac{1}{\varepsilon^2}\int_\Omega \psi^2(x)\left\{\overline{(\mathcal{D}(\nabla_y + 2\pi i\eta_0)\vartheta_{\eta_0}(y))}^{\mathrm{T}}\mathcal{A}(y)\mathcal{D}(\nabla_y + 2\pi i\eta_0)\vartheta_{\eta_0}(y)\right\}\Big|_{y=x/\varepsilon}\mathrm{d}x\\
&\quad + \frac{1}{\varepsilon}\int_\Omega \psi(x)\left\{\overline{(\mathcal{D}(\nabla_y + 2\pi i\eta_0)\vartheta_{\eta_0}(y))}^{\mathrm{T}}\mathcal{A}(y)\mathcal{D}(\nabla_x\psi(x))\vartheta_{\eta_0}(y)\right\}\Big|_{y=x/\varepsilon}\mathrm{d}x\\
&\quad + \frac{1}{\varepsilon}\int_\Omega \psi(x)\left\{\overline{(\mathcal{D}(\nabla_x\psi(x))\vartheta_{\eta_0}(y))}^{\mathrm{T}}\mathcal{A}(y)\mathcal{D}(\nabla_y + 2\pi i\eta_0)\vartheta_{\eta_0}(y)\right\}\Big|_{y=x/\varepsilon}\mathrm{d}x\\
&\quad + \int_\Omega \left\{\overline{(\mathcal{D}(\nabla_x\psi(x))\vartheta_{\eta_0}(y))}^{\mathrm{T}}\mathcal{A}(y)\mathcal{D}(\nabla_x\psi(x))\vartheta_{\eta_0}(y)\right\}\Big|_{y=x/\varepsilon}\mathrm{d}x\\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We begin with estimating I_1. By the definition of $\bar{\mu}$ and η_0, and due to the normalization of ϑ_{η_0}, the mean value of the function in the braces is equal to $\bar{\mu}$. Therefore, by (96), we have

$$\left|I_1 - \frac{\bar{\mu}}{\varepsilon^2}\right| \leqq C.$$

By (95), the mean value of the function

$$\begin{aligned}
Y \ni y \mapsto\ &\overline{(\mathcal{D}(\nabla_y + 2\pi i\eta_0)\vartheta_{\eta_0}(y))}^{\mathrm{T}}\mathcal{A}(y)\mathcal{D}(\nabla_x\psi(x))\vartheta_{\eta_0}(y)\\
&+\overline{(\mathcal{D}(\nabla_x\psi(x))\vartheta_{\eta_0}(y))}^{\mathrm{T}}\mathcal{A}(y)\mathcal{D}(\nabla_y + 2\pi i\eta_0)\vartheta_{\eta_0}(y)
\end{aligned}$$

vanishes. Thus, the contribution of the sum $I_2 + I_3$ vanishes as well while $\varepsilon \to 0$.

Evidently, I_4 is uniformly bounded: $|I_4| \leqq C$.

Summarizing the above estimates for I_q, $q = 1, 2, 3, 4$, and (98), we conclude that

$$-\lambda_1^{\varepsilon,-} \leqq \frac{\bar{\mu}}{\varepsilon^2} + C.$$

The upper bound in (94) has been proved in Theorem 5. The whole proof is completed. □

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