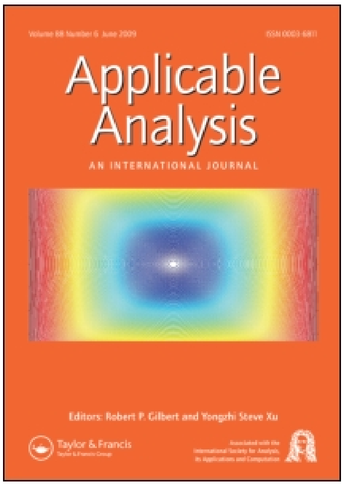

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Dag Lukkassen [a]; Annette Meidell [a]; Andrey L. Piatnitski [ab]; Alexey S. Shamaev [c]
[a] IDER, Narvik University College, 8505 Narvik, Norway [b] P.N. Lebedev Physical Institute of RAS, Moscow 119991, Russian Federation [c] Institute for Problems in Mechanics of RAS, Moscow 119526, Russian Federation

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Twisting a thin periodically perforated elastic rod

Dag Lukkassen[a], Annette Meidell[a], Andrey L. Piatnitski[bc]* and Alexey S. Shamaev[d]

[a]*IDER, Narvik University College, P.O. Box 385, 8505 Narvik, Norway;* [b]*IDER, Narvik University College, 2 Lodve Langes gate, P.O. Box 385, 8505 Narvik, Norway;* [c]*P.N. Lebedev Physical Institute of RAS, Leninski prospect, 53, Moscow 119991, Russian Federation;* [d]*Institute for Problems in Mechanics of RAS, Prospect Vernadskogo, 101, Moscow 119526, Russian Federation*

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We study the asymptotic behaviour of the displacement of a thin periodically perforated rod under the action of forces applied to one of the rod ends, another end of the rod is clamped. We show that, up to boundary layer functions arising in the vicinity of the end points of the rod, the set of solutions forms a finite dimensional space, and that in the interior of the rod any solution only depends on the resultant forces and moments. We also provide the results of numerical computations of the effective torsion rigidity for a hexagonal periodically perforated rod.

Keywords: thin rod; homogenization; periodic structure

AMS Subject Classifications: 35K20; 35Q35; 35R60

1. Introduction

This article deals with the asymptotic behaviour of deformation of a thin elastic periodically perforated rod in the presence of twisting forces. We assume that the deformations are small so that the linearized model applies.

The deformation of a rod of vanishing thickness has been widely studied in the works of mechanical engineers. There are also several rigorous mathematical works on this subject, where, for some particular models, the asymptotic expansion and the limit problem have been justified. A number of these works focused on inhomogeneous rods and rod constructions with periodic microstructure. In particular, in [1] Kozlova and Panasenko studied the vibration of a thin rod with clamped end points under the action of distributed forces and moments of forces. The stationary models describing a thin elastic rod with distributed forces were studied in [2,3] and in the book by Nazarov [4]. Closely related problems were investigated in [5]. A thin curved elastic rod was studied in [6] by means of unfolding technique.

*Corresponding author. Email: andrey@sci.lebedev.ru

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In this work we consider the elasticity system describing a thin periodically perforated rod under the assumption that one end of the rod is clumped, and all the forces and moments are applied to another end of the rod.

It is supposed that the rod thickness and the perforation period are of the same order, a small parameter characterizing these quantities, being denoted by ε.

We will show that a solution of the corresponding elasticity problem admits an asymptotic expansion, as $\varepsilon \to 0$. Moreover, in contrast with the case of forces distributed along the rod [2,3], in the case under considerations the mentioned asymptotic expansion involves only finite number of terms. For the reader convenience we will compare the results of this work with some results from [2,3].

The solution of the original problem will be represented as a finite polynomial series with respect to ε, each term of this series either can be expressed in terms of solutions of auxiliary periodic problems and polynomial functions or is a boundary layer type function. The boundary layer functions describe the behaviour of the rod in the vicinity of its end points. In particular, in the interior points of the rod the solution can be represented as a finite linear combination of the products of polynomials of degree not greater than three and auxiliary periodic functions. The coefficient of this linear combination only depends on the resultant forces and moments applied to the rod end.

The effective equations for the displacements in the tangential and transversal directions are, in general, coupled. However, in the presence of additional symmetries, the limit system of equations is getting decoupled and consists of independent second-order equation for the tangential displacements and fourth-order system for the transversal displacements.

In this work, under some symmetry conditions, we derive the homogenized problem and prove the convergence result. We also show that the boundary layer functions describing the behaviour of solutions in the vicinity of the rod end points are of exponential type.

There is vast literature devoted to thin plates and shells, see, for instance, [7] and the bibliography therein.

General ideas of homogenization of elastic bodies with periodic microstructure were exposed in classical books [8,9].

The paper is organized as follows. In Section 1 we introduce the model studied in this work and specify the assumptions. Section 2 deals with a number of auxiliary cell problems stated in the space of functions being periodic in the axial direction of the rod. Section 3 contains the convergence results. We build an asymptotic expansion for the solution of the problem under consideration, and then justify the convergence. The mentioned asymptotic expansion consists of a locally periodic inner expansion and boundary layer functions in the vicinity of the rod bases. Finally, in Section 4 we expose the results of numerical computations done for a hexagonal rod made of an isotropic homogeneous material, with a periodic set of cylindrical holes. We determine numerically the effective torsion rigidity and other effective characteristics of the rod.

1.1. *Problem set up*

Let B be a 1-periodic in the direction x_1, open, connected set in $\mathbb{R}^3$ with a smooth boundary. We assume that the origin belongs to B and that $|x_2| + |x_3| \leq C$ for all

$x \in B$. For a small $\varepsilon > 0$ we set

$$B_\varepsilon = \left\{ x \in \mathbb{R}^3 : 0 < x_1 < 1, \frac{x}{\varepsilon} \in B \right\},$$

and suppose that B_ε is a connected set. This set is called the perforated rod. Denote

$$S_0^\varepsilon = \{x \in \partial B_\varepsilon : x_1 = 0\}, \quad S_1^\varepsilon = \{x \in \partial B_\varepsilon : x_1 = 1\},$$
$$\Sigma_\varepsilon = \{x \in \partial B_\varepsilon : 0 < x_1 < 1\}.$$

In the set B_ε we consider the following boundary value problem for the elasticity system:

$$\begin{aligned} &\frac{\partial}{\partial x_i}\left(A^{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial}{\partial x_j} \bar{u}_\varepsilon \right) = \bar{f}_\varepsilon(x) \\ &n_k A^{kj}\left(\frac{x}{\varepsilon}\right) \frac{\partial}{\partial x_j} \bar{u}_\varepsilon \Big|_{\Sigma_\varepsilon} = 0 \\ &\bar{u}_\varepsilon \Big|_{S_0^\varepsilon} = 0 \\ &n_1 A^{1j} \frac{\partial}{\partial x_j} \bar{u}_\varepsilon \Big|_{S_1^\varepsilon} = \bar{F}\left(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\right), \end{aligned} \tag{1}$$

where A_{kl}^{ij}, $1 \le i, j, k, l \le 3$, is an elasticity tensor which is defined on B and satisfies the standard symmetry and uniform ellipticity conditions, $n(x) = (n_1(x), n_2(x), n_3(x))^t$ is the exterior unit normal on Σ_ε,

$$\bar{f}_\varepsilon(x) = \begin{pmatrix} f_1(x_1) \\ \varepsilon^2 f_2(x_1) \\ \varepsilon^2 f_2(x_1) \end{pmatrix} + f_4(x_1) \begin{pmatrix} 0 \\ -x_3/\varepsilon \\ x_2/\varepsilon \end{pmatrix},$$

and $f_1(x)$, $f_2(x)$, $f_3(x)$ and $f_4(x)$ are smooth scalar functions. $\bar{u}_\varepsilon = (u_{\varepsilon,1}, u_{\varepsilon,2}, u_{\varepsilon,3})^t$ is the vector of unknown functions to be determined.

Throughout this work we assume that the tensor $a_{kl}^{ij}(\xi)$, extended to the complement of B by 0, is 1-periodic in ξ_1 and possesses the following symmetry conditions:

$$a_{kl}^{ij}(\mathcal{S}_2\xi) = (-1)^{\delta_{2k}+\delta_{2i}+\delta_{2l}+\delta_{2j}} a_{kl}^{ij}(\xi), \quad a_{kl}^{ij}(\mathcal{S}_3\xi) = (-1)^{\delta_{3k}+\delta_{3i}+\delta_{3l}+\delta_{3j}} a_{kl}^{ij}(\xi), \tag{2}$$

where

$$\mathcal{S}_1\xi = (-\xi_1, \xi_2, \xi_3)^t, \quad \mathcal{S}_2\xi = (\xi_1, -\xi_2, \xi_3)^t, \quad \mathcal{S}_3\xi = (\xi_1, \xi_2, -\xi_3)^t.$$

The set B is also invariant with respect to $\mathcal{S}_2$ and $\mathcal{S}_3$. Under the above assumptions, for any $\varepsilon > 0$ problem (1) has a unique solution. Our aim is to describe the limit behaviour of this solution u_ε, as $\varepsilon \to 0$. For presentation simplicity later on we assume that $1/\varepsilon$ is integer.

2. Auxiliary problems

We begin by defining the resultant force and moment for the force $\bar{F}(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon})$ acting on the right base S_1^ε of the rod B_ε. Namely, we set

$$\bar{W} = (\Phi_1, \Phi_2, \Phi_3, M_1, M_2, M_3)^t$$

with

$$\Phi_i = \int_{S_1^\varepsilon} F_i\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big)\mathrm{d}x_2\,\mathrm{d}x_3, \quad i = 1, 2, 3,$$

and

$$M_1 = \int_{S_1^\varepsilon} \Big(x_3 F_2\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big) - x_2 F_3\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big)\Big)\mathrm{d}x_2\,\mathrm{d}x_3,$$
$$M_2 = \int_{S_1^\varepsilon} \Big(x_1 F_3\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big) - x_3 F_1\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big)\Big)\mathrm{d}x_2\,\mathrm{d}x_3,$$
$$M_3 = \int_{S_1^\varepsilon} \Big(x_2 F_1\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big) - x_1 F_2\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big)\Big)\mathrm{d}x_2\,\mathrm{d}x_3.$$

Now we introduce a number of auxiliary 'cell' problems stated in the domain $Q = \{\xi \in B : 0 < \xi_1 < 1\}$. The symbol $\sum$ stands for the lateral boundary of Q: $\sum = \{\xi \in \partial B : 0 < \xi_1 < 1\}$, and $\mathcal{S}_0 = \{\xi \in B : \xi_1 = 0\}$. We will solve these auxiliary problems in the space of 1-periodic in ξ_1 functions. The first problem reads

$$\frac{\partial}{\partial \xi_i}\Big(A^{ij}(\xi)\frac{\partial}{\partial \xi_j}N_1\Big) = -\frac{\partial}{\partial \xi_i}A^{i1}(\xi), \quad \xi \in Q,$$
$$n_k A^{kj}(\xi)\frac{\partial}{\partial \xi_j}N_1\Big|_\Sigma = -n_k A^{k1}.$$

According to [2,3], this problem has a periodic in ξ_1 solution. Notice that $N_1(\xi)$ is a 3×3 matrix. The following three auxiliary problems can be obtained by substituting the formal asymptotic expansion for $\bar{u}_\varepsilon$ in the original equation (1) and by collecting like powers of ε (see Section 3 for the details). These problems read

$$\begin{aligned} &\frac{\partial}{\partial \xi_i}\Big(A^{ij}(\xi)\frac{\partial N_2}{\partial \xi_j}\Big) = -\Big(\frac{\partial}{\partial \xi_i}\Big(A^{i1}(\xi)N_1(\xi)\Big) + A^{1j}(\xi)\frac{\partial N_1}{\partial \xi_i}(\xi) + A^{11}(\xi)h_2\Big), \quad \xi \in Q, \\ &n_k A^{kj}(\xi)\frac{\partial N_2}{\partial \xi_j}\Big|_\Sigma = -n_k A^{k1}N_1, \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\frac{\partial}{\partial \xi_i}\Big(A^{ij}\frac{\partial N_3}{\partial \xi_j}\Big) = -\Big(\frac{\partial}{\partial \xi_i}\Big(A^{i1}N_2\Big) + A^{1j}\frac{\partial N_2}{\partial \xi_i} + A^{11}N_1 + h_3\Big), \quad \xi \in Q, \\ &n_k A^{kj}(\xi)\frac{\partial N_3}{\partial \xi_j}\Big|_\Sigma = -n_k A^{k1}N_2 \end{aligned} \tag{4}$$

and

$$\frac{\partial}{\partial \xi_i}\Big(A^{ij}\frac{\partial N_4}{\partial \xi_j}\Big) = -\Big(\frac{\partial}{\partial \xi_i}\Big(A^{i1}N_3\Big) + A^{1j}\frac{\partial N_3}{\partial \xi_i} + A^{11}N_2 + h_4\Big), \quad \xi \in Q,$$
$$n_k A^{kj}(\xi)\frac{\partial N_4}{\partial \xi_j}\Big|_\Sigma = -n_k A^{k1}N_3; \tag{5}$$

here $N_1(\xi), N_2(\xi), N_3(\xi)$ and $N_4(\xi)$ are 3×3 matrices whose entries are 1-periodic in ξ_1, and h_2, h_3 and h_4 are constant 3×3 matrices. The latter matrices have been introduced in order to make problems (3)–(5) solvable. Writing down the compatibility conditions for problem (3)–(5), one can check that the matrices h_2–h_4 are uniquely determined. The following statement holds [2]:

LEMMA 1 *Under the symmetry conditions* (2) *the matrices* h_2–h_4 *take the form*

$$h_2 = \begin{pmatrix} \hat{c}_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad h_3 = \begin{pmatrix} H & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad h_4 = \begin{pmatrix} 0 & 0 & 0\\ 0 & \hat{c}_2 & 0\\ 0 & 0 & \hat{c}_3\end{pmatrix}.$$

Moreover,

$$\hat{c}_1 = \frac{1}{|Q|}\int_Q\Big[A^{1j}(\xi)\Big(\frac{\partial N_1}{\partial \xi_j}(\xi) + \delta_{1j}\mathrm{E}\Big)\Big]_{11}\mathrm{d}\xi, \quad H = \frac{1}{|Q|}\int_Q\Big[A^{1j}(\xi)\Big(\frac{\partial N_2}{\partial \xi_j}(\xi) + \delta_{1j}N_1(\xi)\Big)\Big]_{11}\mathrm{d}\xi,$$
$$\hat{c}_2 = \frac{1}{|Q|}\int_Q\Big[A^{1j}(\xi)\Big(\frac{\partial N_3}{\partial \xi_j}(\xi) + \delta_{1j}N_2(\xi)\Big)\Big]_{22}\mathrm{d}\xi,$$
$$\hat{c}_3 = \frac{1}{|Q|}\int_Q\Big[A^{1j}(\xi)\Big(\frac{\partial N_3}{\partial \xi_j}(\xi) + \delta_{1j}N_2(\xi)\Big)\Big]_{33}\mathrm{d}\xi,$$

where the symbol E stands for the unit matrix.

Other two auxiliary problems are as follows:

$$\frac{\partial}{\partial \xi_i}\Big(A^{ij}(\xi)\frac{\partial \bar{R}}{\partial \xi_j}\Big) = 0,$$
$$n_k A^{kj}(\xi)\frac{\partial \bar{R}}{\partial \xi_j}\Big|_\Sigma = -n_k A^{kj}\frac{\partial \bar{Z}}{\partial \xi_j} \tag{6}$$

with $\bar{Z} = (0,\ \xi_3\xi_1, -\xi_2\xi_1)^t$, and

$$\frac{\partial}{\partial \xi_i}\Big(A^{ij}(\xi)\frac{\partial \bar{K}_1}{\partial \xi_j}\Big) = -A^{1j}\frac{\partial \bar{G}}{\partial \xi_j} - \frac{\partial\,(A^{1j}\bar{G})}{\partial \xi_j},$$
$$n_k A^{kj}(\xi)\frac{\partial \bar{K}_1}{\partial \xi_j}\Big|_\Sigma = -n_k A^{k1}\bar{G} \tag{7}$$

with $\bar{G} = (0, -\xi_3, \xi_2)^t$. Is it easy to check that the compatibility conditions for both above problems are fulfilled and thus these problems are solvable in the space of 1-periodic in ξ_1 functions.

3. Asymptotic expansion

We write down an approximate solution in the form

$$v_\varepsilon(x) = \begin{pmatrix} C_3x_1 \\ V_1x_1^2 + C_1x_1^3 + V_3x_1x_3 \\ V_2x_1^2 + C_2x_1^3 - V_3x_1x_2 \end{pmatrix} + \varepsilon N_1\Big(\frac{x}{\varepsilon}\Big)\begin{pmatrix} C_3 \\ 2V_1x_1 + 3C_1x_1^2 \\ 2V_2x_1 + 3C_2x_1^2 \end{pmatrix}$$
$$+ \varepsilon V_3 \begin{pmatrix} R_1\big(\frac{x}{\varepsilon}\big) \\ R_2\big(\frac{x}{\varepsilon}\big) \\ R_3\big(\frac{x}{\varepsilon}\big) \end{pmatrix} + \varepsilon^2 N_2\Big(\frac{x}{\varepsilon}\Big)\begin{pmatrix} 0 \\ 2V_1 + 6C_1x_1 \\ 2V_2 + 6C_2x_1 \end{pmatrix} + \varepsilon^3 N_3\Big(\frac{x}{\varepsilon}\Big)\begin{pmatrix} 0 \\ 6C_1 \\ 6C_2 \end{pmatrix}, \quad (8)$$

and define the vector function $\bar{w}_\varepsilon(x)$ on S_1^ε as follows:

$$\bar{w}_\varepsilon(x) = A^{1j}\Big(\frac{x}{\varepsilon}\Big)\frac{\partial}{\partial x_j}\bar{v}_\varepsilon(x)\Big|_{x_1=1}. \quad (9)$$

For any vector function $\bar{\varphi}(x)$ denote

$$\tilde{\Phi}_i(\bar{\varphi}) = \int_{S_1^\varepsilon} \varphi_i(x)\mathrm{d}x_2\,\mathrm{d}x_3, \quad i = 1, 2, 3;$$

and

$$\tilde{M}_1(\bar{\varphi}) = \int_{S_1^\varepsilon} (x_3\varphi_2(x) - x_2\varphi_3(x))\mathrm{d}x_2\,\mathrm{d}x_3,$$
$$\tilde{M}_2(\bar{\varphi}) = \int_{S_1^\varepsilon} (-x_3\varphi_1(x) + x_1\varphi_3(x))\mathrm{d}x_2\,\mathrm{d}x_3,$$
$$\tilde{M}_3(\bar{\varphi}) = \int_{S_1^\varepsilon} (x_2\varphi_1(x) - x_1\varphi_2(x))\mathrm{d}x_2\,\mathrm{d}x_3.$$

Consider the system of equations

$$\tilde{\Phi}_i(\bar{w}_\varepsilon) = \Phi_i, \quad \tilde{M}_i(\bar{w}_\varepsilon) = M_i. \quad (10)$$

One can easily see that this system of equations is a linear system with respect to the unknowns C_1, C_2, C_3 and V_1, V_2, V_3. For brevity we denote

$$L_{ij} = \Big\{A_{1k}\Big(\frac{\partial N_1}{\partial \xi_k} + \delta_{1k}\mathrm{E}\Big)\Big\}_{ij}, \quad P_{ij} = \Big\{A_{1k}\Big(\frac{\partial N_2}{\partial \xi_k} + \delta_{1k}N_1\Big)\Big\}_{ij},$$
$$S_{ij} = \Big\{A_{1k}\Big(\frac{\partial N_3}{\partial \xi_k} + \delta_{1k}N_2\Big)\Big\}_{ij}, \quad Q_i = \Big\{\bar{A}_{1k}\frac{\partial \bar{R}}{\partial \xi_k}\Big\}_i.$$

After straightforward computations, we can find the coefficients of system (10). It reads

$$\begin{pmatrix} \alpha_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_{22} & 0 & \alpha_{24} & 0 & 0 \\ 0 & 0 & \alpha_{33} & 0 & \alpha_{35} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_{46} \\ 0 & 0 & \alpha_{53} & 0 & \alpha_{55} & 0 \\ 0 & \alpha_{62} & 0 & \alpha_{64} & 0 & 0 \end{pmatrix}\begin{pmatrix} C_3 \\ C_1 \\ C_2 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ M_1 \\ M_2 \\ M_3 \end{pmatrix} \quad (11)$$

with

$$\alpha_{11} = \int_{S_1} L_{11}(0,\hat{\xi})\mathrm{d}\hat{\xi}, \qquad \alpha_{22} = 6\varepsilon\int_{S_1} P_{22}(0,\hat{\xi})\mathrm{d}\hat{\xi} + 6\varepsilon^2\int_{S_1} S_{22}(0,\hat{\xi})\mathrm{d}\hat{\xi},$$

$$\alpha_{24} = 2\varepsilon\int_{S_1} P_{22}(0,\hat{\xi})\mathrm{d}\hat{\xi}, \quad \alpha_{33} = 6\varepsilon\int_{S_1} P_{33}(0,\hat{\xi})\mathrm{d}\hat{\xi} + 6\varepsilon^2\int_{S_1} S_{33}(0,\hat{\xi})\mathrm{d}\hat{\xi},$$

$$\alpha_{35} = 2\varepsilon\int_{S_1} P_{33}(0,\hat{\xi})\mathrm{d}\hat{\xi},$$

$$\alpha_{46} = \mu\int_{S_1}(\xi_2^2+\xi_3^2)\mathrm{d}\hat{\xi} + \int_{S_1}\big(Q_2(0,\hat{\xi})\xi_3 - Q_3(0,\hat{\xi})\xi_2\big)\mathrm{d}\hat{\xi},$$

$$\alpha_{53} = 6\varepsilon\int_{S_1}\big(P_{13}(0,\hat{\xi})\xi_3 - P_{33}(0,\hat{\xi})\xi_1\big)\mathrm{d}\hat{\xi} + 6\varepsilon^2\int_{S_1}\big(S_{13}(0,\hat{\xi})\xi_3 - S_{33}(0,\hat{\xi})\xi_1\big)\mathrm{d}\hat{\xi},$$

$$\alpha_{55} = 2\varepsilon\int_{S_1}\big(P_{13}(0,\hat{\xi})\xi_3 - P_{33}(0,\hat{\xi})\xi_1\big)\mathrm{d}\hat{\xi},$$

$$\alpha_{62} = 6\varepsilon\int_{S_1}\big(P_{12}(0,\hat{\xi})\xi_2 - P_{22}(0,\hat{\xi})\xi_1\big)\mathrm{d}\hat{\xi} + 6\varepsilon^2\int_{S_1}\big(S_{12}(0,\hat{\xi})\xi_2 - S_{22}(0,\hat{\xi})\xi_1\big)\mathrm{d}\hat{\xi} \tag{12}$$

and $\hat{\xi} = (\xi_2, \xi_3)$.

Remark 1 Formally speaking, since the forces $F(x_2/\varepsilon, x_3/\varepsilon)$ in (1) are applied at S_1^ε, we have to compute the coefficients α_{ij} at the cross-section $S_{1/\varepsilon} = \{\xi \in B : \xi_1 = 1/\varepsilon\}$. However, taking into account the fact that $\bar{v}_\varepsilon$ satisfies the equation

$$\frac{\partial}{\partial x_i}\left(A^{ij}\Big(\frac{x}{\varepsilon}\Big)\frac{\partial}{\partial x_j}\bar{v}_\varepsilon\right) = 0 \text{ in } B_\varepsilon, \;\; n_k A^{kj}\Big(\frac{x}{\varepsilon}\Big)\frac{\partial}{\partial x_j}\bar{v}_\varepsilon|_{\partial B_c} = 0,$$

we conculde that the quantities $\tilde{\Phi}_i(\bar{w}_\varepsilon)$ and $\tilde{M}_i(\bar{w}_\varepsilon)$ do not change if in (9) we replace the cross-section $\{x_1 = 1\}$ with any other cross-section of B_ε. This yields (12).

LEMMA 2 *The matrix* $\{\alpha_{ij}\}$ *does not degenerate*:

$$\det(\{\alpha_{ij}\}) \neq 0.$$

Proof The statement will be proved later on in this section. ∎

Denote $\mathcal{A} = \{\alpha_{ij}\}^{-1}$. According to the last lemma this matrix is well defined.

Remark 1 If the constants C_1, C_2, C_3 and V_1, V_2, V_3 satisfy the system (10), then all the resultant forces and moments of the difference $\bar{u}_\varepsilon - \bar{v}_\varepsilon$ are equal to zero at S_1^ε.

Assume that $\bar{f}_\varepsilon(x) = 0$ in (1). Then the following statement holds.

THEOREM 1 *Let* $(C_3, C_1, C_2, V_1, V_2, V_3)$ *be a solution of system* (11). *Then the difference between a solution* $\bar{u}_\varepsilon$ *to problem* (1) *and the function* $\bar{v}_\varepsilon$ *defined in* (8) *admits the representation*

$$\bar{u}_\varepsilon(x) - \bar{v}_\varepsilon(x) = \bar{U}_\varepsilon^{(1)}(x) + \bar{U}_\varepsilon^{(2)}(x) + O(e^{-c_0/\varepsilon}),$$

where $c_0 > 0$, $\bar{U}_\varepsilon^{(1)}$ *and* $\bar{U}_\varepsilon^{(2)}$ *are boundary layer functions in the vicinity of the end points of the rod, that is*

$$|\bar{U}_\varepsilon^{(1)}(x)| \leq Ce^{-\kappa x_1/\varepsilon}, \quad |\bar{U}_\varepsilon^{(2)}(x)| \leq Ce^{\kappa(x_1-1)/\varepsilon}$$

for some $\kappa > 0$.

Remark 3 As was shown in [2,3], in the case of non-zero distributed forces $\bar{f}_\varepsilon$ and Dirichlet boundary conditions on the bases of B_ε, the asymptotic expansion of a solution contains infinitely many terms.

Proof Taking into account the Equations (3)–(7), one can easily check that the function $v_\varepsilon(x)$ satisfies the equation and the lateral boundary condition in (1) so that:

$$\begin{aligned} &\frac{\partial}{\partial x_i}\Big(A^{ij}\Big(\frac{x}{\varepsilon}\Big)\frac{\partial}{\partial x_j}(\bar{u}_\varepsilon - \bar{v}_\varepsilon)\Big) = 0 \\ &n_k A^{kj}\Big(\frac{x}{\varepsilon}\Big)\frac{\partial}{\partial x_j}(\bar{u}_\varepsilon - \bar{v}_\varepsilon)\Big|_{\Sigma_\varepsilon} = 0. \end{aligned} \tag{13}$$

However, $\bar{v}_\varepsilon$ does not satisfy the boundary condition imposed on the butt cross sections. We can only assert that

$$\begin{aligned} (\bar{u}_\varepsilon - \bar{v}_\varepsilon)\Big|_{S_0^\varepsilon} &= \varepsilon N_1\Big(\frac{x}{\varepsilon}\Big)\begin{pmatrix} C_3 \\ 0 \\ 0 \end{pmatrix} + \varepsilon V_3 \begin{pmatrix} R_1(\frac{x}{\varepsilon}) \\ R_2(\frac{x}{\varepsilon}) \\ R_3(\frac{x}{\varepsilon}) \end{pmatrix} + \varepsilon^2 N_2\Big(\frac{x}{\varepsilon}\Big)\begin{pmatrix} 0 \\ 2V_1 \\ 2V_2 \end{pmatrix} \\ &+ \varepsilon^3 N_3\Big(\frac{x}{\varepsilon}\Big)\begin{pmatrix} 0 \\ 6C_1 \\ 6C_2 \end{pmatrix} \end{aligned} \tag{14}$$

$$A^{1j}\frac{\partial}{\partial x_j}(\bar{u}_\varepsilon - \bar{v}_\varepsilon)\Big|_{S_1^\varepsilon} = \bar{F}\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big) - \bar{w}_\varepsilon(x), \tag{15}$$

with $\bar{w}_\varepsilon$ defined in (9); it is not difficult to write down an explicit formula for $\bar{w}_\varepsilon$, however, as this formula is very long, we do not present it here. We only note that $\bar{w}_\varepsilon$ has the form $\bar{w}_\varepsilon(x)|_{S_1^\varepsilon} = \varepsilon^0 \bar{w}^0(\frac{x'}{\varepsilon}) + \varepsilon \bar{w}^1(\frac{x'}{\varepsilon}) + \varepsilon^2 \bar{w}^2(\frac{x'}{\varepsilon})$, and that the resultant forces and moments of forces of the functions $(\bar{F}(\frac{x'}{\varepsilon}) - \bar{w}^0(\frac{x'}{\varepsilon}))$, $\bar{w}^1(\frac{x'}{\varepsilon})$ and $\bar{w}^2(\frac{x'}{\varepsilon})$ on the butt end S_1^ε are equal to zero.

Then according to Theorem 3 in [2] there is a solution of problem

$$\begin{aligned} &\frac{\partial}{\partial x_i}\Big(A^{ij}(y)\frac{\partial}{\partial x_j}\bar{L}_\varepsilon^0\Big) = 0 \quad y \in \{y \in B : -\infty < y_1 < 1/\varepsilon\} \\ &n_k A^{kj}(y)\frac{\partial}{\partial x_j}\bar{L}_\varepsilon^0\Big|_\Sigma = 0 \\ &A^{1j}\frac{\partial}{\partial y_j}\bar{L}_\varepsilon^0\Big|_{y_1 = 1/\varepsilon} = \bar{F}(y') - \bar{w}^0(y'), \end{aligned} \tag{16}$$

which satisfies the estimate

$$\|\bar{L}_\varepsilon^0\|_{L^2(\{y \in B : (Y_0 - 1) < y_1 < Y_0\})} \le C\exp(-\kappa(1/\varepsilon - Y_0)) \tag{17}$$

for any $Y_0 \le 1/\varepsilon$; here κ is a positive constant which only depends on the coefficients $A^{ij}(y)$ and domain B.

Similarly, the problems

$$\frac{\partial}{\partial x_i}\left(A^{ij}(y)\frac{\partial}{\partial x_j}\bar{L}^1_\varepsilon\right) = 0 \quad y \in \{y \in B : -\infty < y_1 < 1/\varepsilon\}$$
$$n_k A^{kj}(y)\frac{\partial}{\partial x_j}\bar{L}^1_\varepsilon\Big|_\Sigma = 0$$
$$A^{1j}\frac{\partial}{\partial y_j}\bar{L}^1_\varepsilon\Big|_{y_1=1/\varepsilon} = \bar{w}^1(y')$$

and

$$\frac{\partial}{\partial x_i}\left(A^{ij}(y)\frac{\partial}{\partial x_j}\bar{L}^2_\varepsilon\right) = 0 \quad y \in \{y \in B : -\infty < y_1 < 1/\varepsilon\}$$
$$n_k A^{kj}(y)\frac{\partial}{\partial x_j}\bar{L}^2_\varepsilon\Big|_\Sigma = 0$$
$$A^{1j}\frac{\partial}{\partial y_j}\bar{L}^2_\varepsilon\Big|_{y_1=1/\varepsilon} = \bar{w}^2(y'),$$

have solutions which satisfy the exponential estimates similar to (17).

We proceed with another end of the rod. Consider an auxiliary problem

$$\begin{aligned}&\frac{\partial}{\partial x_i}\left(A^{ij}(y)\frac{\partial}{\partial x_j}\bar{W}^1\right) = 0 \quad y \in \{y \in B : y_1 > 0\}\\ &n_k A^{kj}(y)\frac{\partial}{\partial x_j}\bar{W}^1\Big|_\Sigma = 0\\ &\bar{W}^1\Big|_{y_1=0} = \Xi^1(y'),\end{aligned} \tag{18}$$

with

$$\Xi^1(y') = N_1(0,y')\begin{pmatrix} C_3 \\ 0 \\ 0\end{pmatrix} + \varepsilon V_3\begin{pmatrix} R_1(0,y') \\ R_2(0,y') \\ R_3(0,y')\end{pmatrix}.$$

According to [4,10], the last problem has a solution $\Xi(y)$ which possesses a finite energy. In the class of functions having finite energy this solution is unique and stabilizes at the exponential rate to a rigid displacement as $y_1 \to \infty$. Clearly, functions N_1 and R can be chosen in such a way that they inherit the symmetries of domain B and coefficients A^{ij}_{kl}. Then the solution Ξ also possesses these symmetries. The only rigid displacement compatible with the mentioned symmetries is a constant displacement in the first coordinate direction. Now, by adding a proper constant vector to N_1 we achieve the relation

$$\|\bar{W}^1\|_{L^2(y\in B: Y_0 \le y_1 \le Y_0+1)} \le C\exp(-\kappa Y_0), \quad \kappa > 0. \tag{19}$$

In exactly the same way one can construct exponential boundary layers $\bar{W}^2$ and $\bar{W}^3$ which correct the terms of order ε^2 and ε^3 in the boundary condition (14) at 0.

By construction, the function

$$\bar{v}_\varepsilon + \bar{L}^0_\varepsilon\left(\frac{x}{\varepsilon}\right) + \varepsilon\bar{L}^1_\varepsilon\left(\frac{x}{\varepsilon}\right) + \varepsilon^2\bar{L}^2_\varepsilon\left(\frac{x}{\varepsilon}\right) + \varepsilon\bar{W}^1\left(\frac{x}{\varepsilon}\right) + \varepsilon^2\bar{W}^2\left(\frac{x}{\varepsilon}\right) + \varepsilon^3\bar{W}^3\left(\frac{x}{\varepsilon}\right)$$

satisfies the equation and the lateral boundary conditions in (1) exactly, while the boundary conditions at the butt ends of the rod are satisfied up to exponentially small discrepancies. The desired statement of Theorem 1 follows now from (17), (19) and similar estimates for L^1_ε, L^2_ε, W^2, W^3 and from the fact that the Korn inequality holds in the domain B_ε with a constant which does not exceed $C\varepsilon^{-2}$, see [4,10]. ■

Proof of Lemma 2 Assume, to the contrary, that $\det(\{\alpha_{ij}\}) = 0$. Then, there is a non-trivial vector $(C_1, C_2, C_3, V_1, V_2, V_3)^t$ such that the function $\bar{w}_\varepsilon(x')$ defined by (8) and (9), has zero resultant forces and moments of forces at S^ε_1. Then, in the same way as in the proof of Theorem 1, one can show that there are exponential boundary layers $L^0_\varepsilon(\frac{x}{\varepsilon}) + \varepsilon L^1_\varepsilon(\frac{x}{\varepsilon}) + \varepsilon^2 L^2_\varepsilon(\frac{x}{\varepsilon})$ and $\varepsilon W^1(\frac{x}{\varepsilon}) + \varepsilon^2 W^2(\frac{x}{\varepsilon}) + \varepsilon^3 W^3(\frac{x}{\varepsilon})$ such that function

$$\bar{v}_\varepsilon - \left(L^0_\varepsilon\left(\frac{x}{\varepsilon}\right) + \varepsilon L^1_\varepsilon\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 L^2_\varepsilon\left(\frac{x}{\varepsilon}\right)\right) - \left(\varepsilon W^1\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 W^2\left(\frac{x}{\varepsilon}\right) + \varepsilon^3 W^3\left(\frac{x}{\varepsilon}\right)\right)$$

satisfies, up to an exponentially small discrepancy, the homogeneous equation and boundary conditions in (1). Due to the Korn inequality [4], this implies that $\bar{v}_\varepsilon$ is exponentially small in any interior subdomain of B_ε. This contradicts our assumption that the vector $(C_1, C_2, C_3, V_1, V_2, V_3)^t$ is non-trivial. ■

4. Example of numerical computation of effective torsion rigidity

Let the region $B \subset \mathbf{R}^3$ occupying the rod-structure be a hexagonal bar with cylindrical holes as illustrated in Figure 1, where the material inside is homogeneous and isotropic. As seen from the figure, the set B is periodic in the ξ_1 variable (the longitudinal direction) with respect to some interval $I = (0, \xi^0_1)$.

The set $Y = \{\xi \in B : \xi_1 \in I\}$ corresponds to a period of the rod-structure, and is referred to as the Y-cell. The hexagonal section has side-length d and is centred at the ξ_1-axis. Moreover, the cylindrical hole which is surrounded by the Y-cell, has centre at $(\xi^0_1/2, 0, 0)$, radius r, and is directed upwards parallel with the ξ_2-axis. The boundary ∂Y of Y with outward unit-normal $n = (n_1, n_2, n_3)$ consists of the two

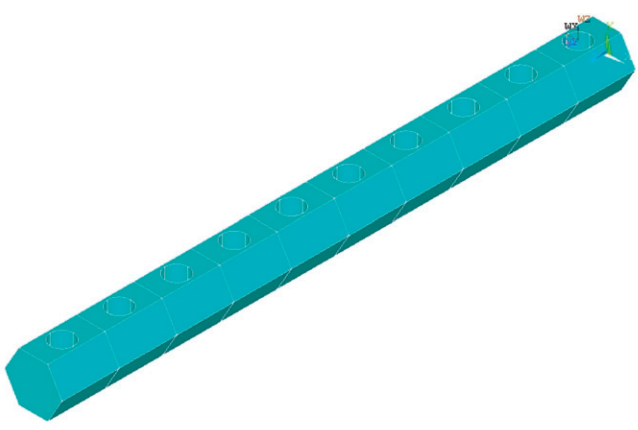

Figure 1. Hexagonal bar with cylindrical holes.

disjoint parts S and Σ given by $S = S(0) \cup S(\xi_1^0)$ and $\Sigma = \partial Y \backslash S$, where $S(t)$ denotes the vertical section $S(t) = \{\xi = (\xi_1, \xi_2, \xi_3) \in \bar{Y}, \xi_1 = t\}$. Moreover, let $\mathbf{H}^1_{\text{per},1}(Y)$ denote the closure in the usual Sobolev space $\mathbf{H}^1(Y)$, of the set $\mathbf{C}^\infty_{\text{per},1}(Y)$ of all smooth vector valued functions $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ which is I-periodic in the ξ_1 variable.

Let $v = (0, \tau\xi_1\xi_3, -\tau\xi_1\xi_2)$ where τ is some constant (the relative twist). In order to calculate the effective torsion rigidity we have to solve the following problem: Find $u = v + w$, where $w \in \mathbf{H}^1_{\text{per},1}(Y)$ such that

$$\int_Y e(\varphi) \cdot \sigma(u) \mathrm{d}\xi = 0 \quad \text{for all } \varphi \in \mathbf{H}^1_{\text{per},1}(Y). \tag{20}$$

The corresponding strain energy W is given by

$$W = \frac{1}{2} \int_Y e(u) \cdot \sigma(u) \mathrm{d}\xi. \tag{21}$$

By (20), we observe that

$$W = \frac{1}{2} \int_Y e(v) \cdot \sigma(u) \mathrm{d}\xi. \tag{22}$$

The resultant torsion moment M about the ξ_1-axis of the stress vector $(\sigma_{13}(u), \sigma_{23}(u), \sigma_{33}(u))$ applied to some arbitrary section $S(\xi_1)$ is given by

$$M = \int_{S(\xi_1)} (-\xi_2\sigma_{31}(u) + \xi_3\sigma_{21}(u)) \mathrm{d}\xi_2 \, \mathrm{d}\xi_3.$$

It is possible to prove that M is constant along the rod-structure (a proof of this fact can be found in [12]). Hence, by (22)

$$M = \frac{1}{\xi_1^0} \int_Y (-\xi_2\sigma_{31}(u) + \xi_3\sigma_{21}(u)) \mathrm{d}\xi = \frac{2}{\xi_1^0 \tau} W. \tag{23}$$

Moreover, due to linearity, M is proportional to τ, i.e. $M = \tau D$, where the constant D is called the effective torsion rigidity. By (23),

$$D = \frac{2}{\xi_1^0 \tau^2} W. \tag{24}$$

Due to the symmetry of the Y-cell about the plane $\xi_1 = \xi_1^0/2$, it is possible to prove that the periodic boundary conditions on $w = (w_1, w_2, w_3)$ can be replaced by Neumann conditions for the displacement-component w_1 and Dirichlet conditions for w_2 and w_3 on S [12]. Accordingly, in the FE-program ANSYS the problem is solved by using 'structural problem' with no body forces and specifying the Dirichlet boundary conditions $u_2 = v_2$ and $u_3 = v_3$ on the two parallel surfaces constituting the set S. This is certainly the same as putting $w_1 = 0$ (or $w_2 = 0$ and $w_3 = 0$) on S. The Neumann boundary condition is automatically imposed by leaving the corresponding displacements on these surfaces unspecified. This gives us a numerical solution u which is unique within an arbitrary translation in ξ_1-direction, i.e. within a constant in the displacement component w_1. In order to obtain a unique solution we may specify u_1 by, e.g. putting $u_1(0, 0, 0) = 0$. Summing up, the problem can be solved by using the following Dirichlet boundary conditions (and leaving all other boundary conditions unspecified):

$$u_3(0,\xi_2,\xi_3) = u_2(0,\xi_2,\xi_3) = 0,$$
$$u_3(\xi_1^0,\xi_2,\xi_3) = -\tau\xi_1^0\xi_2, \quad u_2(\xi_1^0,\xi_2,\xi_3) = \tau\xi_1^0\xi_3,$$
$$u_1(0,0,0) = 0.$$

Note that D is independent of the constant τ. This parameter may therefore be chosen arbitrarily when the only purpose of the computation is to calculate effective properties.

Below we present some numerical result obtained by using ANSYS 9.2. for $\tau = 1$, $\xi_1^0 = 2$, $d = 1$, when Young's modulus $E = 1$ and the Poisson ratio $\nu = 0.3$. The radius r of the hole varies from 0 to 0.5. As easily seen, the volume of the rod-structure per length is $(3-\pi r^2)\sqrt{3}/2$. Thus, the corresponding circular bar with the same volume per length ratio, i.e. with radius

$$R = \sqrt{\frac{\sqrt{3}(3-\pi r^2)}{2\pi}},$$

has effective torsion stiffness

$$D_s = G\frac{\pi}{2}R^4 = \frac{E}{(1+\nu)}\frac{\pi}{4}R^4 = \frac{\pi 3\left(3-\pi r^2\right)^2}{(1+0.3)16\pi^2}.$$

In the table below we have listed D and the shape factor D/D_s for some values of r (Figure 2).

r	0	0.05	0.10	0.15	0.20
D	0.399594	0.397647	0.391483	0.380395	0.365542
D/D_s	0.96710	0.96744	0.96763	0.96559	0.96373

r	0.25	0.30	0.35	0.40	0.45	0.50
D	0.346262	0.322931	0.297007	0.268305	0.237805	0.206581
D/D_s	0.95952	0.95266	0.94594	0.93705	0.92701	0.91749

In the computation we have chosen to use the element called solid95, which is a three-dimensional structural solid element with 20 nodes. Figure 3 shows the mesh of the Y-cell for $r = 0.4$.

In this case the global element size is put equal to 0.2 and the resulting mesh consists of 4617 elements and 7448 nodes.

Figure 4 shows the computed shape of the Y-cell after deformation and Von Mieses stresses.

Note that the shape factor D/D_s is only dependent of the ratio r/d and the Poisson ratio ν. This follows by the fact that the computed value of the effective torsion stiffness D for the above case $d = 1$, $E = 1$ and $\nu = 0.3$ can be used to find the effective torsion stiffness of the rod-structure of the same shape with side-length d, $\xi_1^0 = 2\mathrm{d}$, hole-radius equal to rd, $\nu = 0.3$ and E is arbitrary, just by multiplying with Ed^4. This is quite easy to prove by putting $\xi = yd$ and replacing the solution u of (20) by

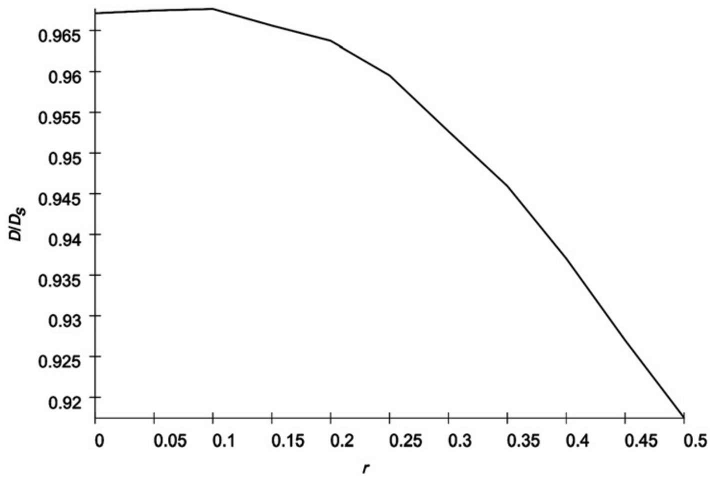

Figure 2. The shape factor D/D_s.

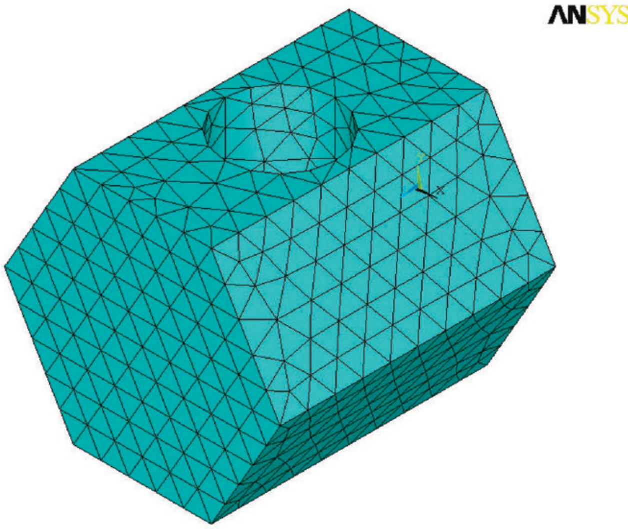

Figure 3. The meshed Y-cell in the case $r = 0.4$.

$u'(\xi) = d^2 u(y)$, v by $v'(\xi) = d^2 v(y)$ $(= v(\xi))$ and Y by $Y' = Yd$. The corresponding energy W' is then

$$W' = \frac{1}{2}\int_{Y'} \big(e_\xi(v') \cdot \sigma_\xi(u')\big)(\xi)\mathrm{d}\xi = \frac{1}{2}d^2 \int_{Y'} \big(e_y(v) \cdot \sigma_y(u)\big)\Big(\frac{\xi}{d}\Big)\mathrm{d}\xi$$
$$= d^5 \frac{1}{2}\int_{Y} \big(e_y(v) \cdot \sigma_y(u)\big)(y)\mathrm{d}y = d^5 W, \qquad (25)$$

where the subscripts ξ and y are used to indicate that the differentiation is performed with respect to ξ and y, respectively. Thus, by (24) the corresponding effective torsion

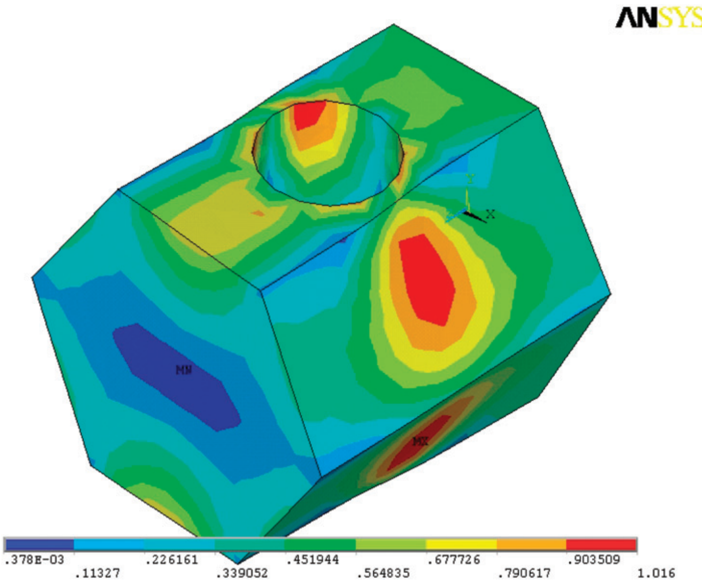

Figure 4. Computed shape of the Y-cell after deformation and Von Mieses stresses.

stiffness D' is given by

$$D' = \frac{2}{\xi_1^0 \tau^2} W' = \frac{2}{2d} d^5 W = d^4 W = d^4 D.$$

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