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Singular Double Porosity Model

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We consider the linear parabolic equation describing the transport of a contaminant in a porous media crossed by a net of infinitely thin fractures. The permeability is very high in the fractures but very low in the porous blocks. We derive the homogenized model corresponding to a net of infinitely thin fractures, by means of the singular measures technique. We assume that these singular measures are supported by hyperplanes of codimension one. We prove in a second step that this homogenized model could be obtained indistinctly either by letting the fracture thickness, in the standard double porosity model, tend to zero, or by homogenizing a model with infinitely thin fractures.

Keywords: Scaling-up; Fractured porous media; Modeling; Homogenization; Double porosity; Singular constructions

AMS Subject Classifications: 35K20; 35Q35; 35R60

1. INTRODUCTION

The so called double porosity model has received a lot of attention both from mathematicians and from engineers, like, for instance, in [4,12]. The model was introduced first for describing the global behavior of fractured porous media by Barenblatt *et al.* [3] and it is since used in a wide range of engineering specialities related to geohydrology, petroleum reservoir engineering, civil engineering or soil science.

In such a model, the "fractured porous medium" is made of two kinds of porous media with different permeability. There is first a set of isolated porous blocks of low permeability (sometimes called matrix), surrounded by a network of high permeability connected porous medium (usually called fractures network).

The double porosity model, like in [2], is assuming that the width of the fractures containing highly permeable porous media (fracture width, in short) is of the same order as the blocks sizes.

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In the present article we consider the case of thin fractures with large transport flux inside so that the flow does not vanish as the thickness is getting infinitely small. The Neumann problem for reticulated structures have been widely discussed in the existing literature, like, for instance, in [10]. The case of thin fractures corresponds to the mathematical model given by the Eq. (22) in Section 3 where the (fracture width)/(block size) ratio is a small parameter δ and where the high permeability of order $1/\delta$ appears as a large coefficient in the equation in the fissure part. Albeit, the case of infinitely thin fissures corresponds to the Eq. (1) below, where the high permeable porous part reduces to (hyper)surfaces on which the permeability appears as a coefficient of order one in front of the Laplacian acting in tangential variables on the (hyper)surface.

The simplest model is related to the behavior of a weakly compressible single phase flow through a porous domain Ω. We assume that Ω is made of a set of porous blocks called matrices $\mathcal{M}^\varepsilon$ with permeability of order ε^2, connected together by a system of infinitely thin fractures $\mathcal{F}^\varepsilon$ with tangential permeability of order 1. This model is described by the following set of equations in $\Omega \subset \mathbb{R}^N$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon^2 k_1 \Delta u = f \text{ in } \Omega \cap \mathcal{M}^\varepsilon \subset \mathbb{R}^N, \text{ the set of porous block;} \\ \dfrac{\partial v}{\partial t} - k_2 \tilde{\Delta} v = \varepsilon^2 k_1 \left[\dfrac{\partial u}{\partial n} \right] \text{ in } \Omega \cap \mathcal{F}^\varepsilon, \text{ the fractures network;} \\ + \text{ conservation of the surface flux through the intersections} \\ \quad \text{of the } (N-1)\text{-facets being faces of codimension 2;} \end{cases} \tag{1}$$

where ε is the normalized microscopic length scale of a typical porous block, $\varepsilon \ll 1$, $\tilde{\Delta}$ is the Laplacian in the tangential variables of the fractures hypersurface, and $[\partial u/\partial n]$ denotes the jump of the normal derivative through the fractures hypersurface.

Most of the mathematical works on that subject were related to the mathematical homogenization theory including the use of the so called two-scale convergence concept (see, for instance, [2,6–8]) both for periodic and for nonperiodic fracture patterns. But behind all those works there was always an assumption on the thickness of the fractures which were supposed to be of the same order ε as the length size of a typical porous block.

The main purpose of the present work is to consider the case of thin and infinitely thin fractures with large transport flux inside, by means of the techniques developed recently in [5,9,12–15] for Sobolev spaces on singular constructions (see Fig. 1 for an example of 2D construction).

In Section 1, we define the geometry and we introduce the singular measure setting.

Section 2 is aimed at obtaining the double porosity homogenized model by using the singular measure framework introduced in Section 1. The main result is given in Theorem 1.

In Section 3 we study the homogenization problem for a double porosity model with fractures of finite width δ and then let δ tend to zero. We prove in Proposition 2 that in this way we get back the previous homogenized model obtained in Theorem 1 of Section 2.

In Theorem 2 we prove that the final homogenized model does not depend on the order of passage to the limit in the equations when ε and δ tends to zero.

For simplicity we assume here that the fractured part only consists of thin and infinitely thin fissures. However, the theory developed in [9] for junctions also allows

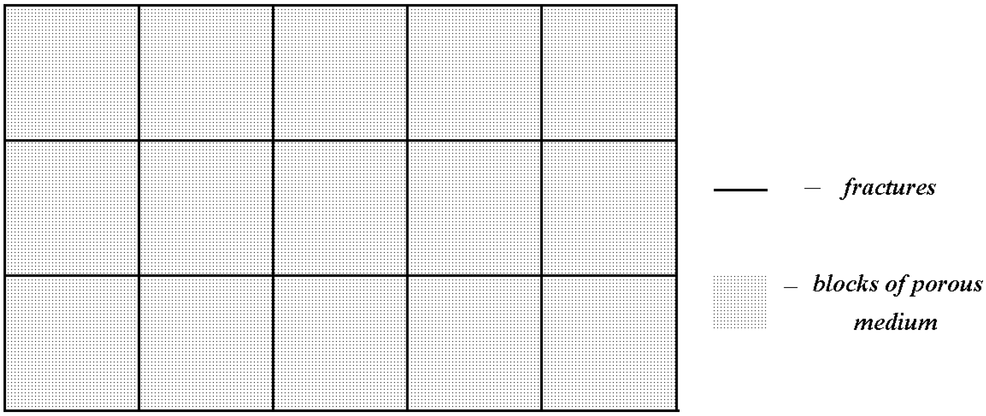

FIGURE 1 Thin fractures network and porous blocks.

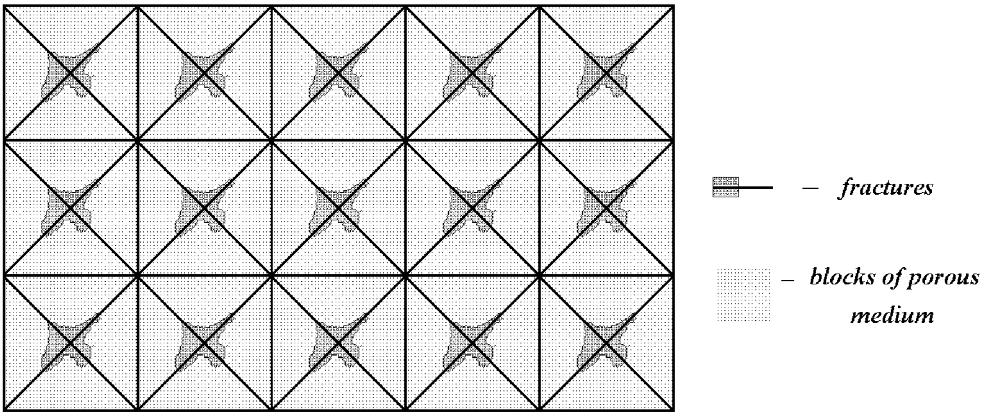

FIGURE 2 Highly permeable thin fractures with junctions, surrounded by blocks of low permeability.

the handling of more complex cases of fractured porous media like, for instance, that shown on Fig. 2, where the matrices still have permeability of order ε^2, but the fractured part consists of highly permeable sets connected by infinitely thin fractures of infinitely high permeability, surrounded by volume distributed porous medium with permeability of order one.

Other homogenization techniques for thin and infinitely thin constructions have been developed in [10,16,18].

2. SETTING OF THE PROBLEM AND DEFINITION OF THE GEOMETRY

We begin by describing the geometry. The matrix part is generated by a finite number of nonintersecting open convex polytopes $G_1, \ldots, G_l$ in $\mathbb{R}^N$, such that $G_i \cap (G_j + k) = \emptyset$ for each $k \in Z^N \setminus 0$, and $\cup_{k \in Z^N} \cup_{j=1}^{l} (\bar{G}_j + k) = \mathbb{R}^N$. Clearly, such a construction is 1-periodic in all the coordinate directions and we may then define the cell of periodicity $Y = [0, 1]^N$, $N \geq 2$, and two periodic sets: the matrix part $\mathcal{M} = \cup_{k \in Z^N} \cup_{j=1}^{l} (G_j + k)$

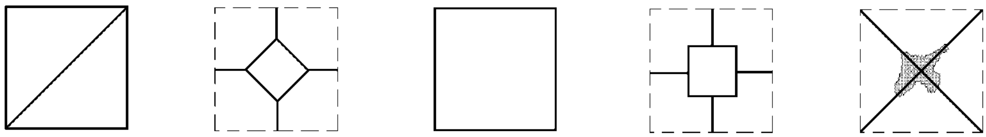

FIGURE 3 Typical fissure configurations in a cell $Y \subset \mathbb{R}^2$, the last picture corresponds to a case with both segments and junctions.

generated by the N-polytopes G_j, and the fissure part $\mathcal{F} = \cup_{k \in Z^N} \cup_{j=1}^{l} (\partial G_j + k)$, generated by the polytopes faces. The set of porous blocks (matrix part of the domain) is then $\mathcal{M}^\varepsilon = \Omega \cap [\varepsilon \cup_{k \in Z^N} \cup_{j=1}^{l} (G_j + k)]$ and its complement in Ω is the fracture part:

$$\mathcal{F}^\varepsilon = \Omega \setminus \mathcal{M}^\varepsilon = \Omega \cap \left[\varepsilon \bigcup_{k \in Z^N} \bigcup_{j=1}^{l} (\partial G_j + k) \right].$$

In the case of $N = 2$, we are then in the situation described in [9] and the fissure part $\mathcal{F}$ is then made of connected segments and junctions as, for instance, in Fig. 3.

In $\mathbb{R}^3$, the simplest example of such a structure is the standard cubic lattice in which the fissures are cube's faces. In what follows we identify the periodic cell Y with the torus $\mathbf{T}^N = \mathbb{R}^N / \mathbb{Z}^N$ and the space of Y-periodic functions with the space of functions on $\mathbf{T}^N$. In the same way, we identify Y-periodic measures with measures on $\mathbf{T}^N$.

We proceed by introducing a singular periodic Borel measure μ which coincides with the standard $(N-1)$-dimensional Lebesgue measure on the fissure part $\mathcal{F}$, that is on any $(N-1)$-dimensional facet Γ of a polytope G_i, μ is the measure product of the Dirac mass in the direction orthogonal to Γ with the usual surface Lebesgue measure on Γ. By the definition of μ we have $\mu(Y \setminus \mathcal{F}) = 0$.

The space $H^1(Y, d\mu)$ is defined as the closure of $C^\infty(Y)$-functions by the norm

$$\|\varphi\|^2_{H^1(Y,\, d\mu)} = \int_Y \varphi^2(y) d\mu(y) + \int_Y |\nabla \varphi(y)|^2 d\mu(y).$$

Further details and the properties of singular Sobolev spaces can be found in [5,13].

Remark 1 As was pointed out in [9] (Proposition 1, Section 1.1), only the tangential component of the gradient of a function $\varphi \in H^1(Y, d\mu)$ on $\mathcal{F}$ is uniquely defined; the normal component can be chosen arbitrarily.

In the sequel, for the sake of simplicity we will restrict ourselves to the Y-periodic fissure structures defined above; an example of such a fissures set is shown in Fig. 3. It is, however, clear that more general structures could also be considered, as soon as they are μ-strongly connected on $\mathbb{R}^N$ (see [5] Sec. 4 and [13] Section 1 for the definition).

3. HOMOGENIZATION PROCEDURE

Let us consider the boundary value problem for Eq. (1) with the homogeneous Dirichlet boundary condition on the boundary $\partial\Omega$ and the initial condition $u(x, 0) = u_0(x)$,

$x \in \Omega$. If we introduce a measure $\mu^\varepsilon(dx) = \varepsilon^N \mu(dx/\varepsilon)$ so that $\mu^\varepsilon(B) = \varepsilon^N \mu((1/\varepsilon)B)$ for any Borel set B then the weak formulation of Problem (1) reads

$$\int_\Omega u^\varepsilon(x,t)v(x)\,dx + \int_\Omega u^\varepsilon(x,t)v(x)\,d\mu^\varepsilon - \int_\Omega u_0(x)v(x)\,dx - \int_\Omega u_0(x)v(x)\,d\mu^\varepsilon$$
$$+ \int_0^t \left[k_2 \int_\Omega \nabla u^\varepsilon \cdot \nabla v\,d\mu^\varepsilon + \varepsilon^2 k_1 \int_\Omega \nabla u \cdot \nabla v\,dx \right] ds$$
$$= \int_0^t \int_\Omega fv\ dx\,ds, \quad \forall v \in H_0^1(\Omega, d\mu^\varepsilon + dx). \tag{2}$$

For each $\varepsilon > 0$ this problem is well-posed, and its solution is an element of the functional space

$$V = \{v\colon v \in L_2(0,T;\ H_0^1(\Omega, d\mu^\varepsilon + dx)),\ \frac{\partial v}{\partial t} \in L_2(0,T;\ H^{-1}(\Omega, d\mu^\varepsilon + dx))$$

where $H^{-1}(\Omega, d\mu^\varepsilon + dx)$ is the dual space to $H_0^1(\Omega, d\mu^\varepsilon + dx)$ with respect to the $L_2(\Omega, d\mu^\varepsilon + dx)$ norm.

In order to deduce the *a priori* estimates we multiply Eq. (1) by u^ε and integrate it over $\Omega \times (0,T)$. After integration by parts this gives

$$\frac{1}{2}\int_\Omega (u^\varepsilon(x,t))^2\,dx + \frac{1}{2}\int_\Omega (u^\varepsilon(x,t))^2\,d\mu^\varepsilon$$
$$+ \varepsilon^2 k_1 \int_0^t \int_\Omega |\nabla u^\varepsilon(x,s)|^2\,dx\,ds + k_2 \int_0^t \int_\Omega |\nabla u^\varepsilon(x,s)|^2\,d\mu^\varepsilon\,ds$$
$$= \frac{1}{2}\int_\Omega (u_0(x))^2\,dx + \frac{1}{2}\int_\Omega (u_0(x))^2\,d\mu^\varepsilon + \int_0^t \int_\Omega f(x,s)u^\varepsilon(x,s)\,dx\,ds \tag{3}$$

One can easily derive from the latter relation the following estimates

$$\|u^\varepsilon\|_{L_2(0,T;\,L_2(\Omega,dx))} \le C \tag{4}$$

$$\|u^\varepsilon\|_{L_2(0,T;\,L_2(\Omega,d\mu^\varepsilon))} \le C \tag{5}$$

$$\varepsilon\|\nabla u^\varepsilon\|_{L_2(0,T;\,L_2(\Omega,dx))} \le C \tag{6}$$

$$\|\nabla u^\varepsilon\|_{L_2(0,T;\,L_2(\Omega,d\mu^\varepsilon))} \le C \tag{7}$$

In order to pass to the limit in matrices set we apply the usual two-scale convergence arguments (see [1,17]) with ad-hoc test functions: the estimates (4) and (6) imply:

$$u^\varepsilon \overset{2}{\rightharpoonup} \tilde{u}(x,t,y) \quad \text{two-scale in } L_2(\Omega \times [0,T]; L_2(Y,dy)) \tag{8}$$

$$\varepsilon\nabla u^\varepsilon \overset{2}{\rightharpoonup} \nabla_y \tilde{u}(x,t,y) \quad \text{two-scale in } L_2(\Omega \times [0,T]; L_2(Y,dy)) \tag{9}$$

with $\tilde{u} \in L_2(\Omega \times [0,T]; H^1_{\text{per}}(Y,dy))$.

We now apply two-scale convergence technique adapted to singular measures; for the definition and basic results on two-scale convergence in the case of singular measures we refer to [5]. Here we recall the definition.

Definition 1 We say that a family of functions $v^\varepsilon(x) \in L_2(\Omega, d\mu^\varepsilon)$, $\|v^\varepsilon\|_{L_2(\Omega, d\mu^\varepsilon)} \le C$, two-scale converges, as $\varepsilon \to 0$, to a function $v^0(x,y) \in L_2(\Omega \times Y, dx \otimes d\mu(y))$ if for any $\psi(x,y) \in C(\Omega, L_2^{\text{per}}(d\mu))$ we have

$$\lim_{\varepsilon\to 0} \int_\Omega v^\varepsilon(x)\psi(x, x/\varepsilon)\, d\mu^\varepsilon = \int_{\Omega\times Y} v^0(x,y)\psi(x,y)\, dx\, d\mu(y).$$

In what follows for brevity we keep the notation

$$v^\varepsilon \overset{2}{\rightharpoonup} v^0(x,y) \quad \text{in } L_2(\Omega;\ L_2(Y, d\mu))$$

for the two-scale convergence with respect to a singular measure μ.

The estimates (5) and (7) imply the existence of a subsequence (still denoted by u^ε) such that

$$u^\varepsilon \overset{2}{\rightharpoonup} u^0(x,t) \quad \text{two-scale in } L_2(\Omega \times [0,T]; L_2(Y, d\mu)) \tag{10}$$

$$\nabla u^\varepsilon \overset{2}{\rightharpoonup} \nabla_x u^0(x,t) + \nabla_y u^1(x,t,y) \quad \text{two-scale in } L_2(\Omega \times [0,T]; L_2(Y, d\mu)) \tag{11}$$

with the periodic measure μ defined above; moreover, $u^0(x,t)$ is an element of $L_2(0,T;\ H_0^1(\Omega, dx))$ and $u_1(x,t,y)$ is a function periodic in y, belonging to $L_2(\Omega\times [0,T]; H^1(Y, d\mu))$.

The next statement describes the structure of elements of $H_0^1(\Omega, dx + d\mu^\varepsilon)$. In what follows all the Sobolev spaces on $\mathcal{F}^\varepsilon$ are related to the standard $(N-1)$-dimensional Lebesgue measure on $\mathcal{F}^\varepsilon$.

LEMMA 1 *The space $H_0^1(\Omega, dx + d\mu^\varepsilon)$ consists, for each $\varepsilon > 0$, of $H_0^1(\Omega, dx)$-functions such that their trace on $\mathcal{F}^\varepsilon$ is a $H^1(\mathcal{F}^\varepsilon)$-function.*

Proof The space $H_0^1(\Omega, dx + d\mu^\varepsilon)$ is continuously embedded in $H_0^1(\Omega, dx)$. Thus, for any $u \in H_0^1(\Omega, dx + d\mu^\varepsilon)$, the trace $u|_{\mathcal{F}^\varepsilon}$ is well-defined as an element of $H^{1/2}(\mathcal{F}^\varepsilon)$. Since u also belongs to $H_0^1(\Omega, d\mu^\varepsilon)$, the gradient of $u|_{\mathcal{F}^\varepsilon}$ w.r.t. the tangential variables is uniquely defined and, moreover, belongs to $L_2(\mathcal{F}^\varepsilon)$ (see [9]). This completes the proof.

It is convenient to introduce the function

$$\hat{u}(x,t,y) = \tilde{u}(x,t,y) - u^0(x,t), \tag{12}$$

with $\tilde{u}(x,t,y)$ and $u^0(x,t)$ defined in (8) and (10). We show that $\hat{u}(x,t,y)$ satisfies homogeneous Dirichlet boundary conditions on $\mathcal{F}$.

LEMMA 2 *The function $\hat{u}$ is an element of $L_2(\Omega \times [0,T]; H^1(Y, dy))$ whose trace on $\mathcal{F}$ is equal to* 0.

Proof The fact that $\hat{u} \in L_2(\Omega \times [0,T]; H^1(Y, dy))$ is a standard result of the usual two-scale method. Let $\psi(y)$ be a smooth periodic function on Y, and denote by $\psi_\gamma(y)$ the product of $\psi(y)$ and the characteristic function of a γ-neighborhood of $\mathcal{F}$, and by $\psi_0(y)$ the trace of $\psi(y)$ on $\mathcal{F}$. From Eq. (10) we have

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega (u^\varepsilon - u^0)\varphi(x,t)\psi_0(x/\varepsilon)\,d\mu^\varepsilon\,dt = 0$$

for any $\varphi \in C_0^\infty(\Omega \times [0,T])$. Considering (6), by the usual continuity of trace arguments, we get

$$\left|\int_0^T\int_\Omega (u^\varepsilon - u^0)\varphi(x,t)\psi_0(x/\varepsilon)\,d\mu^\varepsilon dt - \frac{1}{2\gamma}\int_0^T\int_\Omega (u^\varepsilon - u^0)\varphi(x,t)\psi_\gamma(x/\varepsilon)\,dx\,dt\right| \le C\sqrt{\gamma}$$

uniformly in $\varepsilon > 0$. Passage to the limit, as $\varepsilon \to 0$, gives

$$\overline{\lim_{\varepsilon\to 0}}\left|\frac{1}{2\gamma}\int_0^T\int_\Omega (u^\varepsilon - u^0)\varphi(x,t)\psi_\gamma(x/\varepsilon)\,dx\,dt\right| \le C\sqrt{\gamma}.$$

On the other hand, from (8) and (12) it follows

$$\lim_{\varepsilon\to 0}\frac{1}{2\gamma}\int_0^T\int_\Omega (u^\varepsilon - u^0)\varphi(x,t)\psi_\gamma(x/\varepsilon)\,dx\,dt = \frac{1}{2\gamma}\int_0^T\int_\Omega\int_Y \hat{u}(x,t,y)\psi_\gamma(y)\varphi(x,t)\,dx\,dt\,dy.$$

The last two relations imply

$$\left|\frac{1}{2\gamma}\int_0^T\int_\Omega\int_Y \hat{u}(x,t,y)\psi_\gamma(y)\varphi(x,t)\,dx\,dt\,dy\right| \le C\sqrt{\gamma}.$$

Passing to the limit as $\gamma \to 0$ and using the continuity of trace arguments give

$$\int_0^T\int_\Omega\int_Y \hat{u}(x,t,y)\psi_0(y)\varphi(x,t)\,dx\,dt\,d\mu(y) = 0.$$

Since $\varphi(x,t)$ is an arbitrary function, this implies

$$\lim_{\gamma\to 0}\int_Y \hat{u}(x,t,y)\psi_0(y)\,dy = 0$$

for almost all x and t, and, finally, $\hat{u}|_{\mathcal{F}} = 0$, which completes the proof.

We proceed by characterizing the function $\hat{u}(x,t,y)$. Let us substitute in (2) a test function v of the form $v = \psi(x/\varepsilon)\varphi(x)$ where $\varphi \in C_0^\infty(\Omega)$ and ψ is a smooth

Y-periodic function with $\psi|_{\mathcal{F}} = 0$. Since μ^ε only charges $\mathcal{F}^\varepsilon$, this gives after simple rearrangements

$$\int_\Omega \psi(x/\varepsilon)\varphi(x)u^\varepsilon(x,t)\,dx - \int_\Omega \psi(x/\varepsilon)\varphi(x)u_0(x)\,dx$$
$$+ k_1 \int_0^t \int_\Omega (\varepsilon\nabla u^\varepsilon(x,s))(\varepsilon\nabla\psi(x/\varepsilon))\varphi(x)\,dx\,ds$$
$$+ k_1\varepsilon \int_0^t \int_\Omega (\varepsilon\nabla u^\varepsilon(x,s))\nabla\varphi(x)\psi(x/\varepsilon)\,dx\,ds = \int_0^t \int_\Omega f(x,s)\varphi(x)\psi(x/\varepsilon)\,dx\,ds. \tag{13}$$

Passing to the two-scale limit yields

$$\int_\Omega \int_Y \varphi(x)\psi(y)\hat{u}(x,t,y)\,dx\,dy + \int_\Omega \int_Y \varphi(x)\psi(y)u^0(x,t)\,dx\,dy$$
$$- \int_\Omega \int_Y \varphi(x)\psi(y)u_0(x)\,dx\,dy + \int_0^t \int_\Omega \int_Y \nabla_y\hat{u}(x,s,y)\nabla_y\psi(y)\varphi(x)\,dx\,dy\,ds$$
$$= \int_0^t \int_\Omega \int_Y f(x,s)\varphi(x)\psi(y)\,dx\,dy\,ds.$$

This implies, in turn, for almost every $x \in \Omega$ the relation

$$\int_Y \psi(y)\hat{u}(x,t,y)\,dy + \int_Y \psi(y)u^0(x,t)\,dy - \int_Y \psi(y)u_0(x)\,dy$$
$$+ \int_0^t \int_Y \nabla_y\hat{u}(x,s,y)\nabla_y\psi(y)\,dy\,ds = \int_0^t \int_Y f(x,s)\psi(y)\,dy\,ds. \tag{14}$$

In order to write the problem in a differential form we should show that the initial condition for $\hat{u}$ makes sense.

Taking in (2) the same test function as in (13) and integrating over time interval $[t_1, t_2]$, $0 \le t_1 < t_2 \le T$, using (6) we get

$$\left| \int_\Omega \psi(x/\varepsilon)\varphi(x)u^\varepsilon(x,t_2)\,dx - \int_\Omega \psi(x/\varepsilon)\varphi(x)u^\varepsilon(x,t_1)\,dx \right|$$
$$\le k_1 \left| \int_{t_1}^{t_2} \int_\Omega (\varepsilon\nabla u^\varepsilon(x,s))\{(\varepsilon\nabla\psi(x/\varepsilon))\varphi(x) + (\varepsilon\psi(x/\varepsilon))\nabla\varphi(x)\}\,dx\,ds \right|$$
$$+ \left| \int_{t_1}^{t_2} \int_\Omega f(x,s)\varphi(x)\psi(x/\varepsilon)\,dx\,ds \right| \tag{15}$$
$$\le Ck_1 \|\psi\|_{H^1_{\mathrm{per}}(Y)} \|\nabla\varphi\|_{L_\infty(\Omega)} \int_{t_1}^{t_2} \{\|\varepsilon\nabla u^\varepsilon(s)\|_{L_2(\Omega)} + \|f(s)\|_{L_2(\Omega)}\}\,ds$$
$$\le Ck_1 \|\psi\|_{H^1_{\mathrm{per}}(Y)} \|\nabla\varphi\|_{L_\infty(\Omega)} \sqrt{t_2 - t_1}.$$

By passing to the limit in (13), the two-scale limit $\tilde{u}$ inherits this Hölder continuity property. Namely, we have

$$\left|\int_\Omega\int_Y \psi(y)\varphi(x)\tilde{u}(x,t_2,y)\,dx\,dy - \int_\Omega\int_Y \psi(y)\varphi(x)\tilde{u}(x,t_1,y)\,dx\,dy\right| \\ \leq Ck_1\|\psi\|_{H^1_{\mathrm{per}}(Y)}\|\nabla\varphi\|_{L_\infty(\Omega)}\sqrt{t_2-t_1},$$

and, consequently, the initial condition for $\hat{u}$ makes sense.

Now, the identity (14) can be rewritten in the following differential form, with the Y-periodic function $\hat{u}(x,t,y)$:

$$\begin{cases} \dfrac{\partial}{\partial t}\hat{u} - \Delta_y\hat{u} = f(x,t) - \dfrac{\partial}{\partial t}u^0(x,t) & \text{in } (0,T)\times Y \\ \hat{u}|_{\mathcal{F}} = 0, \qquad \hat{u}|_{t=0} = 0 \end{cases} \tag{16}$$

where x appears as a parameter. Equivalently, in view of (12) one can write

$$\begin{cases} \dfrac{\partial}{\partial t}\tilde{u} - \Delta_y\tilde{u} = f(x,t) & \text{in } (0,T)\times Y \\ \tilde{u}|_{\mathcal{F}} = u^0(x,t), \qquad \tilde{u}|_{t=0} = u_0(x). \end{cases} \tag{17}$$

Our next goal is to describe the structure of the function $u^1(x,t,y)$. Using in (2) a test function of the form $\varepsilon\varphi(x)\psi(x/\varepsilon)$ with $\varphi\in C_0^\infty(\Omega)$ and with $\psi(y)\in C^\infty_{\mathrm{per}}(Y)$, one has after integration by parts

$$\begin{aligned} &\varepsilon\int_\Omega \psi(x/\varepsilon)\varphi(x)u^\varepsilon(x,t)\,dx + \varepsilon\int_\Omega \psi(x/\varepsilon)\varphi(x)u^\varepsilon(x,t)\,d\mu^\varepsilon \\ &\quad - \varepsilon\int_\Omega \psi(x/\varepsilon)\varphi(x)u_0(x)\,dx - \varepsilon\int_\Omega \psi(x/\varepsilon)\varphi(x)u_0(x)\,d\mu^\varepsilon \\ &\quad + \varepsilon^3k_1\int_0^t\int_\Omega \nabla u^\varepsilon(x,s)\nabla[\psi(x/\varepsilon)\varphi(x)]\;dx\;ds \\ &\quad + \varepsilon k_2\int_0^t\int_\Omega \nabla u^\varepsilon(x,s)\psi(x/\varepsilon)\nabla\varphi(x)\,d\mu^\varepsilon\,ds \\ &\quad + k_2\int_0^t\int_\Omega (\nabla u^\varepsilon(x,s))(\varepsilon\nabla\psi(x/\varepsilon))\varphi(x)\,d\mu^\varepsilon\,ds \\ &= \varepsilon\int_0^t\int_\Omega f(x,s)\varphi(x)\psi(x/\varepsilon)\;dx\,ds. \end{aligned}$$

Clearly, all the terms here except the last one on the left-hand side, are vanishing as $\varepsilon\to 0$. Hence, taking the two-scale limit and considering (11), we get

$$\int_Y \nabla_y\psi(y)\nabla_x u^0(x,t)\,d\mu = -\int_Y \nabla_y\psi(y)\nabla_y u^1(x,t,y)\,d\mu. \tag{18}$$

Now, by the usual separation of variables method, we can represent $u^1(x,t,y)$ as follows

$$u^1(x,t,y) = \chi^l(y)\frac{\partial}{\partial x_l}u^0(x,t).$$

where $\chi^l(y)$ are the minimizers in the variational problems:

$$\inf_{\psi\in H^1_{\mathrm{per}}(Y,d\mu)}\int_Y (e^l + \nabla_y\chi^l(y))^2\,d\mu,$$

and where e^l stands for the lth vector of the canonical basis in $\mathbb{R}^N$. These problems are well-posed, see [14], and the functions χ^l are uniquely defined up to an additive constant.

At the last step we determine $u^0(x,t)$. To this end we substitute in (2) a test function $\varphi(x)\in C_0^\infty(\Omega)$. This gives

$$\begin{aligned}
&\int_\Omega \varphi(x)u^\varepsilon(x,t)\,dx + \int_\Omega \varphi(x)u^\varepsilon(x,t)\,d\mu^\varepsilon - \int_\Omega \varphi(x)u_0(x)\,dx - \int_\Omega \varphi(x)u_0(x)\,d\mu^\varepsilon\\
&\quad + \varepsilon k_1\int_0^t\int_\Omega \varepsilon\nabla u^\varepsilon(x,s)\nabla\varphi(x)\,dx\,ds + k_2\int_0^t\int_\Omega \nabla u^\varepsilon(x,s)\nabla\varphi(x)d\mu^\varepsilon\,ds\\
&\quad = \int_0^t\int_\Omega f(x,s)\varphi(x)\,dx\,ds.
\end{aligned}$$

Taking the two-scale limit here and considering (10)–(12) and (17), we get

$$\begin{aligned}
&(1+|\mathcal{F}|)\int_\Omega \varphi(x)u^0(x,t)\,dx + \int_\Omega\int_Y \hat{u}(x,t,y)\,dy\varphi(x)\,dx\\
&\quad - (1+|\mathcal{F}|)\int_\Omega\int_Y \varphi(x)u_0(x)\,dx\,dy\\
&\quad + \int_0^t\int_\Omega\int_Y\left(\delta_{ij} + \frac{\partial}{\partial y_i}\chi^j(y)\right)d\mu\,\frac{\partial}{\partial x_i}u^0(x,t)\frac{\partial}{\partial x_j}\varphi(x)\,dx\,ds\\
&= \int_0^t\int_\Omega\int_Y f(x,s)\varphi(x)\,dx\,dy\,ds, \qquad (19)
\end{aligned}$$

where $|\mathcal{F}|$ stands for the $(N-1)$-dimensional volume of $\mathcal{F}$.

Finally, the Eqs. (12), (17) and (19) lead to the following result:

THEOREM 1 *Let u^ε be a solution of Problem* (1) or (2). *Then when $\varepsilon\to 0$, the solution u^ε two-scale converges in $L_2(\Omega\times[0,T];L_2(Y,d\mu))$, like in* (10) *and* (11), *to u^0, where u^0 is the unique solution of*:

$$\begin{aligned}
&|\mathcal{F}|\frac{\partial}{\partial t}u^0 - \sigma_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u^0 = f(x,t) - \frac{\partial}{\partial t}\int_Y \tilde{u}(x,t,y)\,dy \quad \text{in } \Omega\times[0,T]\\
&\frac{\partial}{\partial t}\tilde{u}(x,t,y) - \Delta_y\tilde{u}(x,t,y) = f(x,t) \quad \text{in } (0,T)\times Y\\
&\tilde{u}(x,t,\cdot)|_{\mathcal{F}} = u^0(x,t), \quad \tilde{u}|_{t=0} = u_0(x)\\
&u^0|_{\partial\Omega} = 0, \quad u^0|_{t=0} = u_0(x) \qquad (20)
\end{aligned}$$

with

$$\sigma_{ij} = \int_Y \left(\delta_{ij} + \frac{\partial}{\partial y_i} \chi^j(y) \right) d\mu. \tag{21}$$

In the same way as in Problem (17) one can verify that the initial conditions in the latter problem make sense. The well-posedness of this system and, in particular, the uniqueness of a solution have been shown in earlier works on double porosity, for instance in [2].

4. THIN STRUCTURES AND SINGULAR CONSTRUCTIONS: COMMUTATIVENESS OF THE DIAGRAM

We consider in this section the standard double porosity model stated in a medium with fissures of positive thickness $\delta > 0$, and show that the "singular" problem considered above can be obtained by passing to the limit in this standard model, as the fissures thickness δ tends to zero. Then we compare the effective problems obtained by letting ε and δ tend to zero in different order, and prove that the corresponding diagram is commutative.

In order to describe the geometry of the fracture part and of the matrices set, we use the same collection of N-polytopes $G_1, G_2, \dots, G_l$ as in the previous section, and denote by $\mathcal{V}_\delta(\partial G_i)$ a δ-neighborhood of ∂G_i, $i = 1, 2, \dots, l$. We define

$$\mathcal{F}_\delta = \bigcup_{j \in Z^N} \bigcup_{i=1}^{l} (\mathcal{V}_\delta(\partial G_i) + j)$$

and

$$\mathcal{M}_\delta = \mathbb{R}^N \setminus \mathcal{F}_\delta$$

and denote by μ^δ the periodic measure whose density is equal to $1/(2\delta)$ in $\mathcal{F}_\delta$ and 0 in $\mathcal{M}_\delta$. This measure μ^δ converges weakly, as $\delta \downarrow 0$, to the measure μ on Y and, respectively, $\mu^{\varepsilon,\delta}$ converges weakly in Ω, as $\delta \downarrow 0$, to the measure μ^ε for each $\varepsilon > 0$. Then we set $\mathcal{F}_\delta^\varepsilon = \Omega \cap \varepsilon\mathcal{F}_\delta$, $\mathcal{M}_\delta^\varepsilon = \Omega \cap \varepsilon\mathcal{M}_\delta$ and $\mu^{\delta,\varepsilon}(B) = \varepsilon^N \mu^\delta(1/\varepsilon B)$ for any Borel set B in $\mathbb{R}^N$.

Now one can state in Ω the following problem, corresponding to the flow in a domain Ω made of two parts, the fissured part $\mathcal{F}_\delta^\varepsilon$ and the matrix set $\mathcal{M}_\delta^\varepsilon$,

$$\begin{aligned} &\gamma^{\varepsilon,\delta}(x) \frac{\partial}{\partial t} u_\delta^\varepsilon = \operatorname{div}(a^{\varepsilon,\delta}(x) \nabla u_\delta^\varepsilon) + f(x,t) \quad \text{in } \Omega \times (0,T), \\ &u_\delta^\varepsilon|_{t=0} = u_0(x) \quad \text{in } \Omega; \qquad u_\delta^\varepsilon|_{\partial\Omega} = 0, \quad t \in (0,T), \end{aligned} \tag{22}$$

where

$$\gamma^{\varepsilon,\delta} = \begin{cases} 1 & \text{on } \mathcal{M}_\delta^\varepsilon, \\ \dfrac{1}{2\delta} & \text{on } \mathcal{F}_\delta^\varepsilon, \end{cases} \qquad a^{\varepsilon,\delta} = \begin{cases} k_1 \varepsilon^2 \mathrm{Id} & \text{on } \mathcal{M}_\delta^\varepsilon, \\ \dfrac{k_2}{2\delta} \mathrm{Id} & \text{on } \mathcal{F}_\delta^\varepsilon. \end{cases}$$

First we are going to pass to the limit, as $\delta \downarrow 0$, in Problem (22) for arbitrary fixed $\varepsilon > 0$.

PROPOSITION 1 *For any fixed $\varepsilon > 0$ the family $\{u^\varepsilon_\delta\}$ of solutions of Problem* (22) *converges in $L_2(0, T;\ H^1_0(\Omega, d\mu^{\varepsilon,\delta} + dx))$, as $\delta \downarrow 0$, to a solution u^ε of Problems* (1) and (2).

Proof First, we write down the weak formulation of (22); this gives

$$\begin{aligned}\frac{1}{2\delta}\int_{\mathcal{F}^\varepsilon_\delta} u^\varepsilon_\delta(x,t)\varphi(x)\,dx + \int_\Omega u^\varepsilon_\delta(x,t)\varphi(x)\,dx - \frac{1}{2\delta}\int_{\mathcal{F}^\varepsilon_\delta} u_0(x)\varphi(x)\,dx - \int_\Omega u_0(x)\varphi(x)\,dx \\ + \frac{k_2}{2\delta}\int_0^t\int_{\mathcal{F}^\varepsilon_\delta} \nabla u^\varepsilon_\delta(x,s)\nabla\varphi(x)\,dx\,ds + k_1\varepsilon^2\int_0^t\int_\Omega \nabla u^\varepsilon_\delta(x,s)\nabla\varphi(x)\,dx\,ds \\ = \int_0^t\int_\Omega f(x,s)\varphi(x)\,dx\,ds \end{aligned} \tag{23}$$

for any $\varphi \in C^\infty_0(\Omega)$. The *a priori* estimates are straightforward; multiplying (22) by u^ε_δ and integrating over $\Omega \times (0, t)$, one gets

$$\|u^\varepsilon_\delta\|_{L_2(\Omega\times[0,T],(d\mu^\varepsilon_\delta+dx)\otimes dt)} \le C, \tag{24}$$

$$\begin{aligned}\varepsilon\|\nabla u^\varepsilon_\delta\|_{L_2(\Omega\times(0,T),dx\otimes dt)} \le C,\\ \|\nabla u^\varepsilon_\delta\|_{L_2(\Omega\times(0,T),d\mu^\varepsilon_\delta\otimes dt)} \le C.\end{aligned} \tag{25}$$

Using the compactness arguments developed in [15], we obtain from the bounds (24), (25) that, up to a subsequence, u^ε_δ converges weakly, as $\delta \downarrow 0$, in $L_2(0, T;\ H^1_0(\Omega, d\mu^\varepsilon_\delta + dx))$ to a function $v^\varepsilon \in L_2(0, T;\ H^1_0(\Omega, d\mu^\varepsilon + dx))$. Passing to the limit in (23) gives

$$\begin{aligned}\int_{\mathcal{F}^\varepsilon} v^\varepsilon(x,t)\varphi(x)\,d\mu^\varepsilon + \int_\Omega v^\varepsilon(x,t)\varphi(x)\,dx - \int_{\mathcal{F}^\varepsilon} u_0(x)\varphi(x)\,d\mu^\varepsilon - \int_\Omega u_0(x)\varphi(x)\,dx \\ + k_2\int_0^t\int_{\mathcal{F}^\varepsilon} \nabla v^\varepsilon(x,s)\nabla\varphi(x)\,d\mu^\varepsilon ds + k_1\varepsilon^2\int_0^t\int_\Omega \nabla v^\varepsilon(x,s)\nabla\varphi(x)\,dx\,ds \\ = \int_0^t\int_\Omega f(x,s)\varphi(x)\,dx\,ds.\end{aligned} \tag{26}$$

This problem coincides with Problem (22) and, thus, in view of the uniqueness of its solution, we have $v^\varepsilon = u^\varepsilon$.

It remains to show that solutions of homogenized δ-problems converge, as $\delta \to 0$, to the solution of Problem (20). For each fixed $\delta > 0$ the two-scale limit problem is well-known (see, for instance [2]), it reads

$$\begin{aligned}|\mathcal{F}_\delta|\frac{\partial}{\partial t}u^0_\delta - \sigma^\delta_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u^0_\delta = f(x,t) - \frac{\partial}{\partial t}\int_{\mathcal{M}_\delta}\tilde u_\delta(x,t,y)\,dy \quad \text{in } \Omega\times(0,T),\\ \frac{\partial}{\partial t}\tilde u_\delta(x,t,y) - \Delta_y\tilde u_\delta(x,t,y) = f(x,t) \quad \text{in } (0,T)\times\mathcal{M}_\delta \\ \tilde u_\delta(x,t,\cdot)|_{\mathcal{F}_\delta} = u^0_\delta(x,t), \qquad \tilde u_\delta|_{t=0} = u_0(x)\\ u^0_\delta|_{\partial\Omega} = 0, \qquad u^0_\delta|_{t=0} = u_0(x),\end{aligned} \tag{27}$$

where $|\mathcal{F}_\delta|$ is the Lebesgue measure of $\mathcal{F}_\delta$ on the torus.

PROPOSITION 2 *The solution* $(u_\delta^0, \tilde{u}_\delta)$ *of Problem* (27) *converges, as* $\delta \downarrow 0$, *in* $L_2(0,T;H_0^1(\Omega)) \times L_2(\Omega \times [0,T];\ H_0^1(Y,dy))$ *to the solution* $(u^0, \tilde{u})$ *of Problem* (20).

Proof As was shown in [10], the effective tensor σ^δ converges, as $\delta \to 0$, to the "singular" effective tensor σ defined by formula (21) in the previous section.

In a standard way one can derive, uniformly in δ, *a priori* estimates for solutions of (27), and select a weakly convergent subsequence

$$u^\delta \underset{\delta\downarrow 0}{\rightharpoonup} v^0 \quad \text{in } L_2(0,T;H_0^1(\Omega)),$$

$$\tilde{u}^\delta \underset{\delta\downarrow 0}{\rightharpoonup} \tilde{v} \quad \text{in } L_2(\Omega \times (0,T));H^1(Y,dy)).$$

Writing down the weak formulation of (27) and passing in it to the limit as $\delta \to 0$, we conclude that the pair of the limit functions $v^0, \tilde{v}$ is necessary a solution of Problem (20). Finally, the convergence of the whole sequence follows from the uniqueness of a solution of Problem (20), and we have $v^0 = u^0$ and $\tilde{v} = \tilde{u}$, which ends the proof of commutativity.

We summarize the above statements in the following theorem.

THEOREM 2 *The diagram*

$$\begin{array}{ccc} A^{\varepsilon,\delta} & \underset{\varepsilon\downarrow 0}{\dashrightarrow} & A^{\mathrm{eff},\delta} \\ \Big\downarrow{\scriptstyle \delta\downarrow 0} & & {\scriptstyle \delta\downarrow 0}\Big\downarrow \\ A^{\varepsilon,\mathrm{sing}} & \underset{\varepsilon\downarrow 0}{\dashrightarrow} & A^{\mathrm{eff},\mathrm{sing}} \end{array} \tag{28}$$

is commutative.

The convergence related to the bottom arrow of the diagram was proved in Section 2 and other convergences were justified in the present section (Fig. 4).

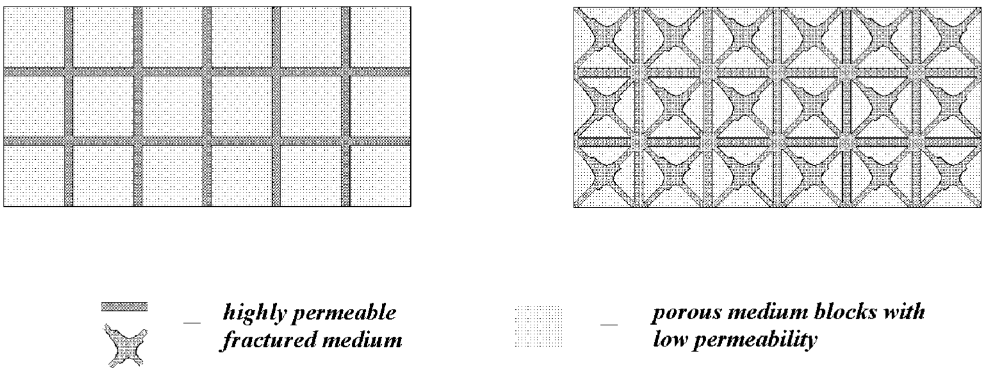

FIGURE 4 Examples of a system of thin channels and porous medium.

Remark 2 Similar technique applies to the case of thin fissures when the fissure thickness δ depends on ε so that $\delta(\varepsilon)$ tends to 0 as $\varepsilon \to 0$. Combining the techniques of this work with the commutativeness result from [10] one can prove, in this case, a statement similar to that of Theorem 1 above.

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References

[1] G. Allaire (1992). Homogenization and two-scale Convergence. *SIAM J. Math. Anal.*, **23**, 1482–1518.
[2] T. Arbogast, J. Douglas and U. Hornung (1990). Derivation of the double porosity model of single phase flow via homogenization theory. *SIAM J. Math. Anal.*, **21**(4), 823–836.
[3] G.I. Barenblatt, I.P. Zheltov and N.I. Kochina (1960). Basic concepts of the theory of seepage of homogeneous liquids in fissured rocks. *Prikl. Math. Mech.*, **24**, 852–864.
[4] J. Bear, C.F. Tsang and G. de Marsily (Eds.) (1993). *Flow and Contaminant Transport in Fractured Rock*. Academic Press, New York – San Francisco – London.
[5] G. Bouchitté and I. Fragalà (2001). Homogenization of thin structures by two-scale method with respect to measure. *SIAM J. Math. Anal.*, **32**(6), 1198–1226.
[6] A. Bourgeat, M. Goncharenko, M. Panfilov and A. Pankratov (1999). A general double porosity model. *C. R. Acad. Sci. Paris Sér. II*, **327**, 1245–1250.
[7] A. Bourgeat, S. Luckhaus and A. Mikelić (1996). Convergence of the homogenization process for a double porosity model of immiscible two phase flow. *SIAM J. Appl. Math.*, **27**(6), 1520–1543.
[8] A. Bourgeat, A. Mikelić and A.L. Piatnitski (1998). Random double porosity model. *C. R. Acad. Sci. Paris Sér. I Math.*, **327**(1), 99–104.
[9] G.A. Chechkin, V.V. Jikov (Zhikov), D. Lukkassen and A.L. Piatnitski (2002). On homogenization on networks and junctions. *Asymptotic Analysis*, **30**(1), 61–80.
[10] D. Cioranescu and J. Saint Jean Paulin (1999). Homogenization of reticulated structures. *Applied Mathematical Sciences*, Volume 136. Springer Verlag, Berlin – New York.
[11] U. Hornung (Ed.) (1997). Homogenization and porous media. *IAM*, Volume 6. Springer Verlag, Berlin – New York.
[12] V.V. Jikov (Zhikov) (1996). Connectedness and homogenization. Examples of fractal conductivity. *Russian Academy of Sciences, Sbornik: Mathematics*, **187**(8), 1109–1147.
[13] V.V. Jikov (Zhikov) (1998). Weighted sobolev spaces. *Russian Academy of Sciences. Sbornik: Mathematics*, **189**(8), 1139–1170.
[14] V.V. Jikov (Zhikov) (1999). On the homogenization technique for variational problems. *Functional Analysis and its Applications*, **33**(1), 11–24.
[15] V.V. Jikov (Zhikov) (2000). On one extension and application of the method of two-scale convergence. *Russian Academy of Sciences. Sbornik: Mathematics*, **191**(7), 31–72.
[16] S.A. Nazarov (1995). General averaging procedure for selfadjoint elliptic systems in many-dimensional domains, including thin ones. *Algebra and Analysis*, **7**(5), 1–92. (English translation: *St. Petersburg Math. J.*, **7**(5), 681–748.
[17] C. Nguetseng (1989). A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.*, **20**, 608–623.
[18] G.P. Panassenko (2000). Method of asymptotic partial decomposition of rod structures. *International Journal for Computational Civil and Structural Engineering*, **1**(2), 57–70.